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GEOMETRY
OF
FOUR DIMENSIONS

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GEOMETRY
OF
FOUR DIMENSIONS

BY

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IN

GRATITUDE

PREFACE.

This book is devoted to the intrinsic geometry of configurations postulated in abstract space. Generally, the space is supposed to be of four dimensions: there are, however, three sections dealing with configurations in multiple space of an unspecific number of dimensions. The treatment is analytical. Throughout, there is a preferential selection of processes which, actually used for space of four dimensions, can be formally extended for use in any multiple space.

The completely comprehensive amplitude, whether the number of its dimensions be four or be any greater integer, is supposed to be uncurved: that is to say, there is the assumption that, along every direction through every point, a straight line can be drawn which lies wholly within the amplitude however far the course of the line be continued in either sense. An equivalent description is provided by the characteristic property that all the geodesics of the amplitude itself shall be straight lines.

Such a space is a purely mathematical conception, as strictly ideal as are the self-consistent mathematical conceptions of the planar space and the solid space assumed in the axioms of Euclid, as strictly ideal also as are the equally self-consistent mathematical conceptions of the elliptic and the hyperbolic triple spaces for which the Euclidean axiom of parallels is waived. Mathematically, there is no impassable bar against adventure into spaces of more than the three dimensions of experience; nor is there any requirement that the spaces shall be absolutely uncurved or be specially curved. The main difficulties consist, of the calculations needed to construct the analytical expression of properties of the configurations, and of geometrical interpretation to be given to novel analytical results. A useful guide to the latter is constituted by the geodesics of the respective configurations of the various types.

The restriction to quadruple space, here adopted, is imposed from a desire to consider, with reasonable fullness, the fundamental possibilities which present themselves in the earliest extension of the number of dimensions beyond the three that are characteristic of average experience.

Within that experience, there is a sufficiently clear perception that (what in common parlance is called) ordinary space is a space of three dimensions. When specific description is desired, the dimensions are often described as length, breadth, height: also, to use another set of indicative terms, as right-and-left, backwards-and-forwards, up-and-down. Measurements in these three directions (but not solely in this group of three directions) are independent of one another, in the sense that no selected two of the three measures can be combined to provide the remaining unselected measure. Further, the three directions in unrestricted combinations suffice for the expression of all analogous measurements.

Contemplative minds often attain intellectual satisfaction when they discern correspondences, between their observations of an external world which they call real, and the results of logical theory which they call abstract in relation to such observations. An occasional tendency to interchange the real and the abstract in such correspondences, as though they are equivalent, can interfere with strictness of argument, can even prove obnoxious to lucidity of the statements in which reasoned thought is expressed. One consequence is not rare: confusion is caused in the presentation of a new theory, launched in the name of science. An obvious illustration is provided by the notion of a fourth dimension. The notion was propounded by the mathematicians: the added dimension, which they have incorporated in an abstract geometry, is coordinate in quality and in possibilities with the three dimensions familiar already in conceptions of triple space. The fourth dimension has been appropriated by some physicists, for what is called a 'natural' geometry, without any requirement as to coordination in quality and in possibilities with the three dimensions familiar to experience. Then an external interest becomes aroused in a section of the unmathematical and unphysical public which, moved by new scientific generalities, can be titillated by a phrase or be attracted by a catch-word: its attention can even be arrested by startling headlines in a newspaper announcing an alleged 'discovery' of a fifth dimension. Attempts, to understand the announcement and to obtain some comprehension of its significance, are unconditional failures. The very notion of dimension, in a mathematical sense, can hardly be formulated in the tabloid shape that is desired. Space is a fundamental notion, in some sense probably common to all minds. Ideas as to its nature, that may be acquired after the earliest vague perception of extension, are founded on conceptions suggested by observations of the external world. Gradually, the specification of extension ceases to be unfamiliar, though only after continued reflection: gradually, some mental grasp of the dimensions of space is attained.

It has long been customary to describe the triple space of human experience as an uncurved (or flat) space. Sometimes the description is tacitly based on an assumption that the linear property is obvious, though there is the usual difficulty of establishing that property when once the obviousness is seriously challenged. Often an argument is based upon a fact that all the observations, obtained by instruments of measurement and subsequently combined by calculation, accord with the assumption. Yet inferences from such observations cannot be deemed accurate, beyond the degree of accuracy of the observations themselves. There is a limit of minuteness below which observations have not been effected; the magnitudes, to which expression is given in theories of the atom and of the electron, as yet defy attempts at direct measurement. Equally beyond the present possibilities of direct measurement are the generous distances assigned in speculations about the magnitude of the universe, being untold millions of light-years. Thus for the very small in theory as for the very large in nature, the degree of accuracy attainable by measurements imposes a limit upon the range of knowledge, which can be acclaimed as the result of observation and of experiment. Of course, for most scientific purposes of the type styled practical, a hypothesis, that our triple space is flat, will and does suffice: it is an adequate working hypothesis. But its efficiency in limited practice does not constitute a proof that the space of our experience is actually uncurved.

On the other hand, mathematical calculations made in the various theories of relativity point towards a suggestion that, on the grand scale, the space of our universe is not uncurved. Such an inference demands consideration. It should, however, be remarked that an inference from far-reaching theories, lavishly propounded at the present epoch, cannot be accepted as established fact, in the absence of any observations which might constitute a qualified establishment or might offer a simple verification of the inference.

Nevertheless, the inference is a legitimate consequence of the theories. Within the restricted range of its hypothetical validity, it entails other consequences, similarly legitimate and similarly restricted in validity. The very notion of curvature, for any amplitude such as a curve or a surface or a region, mathematically implies some uncurved configuration by reference to which the curvature is defined and is estimated; and that implied uncurved configuration either constitutes, or exists in, some ideal space more comprehensive than the curved amplitude. Thus the curvature at any point of a curve, which exists in some Euclidean space of two dimensions or in some Euclidean space of three dimensions, is a mathematical measure of the current deviation of the curve from straightness, the measure being framed by reference to successive straight lines in the two-fold space or in the three-fold space which contains the one-dimensional curve. The curvature at any point of a surface is estimated, initially, by the curvatures of its organic geodesic curves: and all these curvatures are estimated by deviations from straight lines in the Euclidean triple space containing the two-dimensional surface. Later, for reasons which are explicitly mathematical, it is found convenient to adopt a measure (often called the Gauss measure) of curvature of the surface: it is the product of the two principal (maximum and minimum) curvatures of the geodesics through the point of the surface. Such a measure, however, is an incomplete representation of the curvature of the surface: taken alone, it would make the sole measure of a curved developable surface equal to zero and therefore would make the measure the same as for a plane. In actual fact, the Gauss measure is only one of the two principal measures of curvature: another, and different measure, being the sum (instead of the product) of the two principal curvatures of geodesics, must be retained in order to provide for all adequate estimate of the curvature of the surface. But, for the immediate issue, the important conception is that, in framing a mathematical estimate of the curvature of a curved configuration, we require an uncurved space, more extensive dimensionally than the curved configuration in question.

Accordingly, if the objective triple space of the universe is actually found to possess the quality of curvature, whenever and in whatever way that quality may be revealed by measurement, the mathematical conception of that curvature would exact the existence of some further space of ultimate reference. That further space would be more extensive in dimensional range than the three-dimensional objective space of the universe; and, in mathematical conception, it would be characterised by complete linearity.

Thus there would arise a demand, certainly for one dimension, possibly for more than a single dimension, additional to the three dimensions, still possessed by the space of experience, if and after curvature of 'ordinary' space shall have been established. In the mathematical conception, the additional dimension or the additional dimensions would be of the same intrinsic extensional character as the three dimensions already known for space. But, once more, the contribution of another dimension, thus required through the mathematical conception of the presumed curvature of the triple space of observation, is no proof of a corresponding objective existence of another dimension. At the utmost, there is a suggestion of further dimension or dimensions: objective existence, if it is to be accepted in credence, must be established, directly or indirectly, by observation and (probably) through measurement.

It is not always thus, either in bygone discussions or in current theories, that the existence of more than three dimensions has been adumbrated. The mathematician, on the one hand, finds no difficulty in postulating any additional number of dimensions for his abstract space: on the other hand, he does not profess to be dealing with topics other than the properties of abstract space. But since the days of Lagrange, perhaps even from earlier days, the objective existence of at least a fourth dimension has claimed occasional notice from mathematical physicists. Sometimes it has been as passing a fancy as was Lagrange's statement, which gives æsthetic symmetry to his analysis by a suggestion (never again mentioned by him) that time can be regarded as a fourth dimension. In recent days, it has assumed the status of an intellectual conviction, almost amounting to an article in a creed of relativity, that time is a fourth dimension, to be ranked (presumably as coordinate in quality) with the three dimensions customary in common conceptions of space.

A mighty stride is needed if we are to pass, from the direct interpretation of mathematical formulæ which purport to represent relations in nature, forward to an esoteric doctrine that, because certain lengths (as three constituent variables of one kind) and time (as one constituent variable of a different kind) are convenient for a mathematical explanation of the universe, these four constituent variables must compose an irresoluble order of four coordinate dimensions. The term 'dimension,' as used to denote the conception of range in space, seems a misnomer if used equally to denote the conception of range in time.

There is, of course, the established practice of graphical convenience (and also of brevity in description) by which the word 'dimension' is substituted for the word 'variable.' But, in the ordinary exercise of that practice, there is no underlying assumption that the substituted word then denotes an ordered extension, coordinate with the magnitudes similarly represented in an arbitrary illustrative diagram. Two types of instances may be adduced.

Maintained as a variable, time can be conventionally represented along a line in a diagram, simultaneously with variables that denote distances. But other variables, such as temperature, potential, pressure, statistical aggregates, all of them as completely distinct from space-variables as is time, can be (and are) conventionally represented also along lines in diagrams. However convenient such diagrammatic uses may be for mathematical investigation, they do not constitute either time, or temperature, or potential, or pressure, or any statistical aggregate, as a dimension isotelic with the ordered dimensions of triple space.

Again, to cite the purely mathematical use of the diagrammatic practice of substituting the term dimension for the term variable, it is a commonplace that the aggregate of all spheres in Euclidean triple space requires four independent variables for its mathematical expression, these variables being the three coordinates of a centre and the length of a radius: such an aggregate is frequently described as four-dimensional, even though the complete configuration occurs in a three-dimensional comprehending space. Similarly, the mathematical expression of the aggregate of straight lines in the same Euclidean triple space demands four independent variables: their configuration is frequently termed four-dimensional, in consequence of the quadruple variation. In such descriptions, the word dimension is a graphic substitute for the word variable: there is no shadow of a contention that the four dimensions constitute a combination of the customary three dimensions of perceived space with an added fourth spatial dimension. Whatever convenience may arise in practice, and whatever advantage may be derived in their utilisation for the attainment of results, the diagrams are an imagined representation of entities under discussion and do not establish the intrinsic ordering of the nature of those entities.

Certainly in abstract geometry, whatever be the number of dimensions, no discrimination is made among them as regards significance or capacity for representing suitable magnitudes. From the beginning it is assumed, usually without passing hint and without specific mention of the assumption, that the dimensions are coordinate among themselves, not in the special mathematical meaning of the term, but in its customary non-scientific sense. Accurate calculations constitute additional knowledge within the domain limited by the fundamental postulates, explicit and implicit; but a claim for accuracy can justifiably be conceded only within that domain. In framing explanations of the physical universe, new definitions may be imported which can utilise calculations within the domain of abstract geometry; and calculations, thus based on observed phenomena, may lead to new results of a verified or a verifiable character. When this end is attained, the explanatory theory can be regarded as a working hypothesis. But we may not therefore conclude that certain conventions, incidentally assumed for the purposes of the calculation, are raised to the rank of established truths; the safe judgment is that, within the range tested, the working hypothesis is trustworthy. The history of human speculations shews that physical theories, necessarily limited by the degree of accuracy of observations, and modified not infrequently by conceptions otherwise alien to their range, fluctuate with the changing thought of successive generations; they can be revolutionised, they can be abandoned, in the light of new discoveries.

The analytical method, which this treatise employs for the discussion of the intrinsic geometry of the various types of configuration in homaloidal quadruple space, is substantially an amplification of the method devised by Gauss in his treatment of curved surfaces in homaloidal triple space.

There is, however, one supplementary section, which is entirely devoted to the invariants and covariants of configurations. For the analysis necessary in the section, Lie's theory of continuous groups has been applied, in preference to the methods of the absolute differential calculus (the calculus of tensors). Of this calculus, developed by Ricci, Levi-Civita, and others, from earlier investigations by Christoffel, a connected exposition has been given in Levi-Civita's book[*] which contains also several applications to mechanics, optics, and general relativity. The extension, to n-fold space, of the Gauss theory of surfaces was initiated by Riemann[†]; and it has provided a fertile field of research for a multitude of investigators. A systematic account of this extended geometry and of many of its developments has been given by Eisenhart[‡]. It seems practically certain that every method of dealing analytically with the invariants of configurations entails elaborate analysis that apparently is inevitable in some form. What seems to me the relative advantage of the group-method accrues from one characteristic property which is possessed by that method alone. The process requires the integration of a complete Jacobian system of simultaneous partial differential equations of the first order; the requisite integration is achieved through a use of the known algebraical theory of invariantive forms. Every integral of such a system of partial equations is known to be expressible in terms of a specific limited number of algebraically independent integrals. and the adequate aggregate of such integrals is provided, so far as concerns concomitants for quadruple space, by means of binariants and ternariants. The geometrical interpretation of every member of the aggregate, up to each stage, has been obtained; and thus every other covariantive magnitude at that stage, whether actually obtained or not, is expressible algebraically in terms of the interpreted members of the aggregate, and therefore can have its value expressed in terms of the fundamental invariants and covariants. In other words, the aggregate thus established is adequate for the expression of all concomitants; and consequently the group-method provides the means of constructing concomitants directly, and also of selecting, at each stage, a minimum aggregate of independent concomitants in terms of which (and in terms of which alone) all possible concomitants of the configurations can be expressed.

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[*] *Lezioni di calcolo differenziale assoluto*: an English translation was published (1927) by Blackie and Son.

[†] In his dissertation cited later (vol. ii, p. 134, footnote).

[‡] In his book *Riemannian Geometry* (Princeton, 1926).

The book is composed of five chief sections.

The first of these sections, mainly preparatory in scope, is restricted to the uncurved configurations which can occur in quadruple space. they are termed a line, a plane, and a flat, being of one, of two, and of three, dimensions respectively. One outstanding feature of the treatment is a prevalent use of parametric representation. in the case of planes by two parameters, and in the case of flats by three parameters. this representation is of persistent recurrence in the subsequent differential geometry of curved configurations. The section occupies a larger portion of the book than would have been its allotted share, had there been any accessible volume which presented the special kind of treatment of the topics considered.

The second section is devoted to the intrinsic geometry of skew curves in the quadruple space. By the continued application of the analysis in the first section, the whole framework of such a curve, in relation to all the different kinds of its curvatures, is constructed. A chapter is added, outlining the complete aggregate of results, analytical and descriptive, for curves in n-fold space.

The third section is occupied with surfaces existing freely in quadruple space: that is to say, the surfaces are represented analytically solely in that space, without reference to possible three-dimensional regions which, existing themselves in the quadruple space, might contain the surfaces. Some attention is given to curvature properties of general curves on such surfaces. It appears that, except for one set of results belonging specially to any surface in a space which is quadruple (that is, of precisely double the dimensional range of the surface), the main descriptive and intrinsic properties of a surface depend upon the properties of the superficial geodesics. Indeed, it may here be pointed out that, in all the curved configurations, the geodesic lines are the fundamentally important elements, alike for surfaces existing freely in the quadruple space, for curved regions in that space and for surfaces lying within such curved regions: they are as significant as are straight lines in Euclidean geometry.

The fourth section has been assigned to regions, being curved triple spaces within the homaloidal quadruple space: but (with one later exception) only general regions are considered, and no specific attention is paid to particular regions such as are given by the simplest algebraical equations. As part of the discussion of general regions, ample attention is given to surfaces actually existing within a region (but not unrestrictedly existing in quadruple space, as in the third section) and to curves also existing within a region.

Properties of the curvature of triple regions within a homaloidal quadruple space (and, as will be indicated immediately, of primary amplitudes within n-fold homaloidal space) call for a passing remark The consideration of regional geodesics leads to an estimate of linear curvature of the region in any direction at a point, being the circular curvature of the regional geodesic through the direction, and it appears that, at every point, the region is characterised by three principal measures of linear curvature, with three principal directions for geodesics. Upon the circular curvature of regional geodesics is based the estimate of superficial curvature of the region in any orientation at the point; and it appears that there are two distinct measures of superficial curvature. One of these measures is the Riemann measure of curvature of the region: it is an extension of the customary Gauss measure of curvature for a surface in triple Euclidean space. The other measure of superficial curvature of a region seems to have been ignored or to have escaped notice: it is an additive measure of the linear curvatures, and it is the corresponding extension of the so-called 'mean' measure of curvature for a surface in triple Euclidean space. Further, each of these two measures of superficial curvature has its own three principal values, each of which can be expressed solely in terms of the three principal measures of linear curvature of the region; and the orientations for these respective principal values are located by the directions of the geodesics for the principal measures of linear curvature. Finally, at every point, there is a measure of volumetric curvature of the region, characteristic of the region at the point without regard to orientation of any kind. That measure of volumetric curvature is the product of the three principal measures of linear curvature. it bears the same relation to a globe as is borne to a sphere by the Gauss measure for surfaces in triple space.

The results, relating to the different curvatures of a region in quadruple space, practically compel an addition to this fourth section, in the form of an outline of the curvatures of a primary $(n-1)$-fold amplitude in homaloidal n-fold space. Again, it is found that the linear curvature of the amplitudinal geodesic is fundamental in the discussion. There are $n-1$ principal values of that linear curvature; and there are corresponding $n-1$ principal directions in the amplitude, at right angles to one another. There are two measures of superficial curvature at every point of the amplitude—the Riemann measure. and the additive measure, and each of these measures has $\frac{1}{2}(n-1)(n-2)$ principal values, expressible solely in terms of the $n-1$ principal measures of linear curvature, and settled as to orientation by the pair-combinations of directions of those principal measures. There are three measures of volumetric curvature at every point of the amplitude: each of these measures has its principal values, such principal values being expressible in terms of the principal measures of linear curvature, and the regional orientations for the values are settled by the triadic combinations of the directions of the principal measures of linear curvature. There are grades of curvature, of successive spatial dimensions, up to a final (and sole) measure, which is of dimensionality $n-1$ and is the product of the $n-1$ principal measures of linear curvature. For the determination of the principal values of every special grade of measure of curvature, and also for the determination of the respective associated orientations of these several principal values, the principal measures of the linear curvature of geodesics and the corresponding directions of those principal measures are completely sufficient.

To return to configurations within a region in homaloidal quadruple space, it is to be noted that, when kinds and measures of curvature of a surface in the region are considered, two kinds of linear curvature have to be taken into account. One such kind is due to the relative deviation between a geodesic of the surface and a regional geodesic touching that superficial geodesic. The other kind arises through the circular curvature of that regional geodesic touching the surface. The two kinds, in appropriate combinations, give the circular measure, the torsion, and the tilt, of the geodesic on the surface. It is to be remarked that the same kind of combination occurs, though the significance is somewhat obscured, for any curve on a surface in homaloidal triple space; because the Gauss theory takes account of the circular curvature of the superficial geodesic touching the curve (it is the curvature of the normal section of the surface), and it takes account also of the deviation of the curve from that superficial geodesic, such deviation being measured by the geodesic curvature of the curve.

Two chapters have been interpolated between the fourth and the fifth sections. The first of them is concerned with illustrations of the general properties of a region when a particular region—in this instance, an ovoid as represented by the equation of the second degree analogous to that of an ellipsoid—is selected. The second of the two chapters is devoted to minimal problems. Throughout the book, geodesics have an ample share of attention, even if viewed as a minimal problem in a single independent variable: accordingly, in this second chapter, minimal surfaces in free space and in a region (being a problem in two independent variables), and minimal regions (being a problem in three independent variables), are considered, though the problems are not fully resolved for the regional configurations.

The fifth and concluding section deals with the theory of invariantive concomitants of all possible types belonging to the possible general configurations of the various dimensions: curves, surfaces in free space and in a region, and regions. The values of intrinsic magnitudes, though not the topographical relations, of any configuration of any number of dimensions in the quadruple space, must be unaffected by change of site and by change of orientation of the configuration; and they must remain substantially unaffected by every modification or alteration in the parametric representation of the configuration. Accordingly, if it is possible to construct an aggregate of forms, which are invariantive through all changes of site, all changes of orientation, and all changes of parametric representation, and which are sufficiently numerous in each grade as to comprehend all invariantive forms up to that grade, such a constructed aggregate will contain, actually in explicit form or potentially by fitting expression, all invariants and therefore the expression for all intrinsic magnitudes. what remains is the geometrical interpretation of the expressions which are obtained. Now Lie's theory of continuous groups provides the criteria, which are necessary and sufficient to secure that all these requirements of invariance are met; that theory, therefore, is employed. The concurrent analysis, needed in applying the method, consists of the construction and the subsequent integration of the necessary simultaneous partial differential equations of the first order, these constituting a complete Jacobian system. The algebraical identification of an adequate number of simultaneous integrals at each stage is provided by the known results of the theory of the concomitants of binary homogeneous forms and of ternary homogeneous forms; and, at each of the early stages, the constituents of the respective adequate aggregates are obtained and their geometrical interpretation is derived. The net result is an independent establishment, based solely upon the theory of invariantive forms, of the essential and intrinsic geometrical magnitudes of the respective configurations.

The corresponding investigation for n-fold homaloidal space, pursued by the same method with the necessary purely formal extensions, demands the utilisation of a full theory of the simultaneous concomitants of homogeneous forms in $2, 3, \ldots, n-1$, variables and in the associated variables of the different classes corresponding to the sub-spaces of different dimensions within the respective amplitudes.

When the included topics of the book are thus specified, it is only proper to indicate omissions also—at least, those omissions which definitely belong solely to the selected range of geometry. There is no question of dealing with comparatively external subjects, such as the use of multi-dimensional geometry in the theory of rays in heterogeneous media and in the mechanics of theories of relativity. There is no discussion of various attempts to prove the objective existence of a fourth dimension nor any consideration of the arguments which seek to establish time as that fourth dimension. The whole development has been carefully restricted to solely geometrical relations of configurations in a homaloidal space of four dimensions, which are coordinate with one another and are unrestrictedly interchangeable without regard to possible objective significance. Even so, within this range, not a few important subjects have been omitted, of which the following may be mentioned: the general foundations of multiple geometry; the modes of representation (such as conformal and geodesic) of one region in another; the properties of families of orthogonal amplitudes; the Levi-Civita theory of geodesic parallels; and the characteristic features of a curved region which is developable into a region of more restricted curvature such as a flat.

Moreover this recital of omitted subjects does not profess to be complete. One indeed, which seems suitable for investigation, is almost urgent. The square of the arc-element of a surface is represented, analytically, by a homogeneous quadratic differential form in two parameters. The Gauss theory of surfaces shews that, when certain conditions are satisfied by the coefficients of that form and by the coefficients of an associated form, the arc belongs to a surface in homaloidal triple space. But the inference cannot be drawn, in the absence of the associated form or under a failure to satisfy all the conditions: yet, as an isolated datum, the postulated expression for the arc-element remains apparently the same in character for a surface in free quadruple space as for a surface within a curved region in that space and for a surface in any multiple space. The same problem is presented by the postulation of the square of the arc-element in a triple region, as a homogeneous quadratic differential form in three parameters. What is required is a determination of the tests for the smallest number of dimensions of that homaloidal space, which shall contain a configuration characterised by any postulated arc-element represented by means of a quadratic differential form in two parameters or in three parameters respectively.

The main subject of geometry, outside the section of two dimensions and of three dimensions associated with the name of Euclid as the representative of the old Greek geometers, is often divided into the ranges of metageometry and hypergeometry.

Metageometry for the most part deals with the self-consistent geometries of two dimensions and of three dimensions, which emerge when the Euclidean axiom of parallels (or any one of its modern equivalents) is set aside

Hypergeometry deals with the geometry of flat spaces of four or more than four dimensions and of all types of configurations in such spaces. Two main directions have marked the growth of knowledge in hypergeometry which, in effect, began during the last century. The algebraic analysis for n-fold space originated in some of Cayley's early papers[*] in 1844 and 1846. The differential analysis for n-fold space really originated with Riemann's dissertation[†] of 1854, though general vector analysis had made an earlier appearance in Grassmann's *Ausdehnungslehre* of 1844. Some notion of the extent of literature devoted to hypergeometry may be derived from Sommerville's bibliography[‡] published in 1911; and some indication of the amount, published since that date, is provided by the selected papers quoted in Eisenhart's *Riemannian Geometry*. As regards the rudiments of the subject, mention may be made of Schoute's *Mehrdimensionale Geometrie*[§].

But I am not attempting even an outline of the history of the subject. In the comparatively few instances, when previous investigations have been of direct assistance to me, due references are given as a matter of course. Practically, however, while use has been made of the results obtained in the theories of differential equations, of groups, and of algebraic invariants, the whole book has grown in continuous development of its own from first to last, without occasion on my part to select or modify or incorporate the researches of other writers.

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[*] *Coll. Math. Papers*, Vol I, No. 11, No. 50

[†] "Ueber die Hypothesen welche der Geometrie zu Grunde liegen," *Gott. Wiss. Abh.*, Bd. xiii (1866), pp 254–269, and reprinted in the editions of Riemann's *Gesammelte Mathematische Werke*. An English translation is due to Clifford, *Mathematical Papers*, pp 55–69.

[‡] *Bibliography of non-Euclidean Geometry*, University of St Andrews, 1911.

[§] In two volumes, Sammlung Schubert, Leipzig (1902, 1905)

In the production of the book, as on many occasions in past years, I have received the unfailing and responsive assistance of the Staff of the Cambridge University Press. For that assistance, I return them my sincere thanks.

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A. R. F.

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CHAPTER I.

PRELIMINARY NOTIONS.

Amplitudes: dimensions.

1. The geometry, which is here considered, is based upon certain conceptions acquired in the gradual development of the assumptions and definitions of the older geometries of two dimensions and of three dimensions. These conceptions will be stated by title: usually, they will be postulated without attempts at logical foundation or metaphysical analysis, though sometimes definitions are provided which are little more than tautological explanations of the generalised title.

Among such conceptions are *position*, *extent*, *direction*, and (perhaps) *rotation*, to take some initial examples. They are fundamental in one mode of constructing an abstract geometry. In the customary mathematical presentation here adopted, we deal with *relative* position and *relative* direction: that is, the mathematical estimates of position and of direction are constructed in association with some frame of reference. Often, a frame of reference is implicitly assumed without specific description. That frame may itself, in turn, be relative to some other frame unless there is an explicit assumption of definite fixture: but, for the most part, the epithet 'relative' is omitted while it is tacitly implied.

A point is taken as the fundamental entity in the scheme of geometry; as a fundamental entity, it is taken to be irresoluble and to be void of all properties save position. The complete range of an infinitude of all the points, selected for the consideration of a generic body of issues, is called an *amplitude*, and an amplitude is sometimes said to consist of all its points. In an amplitude, a point is determined uniquely by its position. and the same point can belong to distinct amplitudes, being determined (or determinable) uniquely in each of them.

Amplitudes are said to be of a number of dimensions; occasionally, they are said to possess a number of degrees of freedom; occasionally, they are called manifolds, specifically designated by an integer. The respective integers for an amplitude under the various titles are, of course, one and the same.

We have amplitudes of one dimension or amplitudes with one degree of freedom. In ordinary parlance, they may be described as one-fold: the implication being that, in each instance, the range (which may be limited, or which may be unlimited) is one-fold in extent.

We have amplitudes of two dimensions or amplitudes with two degrees of freedom. They may be described as two-fold: the implication being that, in each instance, the range (which may be limited wholly, or may be limited partially, or may be completely unlimited) is two-fold in extent: or, what is a more explicit equivalent implication, that the range can be compounded of a couple of one-fold ranges, neither of which can be resolved or be transformed into the other or can be extended so as to include the other. Moreover, there are alternatives in the selection of the one-fold ranges that compose a two-fold range: each adequate selection must be equivalent to every other adequate selection, as regards the comprehended extent.

And so for amplitudes of dimensions, in number greater than two. In each instance, the specification is made by the appropriate integer n, and the description, by a title such as n-fold. That integer is the number of independent one-fold ranges into which the amplitude can be resolved or from which it can be composed; and the range of the amplitude is the aggregate of the component one-fold ranges.

Ultimately resolved, the range of any amplitude, whatever be its dimensions, consists of the aggregate of its points. Now the usual analytical specification of a point in a geometrical amplitude is effected by the assignment of independent variables, in number equal to the number of dimensions of the amplitude within whose range the point is specified. The amplitude itself may be contained, wholly or partially, within another amplitude of larger dimensions. In the range of that larger amplitude, the analytical specification of the representative point will have a different form, appropriate to the wider range: all the forms, if there be more than one for the same point, must be exactly and completely equivalent to one another for that point. Even within a range, a suitable assignment of independent variables is not unique: thus it may consist of Cartesian coordinates, or polar coordinates, or parametric variables characteristic of the particular amplitude: but the effect must be unique, whatever representation be adopted.

Representation of a point.

2. In a given n-fold amplitude, the variables specifying a typical point are often taken to be $x_1, x_2, \ldots, x_n$. These are not necessarily Cartesian coordinates, if they are, the appropriate amplitude is, mathematically, the simplest among those of n dimensions. But it may be some n-fold amplitude in a larger range of $n+m$ dimensions: in that event, the variables are not Cartesian coordinates but can be any characteristic parameters, as is the fact in much of modern investigation concerned with abstract geometry.

In the ensuing discussions, attention will be concentrated mainly upon four-fold amplitudes, as affording the earliest extension beyond what is popularly described as the space of human experience. Many of the results can be enlarged, by mere formalisation, to n-fold amplitudes: such enlargements will, as a rule, be omitted from consideration and even from explicit mention, unless they clarify the properties of a four-fold amplitude For its most general unrestricted representation in such an amplitude, a point requires four variables: these will be denoted by x, y, z, v These variables are usually taken to be Cartesian coordinates of the four-fold amplitude; but this assumption is a convention, not a necessity.

Curve. straight line.

3. An amplitude of one dimension is usually called a *curve.* Sometimes the word 'line' is adopted as an equivalent of the word 'curve,' for describing such an amplitude. but 'line' is usual, and certainly is more convenient, as the title of a particular kind of curve.

Yet a satisfactory definition of a *straight line,* which here will throughout be called a *line,* is not easy to propound for submission to detailed criticism. Sometimes there is implicit reference to intrinsic properties of the amplitude thus named: sometimes the character is indicated by its relations with an external frame of reference: hence various definitions have been propounded.

Thus a straight line has been defined as a curve which lies evenly between any two of its points. But the virtue of the definition lies in the unexplained adverb 'evenly.' If, for example, evenness is a brief characterisation of the property by which any part of the curve can, without any modification of the curve in shape or size, be changed in position so as to be made to fit any other portion of the curve—such as pushing the curve along itself—the definition would include all great circles on the surface of a three-dimensional sphere and all helices of the same pitch on the surface of a three-dimensional circular cylinder. The adopted implicit interpretation of evenness is, in essence, an assumption concerning what usually are called the curvature and the torsion: the assumption is that both are constant, and (for a line) are zero.

A straight line has alternatively been defined as the curve of shortest length between any two of its points. But length, measured in an amplitude, may be a restricted asset of the amplitude: and the shortest length in an amplitude between two points, as connexible positions in the amplitude, need not be (and often will not be) the shortest length between those two points as connexible positions in a different amplitude. Thus the shortest superficial length between two points on a spherical surface in what is called 'ordinary space' is not the same as the shortest length between those two points measured in that space. The definition is descriptive but inadequate.

Again, a straight line has been defined implicitly by a requirement that, if C is any arbitrarily selected point on such a curve joining two points A and B, the direction CA and the direction CB, estimated along the curve (whether they be in the same sense or in opposite senses, according to the position of C), are independent of the selected point C and are the same for all points C on the curve. It is to be noted that, for certain modes of estimating the direction under this definition, the implied definition would admit the spherical great circle and the cylindrical helix.

Once more, the notion of rotation in the surrounding space has been invoked. A straight line then is defined as a curve such that it could be an axis of rotation for one of the more extensive amplitudes in which it exists, provided the rotation requires no deformation, no stretching, no discontinuity, in its operation. The limitations thus imposed on the rotation seem to demand imported properties, which have ultimately been derived through inferences from the characteristics of a straight line, long previously postulated without reference to rotation.

Perhaps the most direct way of estimating straightness is to be found in a frank assumption of *linearity* as an essential quality or limitation in the undefined conception of direction. As a working definition under this assumption, there is a requirement that, at any point A in a straight line, the direction of the line shall be the same in relation to the most extended frame of reference containing the line, whatever point A be chosen. Such a working definition involves the existence of amplitudes more extensive than the line itself: this existence is assumed, definitely, at almost every stage from the beginning of the investigation of the properties of all curves, and therefore is not postulated merely to facilitate the consideration of straight lines.

4. The mathematical (or analytical) representation of a line is obtained by the expression of the linearity of direction, thus postulated as being unchanged throughout the range of the one-fold amplitude constituting (or constituted by) the line. We are assuming a four-dimensional space through practically all the discussions that follow: and all the configurations, which are considered, exist in this quadruple continuum. A frame of reference is adopted, consisting of four axes OX, OY, OZ, OV, usually supposed to be perpendicular to one another in pairs. They are taken as coordinate axes. The line, joining two points A and B which are given, is made uniquely definite by framing the analytical expressions of the inclinations of the line AB to the lines OX, OY, OZ, OV, in succession.

Surface: plane.

5. An amplitude of two dimensions is called a *surface*. The simplest surface is that which is called a plane surface or, more briefly, a *plane*.

As already explained, amplitudes of two dimensions can be regarded as composed of (or as resoluble into) a couple of completely independent typical amplitudes, each of one dimension. In particular, the plane surface is regarded as composed of straight lines, in the following manner. Let A, B, C, be any three points in the quadruple continuum, limited by the sole excluding restriction that they are not to be collinear. Let P denote a current point on the line AB, where P may range also outside the segment AB in both directions; and let Q denote a current point on the line AC, with a freedom of range also outside the segment AC in both directions. For any position of P, and for any position of Q, each point being chosen independently of the other, let R denote a current point on the line PQ, with a similar unrestricted freedom of range outside the segment PQ of that line in both directions. The locus of R, for all positions on PQ, for all selections of P, and for all selections of Q, is called a *plane*.

No essential limitation is imposed by the adoption of AB and AC as the basic lines; it will appear (p. 15) that the same plane is obtained by the adoption of AB and BC, and by the adoption of AC and BC, as basic lines. Basic lines, or guiding lines, for a plane are not unique.

The plane is described as the plane through A, B, C: or, simply, as the plane ABC.

Region: flat.

6. An amplitude of three dimensions is here called a *region*. (The word 'space' seems the natural successor to 'curve' and 'surface' in the ascending grade of dimensions. There is, however, some convenience in reserving the general word 'space' so that it may be used to denote an amplitude of any number of dimensions: in particular, it will be used to indicate the quadruple continuum—as a space of four dimensions—within which the various curves, surfaces, and regions, exist.)

As a plane is a particular surface, composed of lines in the manner already defined, so a region similarly defined in connection with lines and planes is called a *flat*. (The word 'volume' suggests total content rather than range, just as 'area' suggests the total content of a surface or of some portion of a surface rather than its range.) The title *hyperplane* is also frequently used: but a part-use of the word plane, connoting a two-fold range, harmonises ill with what is a three-fold range. Moreover, this title hyperplane is frequently used to denote the most extensive linear space* of $n-1$ dimensions in a space of n dimensions.

The constructive definition of a flat, by regarding it as composed of (or resoluble into) lines and planes and therefore as composed of (or resoluble into) lines alone, is taken as follows. Let A, B, C, D, be any four points in

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* Clifford used the word *homaloidal* to describe all linear spaces of more than two dimensions.

the quadruple continuum, limited by the sole excluding restriction that the four points are not complanar (and, specially therefore, that no three of the four points are collinear). Let R be any current point in the plane ABC, with a freedom of range over the whole of that plane as given by its preceding construction (§ 5). Let S be any current point in the line AD, with a freedom of range outside the segment AD of that line in both directions. Let T be any current point in the line RS, with a freedom of range outside the segment RS of that line in both directions. Then the locus of T, for all positions on the line RS, for all positions of S on the line AD, and for all positions of R in the plane ABC, is called a *flat*.

As with a plane, so with a flat, no essential limitation is imposed by the adoption of ABC as a basic plane and the line AD as a basic line (or by the adoption of AB, AC, AD, as basic lines, or guiding lines): it will appear (p. 18) that the same flat is obtained by the adoption of ABC with BD or with CD as a basic line, or by the adoption of the plane BCD as a basic plane with BA or CA or DA as a basic line, and so for the other possible alternatives. Basic lines, or guiding lines, for a flat are not unique.

The flat is described as the flat through A, B, C, D. or, simply, as the flat $ABCD$.

7. When the comprehensive abstract space is limited to four dimensions, there are no subsidiary amplitudes of that number of dimensions: every four-dimensional amplitude either is a portion, or is composed of portions, of the completely comprehending space. Consequently, in quadruple space, the only kinds of subsidiary amplitudes demanding consideration are those of one dimension, those of two dimensions, and those of three dimensions. Occasions arise when the variables x, y, z, v, of a point in the space are changed to other variables p, q, r, s, by four relations independent of one another: but such relations are a transformation from one quadruple space into another quadruple space. Under the condition that the relations are independent, no amplitude, differing from the complete quadruple space in essential quality, is thereby constituted.

Similarly in n-fold space, no subsidiary n-fold amplitude is to be recognised. There are $n-1$ distinct types of subsidiary amplitudes existent in that space; and they are of $1, 2, \ldots, n-1$, dimensions respectively, whether homaloidal or curved.

Representation of a point.

8. The only characteristic possession of a point is its position, usually estimated in a frame of reference; a point has no dimensions, that is, its range provides no degree of freedom.

In the frame, there are four axes OX, OY, OZ, OV. There is no intrinsic necessity that they should be perpendicular to one another in all the six pair-combinations; but, unless there is either a manifest assumption or an explicit statement to the contrary, complete orthogonality is to be understood. Also, in manifest agreement with the customary convention for the two-dimensional and the three-dimensional geometries, the several directions *OX*, *OY*, *OZ*, *OV*, will be regarded as positive, and the several directions *XO*, *YO*, *ZO*, *VO*, will be regarded as negative.

Compounded from these four axes in pairs, there are six planes of reference, being *XOY*, *YOZ*, *ZOX*, *XOV*, *YOV*, *ZOV*. These six planes are to be regarded as perpendicular to one another, in all the fifteen pairs; the significance of this statement of property, here made dogmatically, will appear later after a discussion of the angular relation of any two planes, when it will be found, e.g., that the perpendicularity of *XOY* and *YOZ* is different from that of *XOY* and *ZOV*.

Compounded from the four axes in threes, there are four flats of reference, being *OYZV*, *OZVX*, *OVXY*, *OXYZ*. These four flats are to be regarded as perpendicular to one another, in all the six pairs; they are also respectively perpendicular to the four axes *OX*, *OY*, *OZ*, *OV*; and, as in the last instance concerning planes, the significance of these statements will appear later after the discussion of the orientation of flats

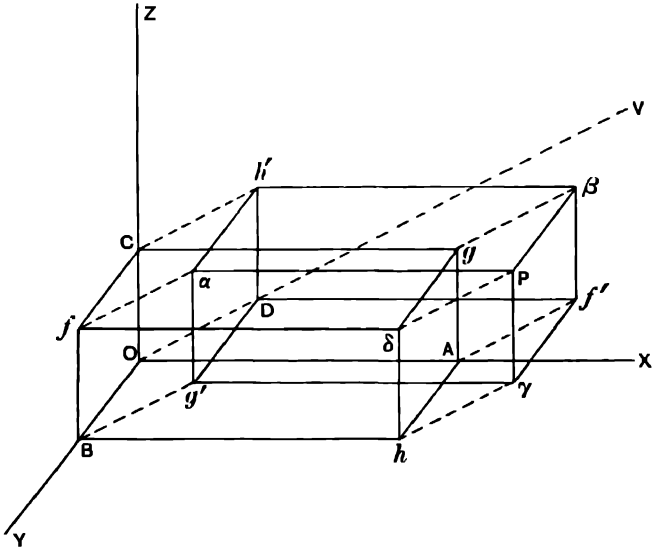

Fig. 1.

9. In the accompanying diagram (Fig 1), the lines *OX*, *OY*, *OZ*, are drawn according to the prevalent convention for three-dimensional space. The fourth axis *OV*, and all lines parallel to *OV*, are represented by dotted lines; and all these parallel directions are imagined to be perpendicular to the three reciprocally perpendicular directions *OX*, *OY*, *OZ*.

The configuration, of which

$$\begin{array}{llll} O, & A, & B, & C, \\ \delta, & f, & g, & h, \end{array}$$

are angular points, is the customary rectangular parallelepiped in the three-dimensional flat $OXYZ$. The configuration, of which

$$\begin{array}{llll} D, & f', & g', & h', \\ P, & \alpha, & \beta, & \gamma, \end{array}$$

are angular points, is the same parallelepiped moved parallel to the axis OV, without change of shape or orientation, all the dotted lengths being equal to OD.

The lines

$$Bh,\ f\delta,\ Cg,\ g'\gamma,\ Df',\ h'\beta,\ \alpha P,$$

are parallel to OX; the lines

$$Cf,\ g\delta,\ Ah,\ h'\alpha,\ Dg',\ f'\gamma,\ \beta P,$$

are parallel to OY; the lines

$$Ag,\ h\delta,\ Bf,\ f'\beta,\ Dh',\ g'\alpha,\ \gamma P,$$

are parallel to OZ; and the lines

$$Af',\ Bg',\ Ch',\ f\alpha,\ g\beta,\ h\gamma,\ \delta P,$$

are parallel to OV.

If the coordinates of P are a, b, c, d, each of the seven specified lines parallel to OX is equal to OA, that is, to a; each of the seven specified lines parallel to OY is equal to OB, that is, to b; each of the seven specified lines parallel to OZ is equal to OC, that is, to c; and each of the seven specified lines parallel to OV is equal to OD, that is, to d.

The coordinates of the various angular points in the diagram, other than P, are as follows:

$$\left.\begin{array}{ll} A \text{ is the point} & a,\ 0,\ 0,\ 0, \\ B \text{ ....} & 0,\ b,\ 0,\ 0, \\ C \text{ ....} & 0,\ 0,\ c,\ 0, \\ D \text{ ....} & 0,\ 0,\ 0,\ d, \end{array}\right\};\qquad \left.\begin{array}{ll} \alpha \text{ is the point} & 0,\ b,\ c,\ d, \\ \beta \text{ ....} & a,\ 0,\ c,\ d, \\ \gamma \text{ ....} & a,\ b,\ 0,\ d, \\ \delta \text{ ....} & a,\ b,\ c,\ 0, \end{array}\right\},$$

and

$$\left.\begin{array}{ll} f \text{ is the point} & 0,\ b,\ c,\ 0, \\ g \text{ ....} & a,\ 0,\ c,\ 0, \\ h \text{ ....} & a,\ b,\ 0,\ 0, \\ f' \text{ ....} & a,\ 0,\ 0,\ d, \\ g' \text{ ....} & 0,\ b,\ 0,\ d, \\ h' \text{ ....} & 0,\ 0,\ c,\ d, \end{array}\right\}.$$

Further, the lines

$$\left.\begin{array}{l} Of,\ A\delta,\ D\alpha,\ f'P, \text{ are parallel to one another} \\ BC,\ hg,\ g'h',\ \gamma\beta,\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\}\text{ and of length } (b^2+c^2)^{\frac{1}{2}};$$

$$\left.\begin{array}{l} Og,\ B\delta,\ D\beta,\ g'P,\ \ldots\ldots\ldots\ldots\ldots \\ CA,\ fh,\ h'f',\ \alpha\gamma,\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\}\text{ and of length } (c^2+a^2)^{\frac{1}{2}},$$

$$\left.\begin{array}{l} Oh,\ C\delta,\ D\gamma,\ h'P,\ \ldots\ldots\ldots\ldots\ldots \\ AB,\ gf,\ f'g',\ \beta\alpha,\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\}\text{ and of length } (a^2+b^2)^{\frac{1}{2}};$$

$$\left.\begin{array}{l} Of',\ B\gamma,\ C\beta,\ fP,\ \ldots\ldots\ldots\ldots\ldots \\ DA,\ g'h,\ h'g,\ \alpha\delta,\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\}\text{ and of length } (a^2+d^2)^{\frac{1}{2}};$$

$$\left.\begin{array}{l} Og',\ C\alpha,\ A\gamma,\ gP,\ \ldots\ldots\ldots\ldots\ldots \\ DB,\ h'f,\ f'h,\ \beta\delta,\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\}\text{ and of length } (b^2+d^2)^{\frac{1}{2}},$$

$$\left.\begin{array}{l} Oh',\ A\beta,\ B\alpha,\ hP,\ \ldots\ldots\ldots\ldots\ldots \\ DC,\ f'g,\ g'f,\ \gamma\delta,\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\}\text{ and of length } (c^2+d^2)^{\frac{1}{2}}.$$

Again, the lines

$$\left.\begin{array}{l} O\alpha \\ AP \end{array}\right\},\quad \left.\begin{array}{l} Bh' \\ h\beta \end{array}\right\},\quad \left.\begin{array}{l} Cg' \\ g\gamma \end{array}\right\},\quad \left.\begin{array}{l} Df \\ f'\delta \end{array}\right\},$$

are of length $(b^2+c^2+d^2)^{\frac{1}{2}}$, each bracketed pair of lines being parallel to one another, the lines

$$\left.\begin{array}{l} O\beta \\ BP \end{array}\right\},\quad \left.\begin{array}{l} Cf' \\ f\gamma \end{array}\right\},\quad \left.\begin{array}{l} Dg \\ g'\delta \end{array}\right\},\quad \left.\begin{array}{l} Ah' \\ h\alpha \end{array}\right\},$$

are of length $(c^2+d^2+a^2)^{\frac{1}{2}}$, in bracketed parallel pairs; the lines

$$\left.\begin{array}{l} O\gamma \\ CP \end{array}\right\},\quad \left.\begin{array}{l} Dh \\ h'\delta \end{array}\right\},\quad \left.\begin{array}{l} Ag' \\ g\alpha \end{array}\right\},\quad \left.\begin{array}{l} Bf' \\ f\beta \end{array}\right\},$$

are of length $(d^2+a^2+b^2)^{\frac{1}{2}}$, in bracketed parallel pairs; and the lines

$$\left.\begin{array}{l} O\delta \\ DP \end{array}\right\},\quad \left.\begin{array}{l} Af \\ f'\alpha \end{array}\right\},\quad \left.\begin{array}{l} Bg \\ g'\beta \end{array}\right\},\quad \left.\begin{array}{l} Ch \\ h'\gamma \end{array}\right\},$$

are of length $(a^2+b^2+c^2)^{\frac{1}{2}}$, in bracketed parallel pairs.

Finally, the eight lines

$$OP,\ A\alpha,\ B\beta,\ C\gamma,\ D\delta,\ ff',\ gg',\ hh';$$

are of length $(a^2+b^2+c^2+d^2)^{\frac{1}{2}}$; and each of them has $\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$, $\frac{1}{2}d$, for its middle point.

In the four-dimensional parallelepiped in the diagram, there are sixteen corners, thirty-two linear edges, twenty-four plane faces, and eight flat regions (or cells).

Parametric representation of points in an amplitude.

10. It may be pointed out, however, that while the method of representing a point by its Cartesian coordinates, each explicitly expressed solely with reference to rectangular axes, is convenient initially, there accrues an advantage, in the intrinsic geometry of configurations, from expressing those coordinates by special reference to the amplitude of which the point is an element.

Thus, when a typical point on a curve is selected, its coordinates x, y, z, v, would be expressed as functions of a parameter; an instance arises in the helix

$$\begin{aligned} x &= a(\sin\beta\cos\delta\cos\gamma s - \sin\delta\sin\gamma s),\\ y &= a(\sin\beta\sin\delta\cos\gamma s + \cos\delta\sin\gamma s),\\ z &= a\cos\beta\cos\gamma s,\\ v &= s\sin\alpha, \end{aligned}$$

where s is the length of arc measured along the curve, and $a, \alpha, \beta, \gamma, \delta$, are inter-related constants.

When a typical point on a surface is selected, x, y, z, v, would be expressed as functions of two parameters; an instance occurs from the spherical surface

$$\left.\begin{aligned} x\cos\alpha\cos\beta + y\cos\alpha\sin\beta + z\sin\alpha &= c\\ x^2 + y^2 + z^2 + v^2 &= a^2 \end{aligned}\right\},$$

for which we may take

$$\begin{aligned} x &= c\cos\alpha\cos\beta + p\sin\alpha\cos\beta + q\sin\beta,\\ y &= c\cos\alpha\sin\beta + p\sin\alpha\sin\beta - q\cos\beta,\\ z &= c\sin\alpha \qquad - p\cos\alpha,\\ v &= (a^2 - c^2 - p^2 - q^2)^{\frac{1}{2}}, \end{aligned}$$

where p and q are the two parameters, varying independently of one another on the surface.

When a typical point in a region is selected, x, y, z, v, would be expressed as functions of three parameters; an instance occurs from the ovoidal region

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} + \frac{v^2}{d} = 1,$$

within which we can take

$$x = \left\{a\frac{(a+p)(a+q)(a+r)}{(a-b)(a-c)(a-d)}\right\}^{\frac{1}{2}},$$

$$y = \left\{b\frac{(b+p)(b+q)(b+r)}{(b-a)(b-c)(b-d)}\right\}^{\frac{1}{2}},$$

$$z = \left\{c\frac{(c+p)(c+q)(c+r)}{(c-a)(c-b)(c-d)}\right\}^{\frac{1}{2}},$$

$$v = \left\{d\frac{(d+p)(d+q)(d+r)}{(d-a)(d-b)(d-c)}\right\}^{\frac{1}{2}},$$

where p, q, r, are the three current parameters of the region (see Chap. XXII).

Representation of a line: direction-cosines.

11. The analytical representation of a (straight) line is obtained simply by using the measure of its inclinations to the respective coordinate axes. In the diagram in § 9 (p. 7), the projections of the point P on the four axes OX, OY, OZ, OV, are A, B, C, D, respectively. We denote by l, m, n, k, the cosines of the angles XOP, YOP, ZOP, VOP, respectively, these four quantities are called the four *direction-cosines* of the line OP. Manifestly,

$$OA = l \,.\, OP, \quad OB = m \,.\, OP, \quad OC = n \,.\, OP, \quad OD = k \,.\, OP,$$

hence, as $OP^2 = OA^2 + OB^2 + OC^2 + OD^2$, we have

$$l^2 + m^2 + n^2 + k^2 = 1,$$

which is a universal relation affecting the four direction-cosines of any line.

Next, suppose the point O to be a, b, c, d, referred to another parallel system of axes: on a line OP through O, suppose the coordinates of P to be x, y, z, v; and denote by l, m, n, k, the direction-cosines of the line *measured from O towards P* along the line. (If a point Q be taken on PO produced through O, the direction-cosines l', m', n', k', of OQ are equal to $-l$, $-m$, $-n$, $-k$.) Also, denote the distance OP by r. Then we have

$$x - a = lr, \quad y - b = mr, \quad z - c = nr, \quad v - d = kr,$$

while

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 + (v-d)^2,$$

and we usually assume, as a convention, that, in deducing a value for r, the positive sign is given to the square root of the last expression. Further,

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = \frac{v-d}{k}.$$

Here, we can regard a, b, c, d, as a fixed point on the line, and x, y, z, v, as a current point on the line; and therefore these three equations can be regarded as *the equations of the line*. If, moreover we take

$$l = \lambda P, \quad m = \mu P, \quad n = \nu P, \quad k = \kappa P,$$

where P is any non-zero magnitude, the equations become

$$\frac{x-a}{\lambda} = \frac{y-b}{\mu} = \frac{z-c}{\nu} = \frac{v-d}{\kappa},$$

where λ, μ, ν, κ, are not now actually equal to, but are only proportional to, the direction-cosines of the line. The latter form is chosen as the typical form of the (three) equations of a line.

Manifestly, the coordinates of any point on a line through a, b, c, d, with direction-cosines l, m, n, k, are

$$x = a + lr, \quad y = b + mr, \quad z = c + nr, \quad v = d + kr,$$

that is, along the direction l, m, n, k, as measured from a, b, c, d. For the rest of the whole line, in the opposite direction from a, b, c, d, we may either reverse the signs of l, m, n, k, or admit negative values of r. When we take r as a parametric variable along the line, we can regard these four equations as the equations of the line. The equations can also be taken in the form

$$x = a + \lambda R, \quad y = b + \mu R, \quad z = c + \nu R, \quad v = d + \kappa R;$$

and then the distance along the line from the fixed point up to any position of the current point is

$$(\lambda^2 + \mu^2 + \nu^2 + \kappa^2)^{\frac{1}{2}} R,$$

where R is the parameter of the current point.

Later (§ 17), it will appear that the three equations of a line need not necessarily occur in this form and that, in any form, three independent equations are necessary and sufficient for the mathematical specification of the line.

12. Next, let a', b', c', d', be any point on the line, distinct from the point a, b, c, d. As the coordinates must satisfy the equations, we must have

$$\frac{a'-a}{\lambda} = \frac{b'-b}{\mu} = \frac{c'-c}{\nu} = \frac{d'-d}{\kappa};$$

and therefore, on the elimination of λ, μ, ν, κ, from the equations of the line, we have

$$\frac{x-a}{a'-a} = \frac{y-b}{b'-b} = \frac{z-c}{c'-c} = \frac{v-d}{d'-d},$$

which are the equations of a (straight) line joining the points a', b', c', d', and a, b, c, d.

Take a point x_1, y_1, z_1, v_1, in the quadruple space, such that

$$x_1 = \frac{\alpha a + \beta a'}{\alpha + \beta}, \quad y_1 = \frac{\alpha b + \beta b'}{\alpha + \beta}, \quad z_1 = \frac{\alpha c + \beta c'}{\alpha + \beta}, \quad v_1 = \frac{\alpha d + \beta d'}{\alpha + \beta},$$

where α and β are any magnitudes. Then

$$\frac{x_1 - a}{a'-a} = \frac{\beta}{\alpha+\beta}, \quad \frac{y_1 - b}{b'-b} = \frac{\beta}{\alpha+\beta}, \quad \ldots,$$

and so for the others: hence

$$\frac{x_1 - a}{a'-a} = \frac{y_1 - b}{b'-b} = \frac{z_1 - c}{c'-c} = \frac{v_1 - d}{d'-d},$$

and therefore the point x_1, y_1, z_1, v_1, lies on the line, whatever values be assigned to α and β. Conversely, any point on the line joining a', b', c', d', to a, b, c, d, can be represented by the four expressions for x_1, y_1, z_1, v_1, above given. If we write

$$\frac{\alpha}{1-\gamma} = \frac{\beta}{\gamma} = \frac{\alpha+\beta}{1},$$

so that γ is a parametric variable taking the place of the ratio $\alpha : \beta$, any point on the line is given by

$$x_1 = a + \gamma(a'-a), \quad y_1 = b + \gamma(b'-b), \quad z_1 = c + \gamma(c'-c), \quad v_1 = d + \gamma(d'-d).$$

The parametric variable γ can range from $-\infty$ to $+\infty$. As it ranges from $-\infty$ to 0, the point ranges along that part of the line between its distant extremity and a, b, c, d, which does not enclose the point a', b', c', d'. As γ ranges from 0 to 1, the point ranges along the portion of the line between a, b, c, d, and a', b', c', d'. As γ ranges from 1 to $+\infty$, the point ranges along that part of the line between its distant extremity and a', b', c', d', which does not enclose the point a, b, c, d.

Ex. 1. Verify that the lines in each pair

$$O\alpha \text{ and } AP, \quad O\beta \text{ and } BP, \quad O\gamma \text{ and } CP, \quad O\delta \text{ and } DP,$$

in the diagram (p. 7) have the same direction-cosines; likewise the lines in each pair

$$\begin{array}{lll} \alpha h \text{ and } h'A, & \beta f \text{ and } f'B, & \gamma g \text{ and } g'C, \\ \delta f' \text{ and } fD, & \delta g' \text{ and } gD, & \delta h' \text{ and } hD, \\ Of \text{ and } f'P, & Og \text{ and } g'P, & Oh \text{ and } h'P. \end{array}$$

Obtain other lines parallel (§ 18) to any pair of these specified lines in the diagram.

Ex. 2. Shew that, if three points a_1, b_1, c_1, d_1; a_2, b_2, c_2, d_2; a_3, b_1, c_1, d_1; are not collinear, any three independent equations of the set

$$\left\| \begin{array}{ccccc} a_1, & b_1, & c_1, & d_1, & 1 \\ a_2, & b_2, & c_2, & d_2, & 1 \\ a_3, & b_3, & c_3, & d_3, & 1 \end{array} \right\| = 0$$

must not be satisfied simultaneously.

Representation of a plane.

13. The analytical representation of a plane can be deduced immediately from the definition already given (§ 5). Let the three points A, B, C, there specified, be a_1, b_1, c_1, d_1; a_2, b_2, c_2, d_2, a_3, b_3, c_3, d_3. A current point P on AB is (§ 12)

$$a_1 + \alpha(a_2 - a_1), \quad b_1 + \alpha(b_2 - b_1), \quad c_1 + \alpha(c_2 - c_1), \quad d_1 + \alpha(d_2 - d_1),$$

where α is a parameter that can range from $-\infty$ to $+\infty$. A current point Q on AC is (§ 12)

$$a_1 + \beta(a_3 - a_1), \quad b_1 + \beta(b_3 - b_1), \quad c_1 + \beta(c_3 - c_1), \quad d_1 + \beta(d_3 - d_1),$$

where β is another (and independent) parameter, with the same range of variation as α. Then a current point R on PQ is (§ 12)

$$x = a_1 + \alpha(a_2 - a_1) + \gamma\left[\{a_1 + \beta(a_3 - a_1)\} - \{a_1 + \alpha(a_2 - a_1)\}\right],$$

where γ is another (and also independent) parameter, with the same range of variation as α and β: that is,

$$x = a_1 + \lambda'(a_2 - a_1) + \mu'(a_3 - a_1),$$

and, similarly,

$$\begin{aligned} y &= b_1 + \lambda'(b_2 - b_1) + \mu'(b_3 - b_1), \\ z &= c_1 + \lambda'(c_2 - c_1) + \mu'(c_3 - c_1), \\ v &= d_1 + \lambda'(d_2 - d_1) + \mu'(d_3 - d_1), \end{aligned}$$

where $\lambda' = \alpha - \gamma\alpha$, $\mu' = \gamma\beta$. Thus λ' and μ' are two parameters: they range, independently of one another, from $-\infty$ to $+\infty$: and, when α, β, γ, are omitted from further consideration, λ' and μ' can be taken as two parameters, with full variation between $-\infty$ and $+\infty$, and independent of one another. Thus x, y, z, v, satisfy two equations

$$\left\| \begin{matrix} x-a_1, & y-b_1, & z-c_1, & v-d_1 \\ a_2-a_1, & b_2-b_1, & c_2-c_1, & d_2-d_1 \\ a_3-a_1, & b_3-b_1, & c_3-c_1, & d_3-d_1 \end{matrix} \right\| = 0.$$

But the locus of R, which is the point x, y, z, v, is the plane ABC. Hence a plane is represented analytically by two equations, each of the first degree in the current coordinates x, y, z, v.

Now let l, m, n, k, be the direction-cosines of the line AB in the direction from A to B; and let l', m', n', k', be those of the line AC in the direction from A to C. Also let $AB = r$, $AC = r'$; and write $\lambda' r = \lambda$, $\mu' r' = \mu$. Then the coordinates of the current point R in the plane ABC are

$$\begin{aligned} x &= a_1 + \lambda l + \mu l', \\ y &= b_1 + \lambda m + \mu m', \\ z &= c_1 + \lambda n + \mu n', \\ v &= d_1 + \lambda k + \mu k'; \end{aligned}$$

and the two equations of the plane become

$$\left\| \begin{matrix} x-a_1, & y-b_1, & z-c_1, & v-d_1 \\ l\ , & m\ , & n\ , & k \\ l'\ , & m'\ , & n'\ , & k' \end{matrix} \right\| = 0.$$

In this form, the equations of the plane place into evidence the distinguishing properties: (i), that the plane passes through the point a_1, b_1, c_1, d_1; and (ii), that the plane contains the two lines through a_1, b_1, c_1, d_1, with the respective direction-cosines l, m, n, k; l', m', n', k'. Further, $x - a_1$, $y - b_1$, $z - c_1$, $v - d_1$, are equal to LR', MR', NR', KR', where R' is the distance AR, and L, M, N, K, are the direction-cosines of AR measured in the direction from A to R; hence

$$\begin{aligned} L &= \rho l + \sigma l', \\ M &= \rho m + \sigma m', \\ N &= \rho n + \sigma n', \\ K &= \rho k + \sigma k', \end{aligned}$$

where $\lambda = \rho R'$, $\mu = \sigma R'$. Thus any variable direction through the point A in the plane is represented analytically in terms of two directions of reference through that point, by means of two parameters ρ and σ. It will appear immediately that these two parameters, ρ and σ, are not independent of one another.

The equations of the plane are satisfied by taking $x, y, z, v, = a_1, b_1, c_1, d_1$, or $= a_2, b_2, c_2, d_2$, or $= a_3, b_3, c_3, d_3$, verifying the fact that the plane passes through A, B, C.

In constructing these equations, the point A was chosen as an initial point of reference. But

$$\begin{aligned}
a_1 + \lambda' (a_2 - a_1) + \mu' (a_3 - a_1) &= a_2 + \lambda'' (a_1 - a_2) + \mu'' (a_3 - a_2),\\
b_1 + \lambda' (b_2 - b_1) + \mu' (b_3 - b_1) &= b_2 + \lambda'' (b_1 - b_2) + \mu'' (b_3 - b_2),\\
c_1 + \lambda' (c_2 - c_1) + \mu' (c_3 - c_1) &= c_2 + \lambda'' (c_1 - c_2) + \mu'' (c_3 - c_2),\\
d_1 + \lambda' (d_2 - d_1) + \mu' (d_3 - d_1) &= d_2 + \lambda'' (d_1 - d_2) + \mu'' (d_3 - d_2),
\end{aligned}$$

provided $\lambda'' = 1 - \lambda' - \mu'$, $\mu'' = \lambda''$; and the right-hand sides of the four equations are new expressions for the coordinates x, y, z, v, of R, which consequently satisfy the two equations

$$\left\| \begin{matrix} x - a_2, & y - b_2, & z - c_2, & v - d_2 \\ a_1 - a_2, & b_1 - b_2, & c_1 - c_2, & d_1 - d_2 \\ a_3 - a_2, & b_3 - b_2, & c_3 - c_2, & d_3 - d_2 \end{matrix} \right\| = 0,$$

being the equivalent to the two equations of the plane ABC. Also, if l'', m'', n'', k'', be the direction-cosines of the line BC in the direction BC, these two equations have the form

$$\left\| \begin{matrix} x - a_2, & y - b_2, & z - c_2, & v - d_2 \\ l\ , & m\ , & n\ , & k \\ l''\ , & m''\ , & n''\ , & k'' \end{matrix} \right\| = 0.$$

In these equations, B manifestly is the initial point of reference.

Similarly, if C is chosen as the initial point of reference, the two equations of the plane are obtainable in the forms

$$\left\| \begin{matrix} x - a_3, & y - b_3, & z - c_3, & v - d_3 \\ l'\ , & m'\ , & n'\ , & k' \\ l''\ , & m''\ , & n''\ , & k'' \end{matrix} \right\| = 0,$$

$$\left\| \begin{matrix} x - a_3, & y - b_3, & z - c_3, & v - d_3 \\ a_1 - a_3, & b_1 - b_3, & c_1 - c_3, & d_1 - d_3 \\ a_2 - a_3, & b_2 - b_3, & c_2 - c_3, & d_2 - d_3 \end{matrix} \right\| = 0.$$

The general conclusion therefore is that a plane, determined by the requirements of passing through a point a, b, c, d, and containing two directions l, m, n, k; l', m', n', k'; is represented by the two equations

$$\left\| \begin{matrix} x - a, & y - b, & z - c, & v - d \\ l\ , & m\ , & n\ , & k \\ l'\ , & m'\ , & n'\ , & k' \end{matrix} \right\| = 0,$$

or by the four equations

$$\left.\begin{aligned} x &= a + \lambda l + \mu l' \\ y &= b + \lambda m + \mu m' \\ z &= c + \lambda n + \mu n' \\ v &= d + \lambda k + \mu k' \end{aligned}\right\},$$

where λ and μ are two independent parameters, capable of varying separately between $-\infty$ and $+\infty$. And it is an immediate corollary that any three directions in a plane satisfy the relations

$$\begin{Vmatrix} l, & m, & n, & k \\ l', & m', & n', & k' \\ l'', & m'', & n'', & k'' \end{Vmatrix} = 0.$$

In passing, we remark that (as will be seen later) there are other forms of equations of a plane: but all of them are equivalent to two (and only two) independent equations, linear in the current variables and involving no other variable quantities such as parameters.

Ex. 1. Shew that the plane can be represented by the (two) independent equations of the set

$$\begin{Vmatrix} x, & y, & z, & v, & 1 \\ a_1, & b_1, & c_1, & d_1, & 1 \\ a_2, & b_2, & c_2, & d_2, & 1 \\ a_3, & b_3, & c_3, & d_3, & 1 \end{Vmatrix} = 0.$$

Ex. 2. Prove that, if the equations

$$\begin{Vmatrix} x-a_1, & y-b_1, & z-c_1, & v-d_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \end{Vmatrix} = 0,$$

$$\begin{Vmatrix} x-a_2, & y-b_2, & z-c_2, & v-d_2 \\ l_3, & m_3, & n_3, & k_3 \\ l_1, & m_1, & n_1, & k_1 \end{Vmatrix} = 0,$$

$$\begin{Vmatrix} x-a_3, & y-b_3, & z-c_3, & v-d_3 \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{Vmatrix} = 0,$$

are to represent one and the same plane, the necessary and sufficient conditions are that the relations

$$\begin{Vmatrix} a_p-a_q, & b_p-b_q, & c_p-c_q, & d_p-d_q \\ l_r, & m_r, & n_r, & k_r \\ l_s, & m_s, & n_s, & k_s \end{Vmatrix} = 0$$

shall be satisfied for all the combinations p, q, $=1, 2, 3$, and r, s, $=1, 2, 3$.

To how many independent relations are these conditions equivalent?

Representation of a flat.

14. The analytical representation of a flat can be deduced from the definition (§ 6), in the same manner as was the analytical representation of a plane. Let the non-collinear points A, B, C, determining a plane ABC, be a_r, b_r, c_r, d_r, for $r = 1, 2, 3$, as before; and let a fourth point D, not lying in the plane ABC, be a_4, b_4, c_4, d_4. A current point R in the plane ABC is given by

$$\begin{aligned} x' &= a_1 + \lambda' (a_2 - a_1) + \mu' (a_3 - a_1), \\ y' &= b_1 + \lambda' (b_2 - b_1) + \mu' (b_3 - b_1), \\ z' &= c_1 + \lambda' (c_2 - c_1) + \mu' (c_3 - c_1), \\ v' &= d_1 + \lambda' (d_2 - d_1) + \mu' (d_3 - d_1). \end{aligned}$$

A current point S in the line AD is given by

$$\begin{aligned} x'' &= a_1 + \gamma (a_4 - a_1), \\ y'' &= b_1 + \gamma (b_4 - b_1), \\ z'' &= c_1 + \gamma (c_4 - c_1), \\ v'' &= d_1 + \gamma (d_4 - d_1). \end{aligned}$$

Hence a current point T on the line RS is given by

$$\begin{aligned} x &= x' + \delta (x'' - x') \\ &= a_1 + \rho (a_2 - a_1) + \sigma (a_3 - a_1) + \tau (a_4 - a_1), \\ y &= b_1 + \rho (b_2 - b_1) + \sigma (b_3 - b_1) + \tau (b_4 - b_1), \\ z &= c_1 + \rho (c_2 - c_1) + \sigma (c_3 - c_1) + \tau (c_4 - c_1), \\ v &= d_1 + \rho (d_2 - d_1) + \sigma (d_3 - d_1) + \tau (d_4 - d_1), \end{aligned}$$

where

$$\rho = \lambda' - \delta\lambda', \quad \sigma = \mu' - \delta\mu', \quad \tau = \delta\gamma.$$

The parameters λ' and μ' are general and they are independent of one another, the parameter γ is general, and likewise δ. Hence ρ, σ, τ, are three parameters, independent of one another, each capable of variation by itself from $-\infty$ to $+\infty$. Accordingly, the coordinates of T satisfy the single equation

$$\begin{vmatrix} x - a_1, & y - b_1, & z - c_1, & v - d_1 \\ a_2 - a_1, & b_2 - b_1, & c_2 - c_1, & d_2 - d_1 \\ a_3 - a_1, & b_3 - b_1, & c_3 - c_1, & d_3 - d_1 \\ a_4 - a_1, & b_4 - b_1, & c_4 - c_1, & d_4 - d_1 \end{vmatrix} = 0,$$

which therefore is the (single) equation of a flat.

Now let l, m, n, k, be the direction-cosines of AB; l', m', n', k', those of

AC; l'', m'', n'', k'', those of *AD*. Then the coordinates of a current point *T* in the flat *ABCD* are

$$\begin{aligned} x &= a_1 + \lambda l + \mu l' + \nu l'' , \\ y &= b_1 + \lambda m + \mu m' + \nu m'', \\ z &= c_1 + \lambda n + \mu n' + \nu n'' , \\ v &- d_1 + \lambda k + \mu k' + \nu k'' , \end{aligned}$$

where λ, μ, ν, are three independent parameters; and the equation of the flat becomes

$$\begin{vmatrix} x-a_1, & y-b_1, & z-c_1, & v-d_1 \\ l \ , & m \ , & n \ , & k \\ l' \ , & m' \ , & n' \ , & k' \\ l'' \ , & m'' \ , & n'' \ , & k'' \end{vmatrix} = 0.$$

From the first form of the equation of the flat, we verify at once that the flat passes through the four points *A*, *B*, *C*, *D*. From the second form, we infer that, if *L*, *M*, *N*, *K*, are the direction-cosines of any line in the flat,

$$\begin{aligned} L &= \rho l + \sigma l' + \tau l'' , \\ M &= \rho m + \sigma m' + \tau m'', \\ N &= \rho n + \sigma n' + \tau n'' , \\ K &= \rho k + \sigma k' + \tau k'' \end{aligned}$$

Alternative (or apparently alternative) forms of the equation of the flat are obtained when different arrangements of the four points *A*, *B*, *C*, *D*, are taken so as to adopt a basic plane different from *ABC* with a different basic line *AD*. They are equivalent to one another, and all of them are equivalent to the single symmetrical form

$$\begin{vmatrix} x \ , & y \ , & z \ , & v \ , & 1 \\ a_1, & b_1, & c_1, & d_1, & 1 \\ a_2, & b_2, & c_2, & d_2, & 1 \\ a_3, & b_3, & c_3, & d_3, & 1 \\ a_4, & b_4, & c_4, & d_4, & 1 \end{vmatrix} = 0.$$

What seems the most convenient form of the equation of a flat is given by the foregoing expressions for the four coordinates of a current point in terms of the three independent parameters λ, μ, ν.

It thus appears

(i) that a line can be drawn through two, and not through more than two, arbitrary points, conditions are necessary if a third assigned point is on the line:

(ii) that a plane can be drawn through three, and not through more than three, arbitrary points; conditions are necessary if a fourth assigned point is in the plane:

(iii) that a flat can be drawn through four, and not through more than four, arbitrary points: a condition is necessary if a fifth assigned point lies within the flat.

Also, a line is represented by three equations, a plane by two equations, and a flat by one equation, each such equation being of the first degree in the variables. Alternative forms of the three equations of a line, and of the two equations of a plane, have yet to be indicated.

Ex. 1. Obtain the equations of the four (parallel) planes ABC, $f'g'h'$, fgh, $\alpha\beta\gamma$, in the figure on p 7, as follows

$$ABC, \quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1, \quad v=0,$$

$$f'g'h', \quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1, \quad v=d;$$

$$fgh, \quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=2, \quad v=0,$$

$$\alpha\beta\gamma, \quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=2, \quad v=d$$

Obtain also the similar equations of the sets of four (parallel) planes

(i) BCD, $f'gh$, $fg'h'$, $\beta\gamma\delta$
(ii) CDA, $g'hf$, $gh'f'$, $\gamma\delta\alpha$
(iii) DAB, $h'fg$, $hf'g'$, $\delta\alpha\beta$.

Ex 2 Verify that the six points f, g, h, f', g', h', lie in the flat

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{v}{d}=2;$$

that the (parallel) flats $O\alpha\beta\gamma$, $PABC$, have the respective equations

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-2\frac{v}{d}=0,$$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-2\frac{v}{d}=1;$$

and that the (parallel) flats $ABC\alpha\beta\gamma$, $Pfgh$, have the respective equations

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-\frac{v}{d}=1,$$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-\frac{v}{d}=2.$$

Types of amplitude in n-fold homaloidal space are limited in number.

15. In confirmation of an inference that, in a quadruple continuum, no amplitude can exist which, being represented by a linear equation or by linear equations, is required to contain more than four arbitrarily assumed points, the following argument can be adduced

Consider a general amplitude of linear type, existing in any n-fold space: let it be represented by r independent equations, each of them linear. These independent equations can be resolved so as to express r of the variables in terms of the remainder: let the resolved expression be

$$x_s = \alpha_s + a_{s,r+1} x_{r+1} + a_{s,r+2} x_{r+2} + \dots + a_{s,n} x_n,$$

for $s = 1, \dots, r$. The number of constants α_s and $a_{s,m}$ in each equation is $n - r + 1$: thus the expression of the amplitude contains $r(n - r + 1)$ constants in all; and these, in the most general instance, can be regarded as unconnected by relations. Let the amplitude be required to contain m arbitrarily assumed points. The equations of the amplitude must be satisfied by the coordinates of each point: and therefore there must be r relations to be satisfied in connection with each point. Consequently there will be mr relations in all, arising out of the requirement imposed on the amplitude: and each such relation is linear in the constants α_s and $a_{s,m}$. Hence, in general,

$$mr \leqslant r(n - r + 1),$$

and therefore

$$m \leqslant n - r + 1.$$

For the analytical expression of an amplitude, there must be one equation at least: that is,

$$r - 1 \geqslant 0.$$

Hence

$$m \leqslant n - (r - 1) \leqslant n:$$

so that no linear amplitude can be made to pass through more than n arbitrarily assumed points. In particular, no linear amplitude in a quadruple continuum can be required to pass through more than four arbitrary points.

Further, the number r of independent equations (whether linear or not, separately or in independent and equivalent combinations), which are the analytical expression of an amplitude of any number of degrees of freedom existing in any n-fold space, is not greater than n. If it were greater than n, there would then be a number of relations among the variables greater than the number of variables: in that event, the relations could not be independent and simultaneous. If the number be equal to n, the n equations among the n variables can be imagined as resolved: the resolution would provide an aggregate of sets of values, each set providing constant values for $x_1, \dots, x_n$, that is, determining a point; and therefore the n relations among the n variables would provide a number of points, each point being in itself of no dimensions. In all other instances, r is less than n; and so we have, as providing possible amplitudes with ranges,

$$n > r \geqslant 1.$$

It follows immediately that, in a quadruple continuum, the only amplitudes, which arise for consideration, are curves, surfaces, and regions, with lines, planes, and flats, as the simplest instances of the respective types.

CHAPTER II.

LINES.

Lemmas on determinants.

16. Certain lemmas on determinants will be required in this chapter and, occasionally, at some later stages: they are given here so that, when necessary, they may be quoted without any subsequent interruption of a discussion.

Let sets of quantities, each set containing the same number of quantities, be denoted by

$$\begin{array}{l} x_1,\ y_1,\ z_1,\ v_1,\ w_1,\ \ldots, \\ x_2,\ y_2,\ z_2,\ v_2,\ w_2,\ \ldots, \\ x_3,\ y_3,\ z_3,\ v_3,\ w_3,\ \ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots; \end{array}$$

and for all values of integers i and j specifying the sets, let

$$A_{ij} = x_i x_j + y_i y_j + z_i z_j + v_i v_j + w_i w_j + \ldots.$$

Then the results, which will most often be required, are as follows

$$\Sigma \begin{vmatrix} x_1, & y_1 \\ x_2, & y_2 \end{vmatrix}^2 = \begin{vmatrix} A_{11}, & A_{12} \\ A_{12}, & A_{22} \end{vmatrix},$$

$$\Sigma \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}^2 = \begin{vmatrix} A_{11}, & A_{12}, & A_{13} \\ A_{12}, & A_{22}, & A_{23} \\ A_{13}, & A_{23}, & A_{33} \end{vmatrix},$$

$$\Sigma \begin{vmatrix} x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ x_3, & y_3, & z_3, & v_3 \\ x_4, & y_4, & z_4, & v_4 \end{vmatrix}^2 = \begin{vmatrix} A_{11}, & A_{12}, & A_{13}, & A_{14} \\ A_{12}, & A_{22}, & A_{23}, & A_{24} \\ A_{13}, & A_{23}, & A_{33}, & A_{34} \\ A_{14}, & A_{24}, & A_{34}, & A_{44} \end{vmatrix};$$

and so on: where, in the first summation, all possible similar determinants of two rows are constructed out of the first two sets of quantities: in the second summation, all possible similar determinants of three rows are constructed out of the first three sets of quantities: and so forth.

Similar results that will be used are:

$$\Sigma \begin{vmatrix} x_1, & y_1 \\ x_2, & y_2 \end{vmatrix} \begin{vmatrix} x_1, & y_1 \\ x_3, & y_3 \end{vmatrix} = \begin{vmatrix} A_{11}, & A_{12} \\ A_{13}, & A_{23} \end{vmatrix}$$

$$\Sigma \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_4, & y_4, & z_4 \end{vmatrix} = \begin{vmatrix} A_{11}, & A_{12}, & A_{13} \\ A_{12}, & A_{22}, & A_{23} \\ A_{14}, & A_{24}, & A_{34} \end{vmatrix};$$

and so on.

Also, it is convenient to use the expansion of the determinant

$$\begin{vmatrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \\ L, & M, & N, & D \end{vmatrix}$$

in the form

$$\begin{aligned} &D(ABC + 2FGH - AF^2 - BG^2 - CH^2) \\ &- BCL^2 - CAM^2 - ABN^2 + 2AFMN + 2BGNL + 2CHLM \\ &+ F^2L^2 + G^2M^2 + H^2N^2 - 2GHMN - 2HFNL - 2FGLM, \end{aligned}$$

in connection with theory of curves in four-fold space (Chap. VIII).

Equations of a line.

17. We have seen that a straight line may be represented by the equations

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = \frac{v-d}{k} = r,$$

where l, m, n, k, are the direction-cosines of the line measured from a given point a, b, c, d, on the line towards the current point x, y, z, v, on the line; r denotes the distance from the fixed point to the current point; and l, m, n, k, satisfy the permanent relation $l^2 + m^2 + n^2 + k^2 = 1$.

Also, if a', b', c', d', be another point on the line, the equations may be taken in the form

$$\frac{x-a}{a'-a} = \frac{y-b}{b'-b} = \frac{z-c}{c'-c} = \frac{v-d}{d'-d} = \frac{r}{D},$$

where D denotes the distance between the points a, b, c, d, and a', b', c', d'. In this form, the direction-cosines of the line, measured from a, b, c, d, towards a', b', c', d', are

$$\frac{l}{a'-a} = \frac{m}{b'-b} = \frac{n}{c'-c} = \frac{k}{d'-d} = \frac{1}{D},$$

while

$$r = \{(x-a)^2 + (y-b)^2 + (z-c)^2 + (v-d)^2\}^{\frac{1}{2}},$$

$$D = \{(a'-a)^2 + (b'-b)^2 + (c'-c)^2 + (d'-d)^2\}^{\frac{1}{2}},$$

a positive sign being taken for the latter radical. Manifestly three linear equations, when taken in the form

$$\frac{x-a}{\lambda} = \frac{y-b}{\mu} = \frac{z-c}{\nu} = \frac{v-d}{\kappa},$$

are sufficient to specify the line, the direction-cosines then being

$$\lambda Q^{-\frac{1}{2}}, \quad \mu Q^{-\frac{1}{2}}, \quad \nu Q^{-\frac{1}{2}}, \quad \kappa Q^{-\frac{1}{2}},$$

where Q denotes $\lambda^2 + \mu^2 + \nu^2 + \kappa^2$. And these three linear equations are independent of one another.

The foregoing form is the most generally useful form for the equations of a line. There are other forms, equivalent, always linear, and always consisting of three members. Of these, the set

$$\left.\begin{aligned} L_1x + M_1y + N_1z + K_1v &= a_1 \\ L_2x + M_2y + N_2z + K_2v &= a_2 \\ L_3x + M_3y + N_3z + K_3v &= a_3 \end{aligned}\right\},$$

assumed to be independent of one another, occurs occasionally. It is easily reducible to the form already given. Let a, b, c, d, be any simultaneous values of x, y, z, v, satisfying the equations; such a combination can be chosen in an unlimited number of ways. Then the equations may be written

$$\begin{aligned} L_1(x-a) + M_1(y-b) + N_1(z-c) + K_1(v-d) &= 0, \\ L_2(x-a) + M_2(y-b) + N_2(z-c) + K_2(v-d) &= 0, \\ L_3(x-a) + M_3(y-b) + N_3(z-c) + K_3(v-d) &= 0; \end{aligned}$$

and therefore, if

$$\frac{\lambda}{\begin{vmatrix} M_1, & N_1, & K_1 \\ M_2, & N_2, & K_2 \\ M_3, & N_3, & K_3 \end{vmatrix}} = \frac{\mu}{-\begin{vmatrix} N_1, & K_1, & L_1 \\ N_2, & K_2, & L_2 \\ N_3, & K_3, & L_3 \end{vmatrix}} = \frac{\nu}{\begin{vmatrix} K_1, & L_1, & M_1 \\ K_2, & L_2, & M_2 \\ K_3, & L_3, & M_3 \end{vmatrix}}$$

$$= \frac{\kappa}{-\begin{vmatrix} L_1, & M_1, & N_1 \\ L_2, & M_2, & N_2 \\ L_3, & M_3, & N_3 \end{vmatrix}} = \frac{Q^{\frac{1}{2}}}{\Theta^{\frac{1}{2}}},$$

where

$$\Theta = \Sigma \begin{vmatrix} M_1, & N_1, & K_1 \\ M_2, & N_2, & K_2 \\ M_3, & N_3, & K_3 \end{vmatrix}^2 = \begin{vmatrix} \Sigma L_1^2\,, & \Sigma L_1L_2, & \Sigma L_1L_3 \\ \Sigma L_1L_2, & \Sigma L_2^2\,, & \Sigma L_2L_3 \\ \Sigma L_1L_3, & \Sigma L_2L_3, & \Sigma L_3^2 \end{vmatrix},$$

the three equations can be resolved into the canonical form

$$\frac{x-a}{\lambda} = \frac{y-b}{\mu} = \frac{z-c}{\nu} = \frac{v-d}{\kappa}.$$

The direction-cosines of the line, being $\lambda Q^{-\frac{1}{2}}$, $\mu Q^{-\frac{1}{2}}$, $\nu Q^{-\frac{1}{2}}$, $\kappa Q^{-\frac{1}{2}}$, are expressible in terms of the coefficients L, M, N, K; and it is to be noticed that they satisfy the relations

$$\begin{aligned} \lambda L_1 + \mu M_1 + \nu N_1 + \kappa K_1 &= 0, \\ \lambda L_2 + \mu M_2 + \nu N_2 + \kappa K_2 &= 0, \\ \lambda L_3 + \mu M_3 + \nu N_3 + \kappa K_3 &= 0, \end{aligned}$$

the significance of each of these relations appearing later (§§ 44, 45).

Moreover, it is to be remarked that each of the three new equations which, combined, give the line, is of the form

$$Lx + My + Nz + Kv = p,$$

that is, it represents a flat. We therefore infer that, in four-fold space, *a line is the intersection of three flats.* It will be seen later that, in general in four-fold space, a line is not the intersection of two planes, contrary to the property of three-fold space: for, in four-fold space, two general planes intersect only in a point (§ 29).

Again, it will be established (if it is not already obvious) that any two of the flats determine a plane, by means of their equations; and therefore we can regard a line as the intersection of a flat and a plane. But, in this mode of regarding a line, there is the exclusive limitation that the three equations, representing flats, must be linearly independent: the geometrical limitation, for the purpose of giving a line as the intersection of a flat and a plane, is that the plane must not lie in the flat

Parallel lines.

18. A line has been defined (§§ 3, 11) in connection with its property of uniform direction. We therefore define two lines as *parallel*, when they have the same direction, that is, when they are characterised by the same direction-cosines

Let two lines, having the same direction-cosines l, m, n, k, be

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = \frac{v-d}{k},$$

and

$$\frac{x-a'}{l} = \frac{y-b'}{m} = \frac{z-c'}{n} = \frac{v-d'}{k}.$$

The point a', b', c', d', must not lie on the former; the point a, b, c, d, must not lie on the latter: otherwise, the two lines would be the same Consequently, the three equations

$$\frac{a'-a}{l} = \frac{b'-b}{m} = \frac{c'-c}{n} = \frac{d'-d}{k}$$

must not simultaneously be satisfied (that is, one may be satisfied, two may be satisfied, but not all three) Now the two equations

$$\left\| \begin{matrix} x-a, & y-b, & z-c, & v-d \\ a'-a, & b'-b, & c'-c, & d'-d \\ l\ , & m\ , & n\ , & k \end{matrix} \right\| = 0$$

represent a plane passing through the point a, b, c, d; the plane contains the line through that point in the direction given by l, m, n, k: and it contains the line joining a', b', c', d', to a, b, c, d. But the equations are equivalent to

$$\left\| \begin{matrix} x-a', & y-b', & z-c', & v-d' \\ a'-a, & b'-b, & c'-c, & d'-d \\ l\ , & m\ , & n\ , & k \end{matrix} \right\| = 0,$$

which manifestly represent the same plane; and they shew that the plane contains the line through a', b', c', d', in the direction given by l, m, n, k. Thus a plane can be drawn through the two lines.

If the two lines can meet at a finite distance from the points a, b, c, d, and a', b', c', d', let the distance of the meeting point be r from the former and r' from the latter. Then the coordinates of that meeting point are

$$a+lr, \quad b+mr, \quad c+nr, \quad d+kr,$$

because it lies on the first line, and they are

$$a'+lr', \quad b'+mr', \quad c'+nr', \quad d'+kr',$$

because it lies on the second line, while r and r' are presumed finite. Accordingly, we should have

$$a+lr=a'+lr', \quad b+mr=b'+mr', \quad c+nr=c'+nr', \quad d+kr=d'+kr'.$$

But for finite quantities r and r', these equations are not simultaneously satisfied, and therefore the assumption cannot be justified.

Consequently, the two lines do not meet at a finite distance, however great; and they lie in one and the same plane. They therefore are characterised by the property which, under the ancient definition, is called *parallelism*.

Ex. Shew that the two lines

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = \frac{v-d}{k},$$

and

$$\left. \begin{aligned} L_1x+M_1y+N_1z+K_1v&=a_1 \\ L_2x+M_2y+N_2z+K_2v&=a_2 \\ L_3x+M_3y+N_3z+K_3v&=a_3 \end{aligned} \right\},$$

are distinct and parallel, if the relations

$$\frac{l}{\begin{vmatrix} M_1, & N_1, & K_1 \\ M_2, & N_2, & K_2 \\ M_3, & N_3, & K_3 \end{vmatrix}} = -\frac{m}{\begin{vmatrix} N_1, & K_1, & L_1 \\ N_2, & K_2, & L_2 \\ N_3, & K_3, & L_3 \end{vmatrix}} = \frac{n}{\begin{vmatrix} K_1, & L_1, & M_1 \\ K_2, & L_2, & M_2 \\ K_3, & L_3, & M_3 \end{vmatrix}} = -\frac{k}{\begin{vmatrix} L_1, & M_1, & N_1 \\ L_2, & M_2, & N_2 \\ L_3, & M_3, & N_3 \end{vmatrix}}$$

are satisfied, while the three quantities

$$L_ra+M_rb+N_rc+K_rd-a_r$$

(for $r=1, 2, 3$) do not simultaneously vanish.

Inclination of two lines.

19. When two lines meet in a point, a plane can be drawn through the point containing the two lines; and their inclination can be derived from the properties of a linear triangle in the plane. Let the two lines through A, the point a, b, c, d, be

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}=\frac{v-d}{k},$$

$$\frac{x-a}{l'}=\frac{y-b}{m'}=\frac{z-c}{n'}=\frac{v-d}{k'},$$

where l, m, n, k, and l', m', n', k', are the direction-cosines of the lines. On the former, take any point P, distant r from A: on the latter, take any point P', distant r' from A: and join the points P and P' by a line, which necessarily lies in the plane, and which completes the triangle PAP'. If θ denote the angle PAP', we have

$$\begin{aligned}PP'^2 &= AP^2+AP'^2-2AP\ AP'\cos\theta\\ &= r^2+r'^2-2rr'\cos\theta.\end{aligned}$$

The coordinates of P are $a+lr$, $b+mr$, $c+nr$, $d+kr$: those of P' are $a+l'r'$, $b+m'r'$, $c+n'r'$, $d+k'r'$, and therefore

$$\begin{aligned}PP'^2 &= (lr-l'r')^2+(mr-m'r')^2+(nr-n'r')^2+(kr-k'r')^2\\ &= r^2(l^2+m^2+n^2+k^2)+r'^2(l'^2+m'^2+n'^2+k'^2)\\ &\qquad -2rr'(ll'+mm'+nn'+kk')\\ &= r^2+r'^2-2rr'(ll'+mm'+nn'+kk').\end{aligned}$$

Hence

$$\cos\theta = ll'+mm'+nn'+kk'.$$

For given values of l, m, n, k, and l', m', n', k', there are two values of θ, being θ and $-\theta$: so that, by this formula, no convention for the positive value of θ is required.

Further, we have

$$\begin{aligned}\sin^2\theta &= 1-\cos^2\theta\\ &= (l^2+m^2+n^2+k^2)(l'^2+m'^2+n'^2+k'^2)-(ll'+mm'+nn'+kk')^2\\ &= (mn'-nm')^2+(nl'-ln')^2+(lm'-ml')^2\\ &\quad +(lk'-kl')^2+(mk'-km')^2+(nk'-kn')^2\\ &= \Sigma\,(lm'-ml')^2,\end{aligned}$$

with the conventional significance for the summation-sign; hence

$$\sin\theta = \{\Sigma\,(lm' - ml')^2\}^{\frac{1}{2}}.$$

A convention, concerning the positive direction in which θ is to be measured, is required in order to settle the sign of the radical.

When the equations of the lines occur in the forms

$$\frac{x-a}{\lambda} = \frac{y-b}{\mu} = \frac{z-c}{\nu} = \frac{v-d}{\kappa},$$

$$\frac{x-a}{\lambda'} = \frac{y-b}{\mu'} = \frac{z-c}{\nu'} = \frac{v-d}{\kappa'},$$

their inclination θ is given by

$$\cos\theta = \frac{\lambda\lambda' + \mu\mu' + \nu\nu' + \kappa\kappa'}{(\lambda^2 + \mu^2 + \nu^2 + \kappa^2)^{\frac{1}{2}}\,(\lambda'^2 + \mu'^2 + \nu'^2 + \kappa'^2)^{\frac{1}{2}}};$$

and then

$$\sin\theta = \frac{\{\Sigma\,(\lambda\mu' - \mu\lambda')^2\}^{\frac{1}{2}}}{(\lambda^2 + \mu^2 + \nu^2 + \kappa^2)^{\frac{1}{2}}\,(\lambda'^2 + \mu'^2 + \nu'^2 + \kappa'^2)^{\frac{1}{2}}}.$$

Ex. Verify that $\sin^2\theta$ can be expressed in the forms

$$\{(mn' - nm') \pm (lk' - kl')\}^2 + \{(nl' - ln') \pm (mk' - km')\}^2 + \{(lm' - ml') \pm (nk' - kn')\}^2,$$

where the positive sign can be taken throughout, and the negative sign can be taken throughout.

20. The inclination of two straight lines

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = \frac{v-d}{k},$$

$$\frac{x-a'}{l'} = \frac{y-b'}{m'} = \frac{z-c'}{n'} = \frac{v-d'}{k'},$$

which are assumed not to meet, can be derived through the preceding result.

Let a'', b'', c'', d'', be any arbitrary point, and through this point let lines be drawn parallel to the postulated lines; their equations are

$$\frac{x-a''}{l} = \frac{y-b''}{m} = \frac{z-c''}{n} = \frac{v-d''}{k},$$

$$\frac{x-a''}{l'} = \frac{y-b''}{m'} = \frac{z-c''}{n'} = \frac{v-d''}{k'},$$

respectively. The inclination of the latter lines is (§ 19) given by

$$\cos\theta = ll' + mm' + nn' + kk',$$

$$\sin\theta = \{(lm' - ml')^2\}^{\frac{1}{2}}.$$

This inclination θ, as it does not involve a'', b'', c'', d'', is independent of the position of the arbitrary point; and it depends only upon the direction-cosines of the two initial lines. It is called the inclination of those initial lines.

When the equations of the given lines involve quantities λ, μ, ν, κ, λ', μ', ν', κ'; which are only proportional to the direction-cosines and are not their actual values, the formulæ for $\cos\theta$ and $\sin\theta$ are modified exactly as in § 19.

Conditions of parallelism of two lines.

21. In Euclidean geometry, whether of two dimensions or of three dimensions, two lines are parallel to one another, if their inclination is zero when the directions are taken in the same sense, and if it is π when the directions are taken in opposite senses. To settle whether this condition for parallelism is sufficient under the preceding condition, we suppose that

$$\sin^2\theta = 0,$$

that is,

$$\Sigma\,(lm' - ml')^2 = 0.$$

As we are dealing with real lines, so that each of the six terms on the left-hand side is a real non-negative square, we must have each such square equal to zero. Thus

$$mn' - nm' = 0,\quad nl' - ln' = 0,\quad lm' - ml' = 0,$$

$$lk' - kl' = 0,\quad mk' - km' = 0,\quad nk' - kn' = 0,$$

all of which are satisfied by

$$\frac{l'}{l} = \frac{m'}{m} = \frac{n'}{n} = \frac{k'}{k} = \pm\frac{(l'^2 + m'^2 + n'^2 + k'^2)^{\frac{1}{2}}}{(l^2 + m^2 + n^2 + k^2)^{\frac{1}{2}}},$$

where a positive sign is attributed to each radical. When l, m, n, k, and l', m', n', k', are the actual direction-cosines, the last fraction becomes ± 1, and the direction-cosines of the one line are then equally the direction-cosines of the other line. that is, according to the definition of § 18, the two lines are parallel. Thus we infer that two lines are parallel if their inclination is zero or π.

Further, the necessary and sufficient analytical conditions for the parallelism of the two lines is

$$\frac{l'}{l} = \frac{m'}{m} = \frac{n'}{n} = \frac{k'}{k}.$$

If the quantities λ, μ, ν, κ, and λ', μ', ν', κ'; occur in the equations of the two lines, these quantities being only proportional to the direction-cosines and not being their actual values, the necessary and sufficient analytical conditions for parallelism of the lines are

$$\frac{\lambda'}{\lambda}=\frac{\mu'}{\mu}=\frac{\nu'}{\nu}=\frac{\kappa'}{\kappa}.$$

Condition of perpendicularity of two lines.

22. As two lines are perpendicular to one another when their inclination is $\frac{1}{2}\pi$, the one single necessary and sufficient condition that the two lines should be *perpendicular* is

$$ll'+mm'+nn'+kk'=0,$$

or, when the direction-parameters are not actual direction-cosines,

$$\lambda\lambda'+\mu\mu'+\nu\nu'+\kappa\kappa'=0.$$

It is to be noted, in passing, that while only a single line can be drawn through a point parallel to a given line, the same limitation to uniqueness does not arise for a line drawn through a point perpendicular to a given line. In fact, if such a line

$$\frac{x-a'}{l'}=\frac{y-b'}{m'}=\frac{z-c'}{n'}=\frac{v-d'}{k'},$$

be perpendicular to a given line

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}=\frac{v-d}{k},$$

the only limitation, upon the ratios $l':m'.n':k'$, is the condition

$$ll'+mm'+nn'+kk'=0.$$

Thus the coordinates of any point on the perpendicular line satisfy the equation

$$l(x-a')+m(y-b')+n(z-c')+k(v-d')=0,$$

that is, the perpendicular line lies in a flat. And it is easy to infer that any line in this flat is perpendicular to the given line*.

Ex. 1. *Required the direction-cosines of a line perpendicular to three given lines*

Let the three lines have direction-cosines l_1, m_1, n_1, k_1; l_2, m_2, n_2, k_2; and l_3, m_3, n_3, k_3, respectively; and write

$$\cos\alpha=l_2l_3+m_2m_3+n_2n_3+k_2k_3,$$
$$\cos\beta=l_3l_1+m_3m_1+n_3n_1+k_3k_1,$$
$$\cos\gamma=l_1l_2+m_1m_2+n_1n_2+k_1k_2.$$

---

* As to this result, see §§ 44, 45, *post*.

Let the required direction-cosines of a line perpendicular to all these three lines be l, m, n, k; then

$$ll_1+mm_1+nn_1+kk_1=0,$$

$$ll_2+mm_2+nn_2+kk_2=0,$$

$$ll_3+mm_3+nn_3+kk_3=0,$$

and therefore

$$\frac{l}{\begin{vmatrix} m_1, & n_1, & k_1 \\ m_2, & n_2, & k_2 \\ m_3, & n_3, & k_3 \end{vmatrix}}=\frac{m}{-\begin{vmatrix} n_1, & k_1, & l_1 \\ n_2, & k_2, & l_2 \\ n_3, & k_3, & l_3 \end{vmatrix}}=\frac{n}{\begin{vmatrix} k_1, & l_1, & m_1 \\ k_2, & l_2, & m_2 \\ k_3, & l_3, & m_3 \end{vmatrix}}=\frac{k}{-\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}}=\frac{1}{D},$$

on taking $\frac{1}{D}$ as the common value of the four fractions Now

$$l^2+m^2+n^2+k^2=1\,;$$

hence

$$D^2=\Sigma\begin{vmatrix} m_1, & n_1, & k_1 \\ m_2, & n_2, & k_2 \\ m_3, & n_3, & k_3 \end{vmatrix}^2$$

$$=\begin{vmatrix} \Sigma l_1^2\,, & \Sigma l_1l_2, & \Sigma l_1l_3 \\ \Sigma l_1l_2, & \Sigma l_2^2\,, & \Sigma l_2l_3 \\ \Sigma l_1l_3, & \Sigma l_2l_3, & \Sigma l_3^2 \end{vmatrix}$$

$$=\begin{vmatrix} 1\;, & \cos\gamma, & \cos\beta \\ \cos\gamma, & 1\;, & \cos\alpha \\ \cos\beta, & \cos\alpha, & 1 \end{vmatrix}$$

$$=1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+\qquad\qquad\beta\cos\gamma$$

We choose the positive sign for the square root of the right-hand side. thus

$$D=(1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma)^{\frac{1}{2}}.$$

Consequently, the values of l, m, n, k, are known

If the given lines are perpendicular, $\alpha=\frac{1}{2}\pi$, $\beta=\frac{1}{2}\pi$, $\gamma=\frac{1}{2}\pi$, and then $D=1$. We then should have, with the chosen positive sign,

$$l=\begin{vmatrix} m_1, & n_1, & k_1 \\ m_2, & n_2, & k_2 \\ m_3, & n_3, & k_3 \end{vmatrix},\quad m=-\begin{vmatrix} n_1, & k_1, & l_1 \\ n_2, & k_2, & l_2 \\ n_3, & k_3, & l_3 \end{vmatrix},$$

$$n=\begin{vmatrix} k_1, & l_1, & m_1 \\ k_2, & l_2, & m_2 \\ k_3, & l_3, & m_3 \end{vmatrix},\quad k=-\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix};$$

and therefore

$$\begin{vmatrix} l\;, & m\;, & n\;, & k \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \end{vmatrix}=l^2+m^2+n^2+k^2=1,$$

a relation among the direction-cosines of four lines which are perpendicular to one another in pairs. We shall return later (§ 25) to frames of four perpendicular lines.

Ex 2. *Obtain the direction-cosines of a line making angles* θ, ϕ, ψ, *respectively, with the three lines* l_1, m_1, n_1, k_1; l_2, m_2, n_2, k_2; l_3, m_3, n_3, k_3.

Draw the line perpendicular to the three given lines, its direction-cosines l, m, n, k, are known, from the result of the preceding example. Let ω be the angle made with this perpendicular by the required line, the direction-cosines of which we denote by L, M, N, K. Then

$$\begin{aligned} Ll + Mm + Nn + Kk &= \cos\omega, \\ Ll_1 + Mm_1 + Nn_1 + Kk_1 &= \cos\theta, \\ Ll_2 + Mm_2 + Nn_2 + Kk_2 &= \cos\phi, \\ Ll_3 + Mm_3 + Nn_3 + Kk_3 &= \cos\psi \end{aligned}$$

The determinant of the coefficients on the left-hand side is

$$\begin{vmatrix} l, & m, & n, & k \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \end{vmatrix}$$

$$= l \,.\, Dl + m \,.\, Dm + n \,.\, Dn + k \,.\, Dk = D$$

Hence

$$DL = \begin{vmatrix} \cos\omega, & m, & n, & k \\ \cos\theta, & m_1, & n_1, & k_1 \\ \cos\phi, & m_2, & n_2, & k_2 \\ \cos\psi, & m_3, & n_3, & k_3 \end{vmatrix} = Dl\cos\omega + P_l\cos\theta + Q_l\cos\phi + R_l\cos\psi,$$

$$DM = Dm\cos\omega + P_m\cos\theta + Q_m\cos\phi + R_m\cos\psi,$$

$$DN = Dn\cos\omega + P_n\cos\theta + Q_n\cos\phi + R_n\cos\psi,$$

$$DK = Dk\cos\omega + P_k\cos\theta + Q_k\cos\phi + R_k\cos\psi,$$

where

$$-P_l = \begin{vmatrix} m, & n, & k \\ m_2, & n_2, & k_2 \\ m_3, & n_3, & k_3 \end{vmatrix}, \quad P_m = \begin{vmatrix} n, & k, & l \\ n_2, & k_2, & l_2 \\ n_3, & k_3, & l_3 \end{vmatrix},$$

$$-P_n = \begin{vmatrix} k, & l, & m \\ k_2, & l_2, & m_2 \\ k_3, & l_3, & m_3 \end{vmatrix}, \quad P_k = \begin{vmatrix} l, & m, & n \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix};$$

$$Q_l = \begin{vmatrix} m, & n, & k \\ m_1, & n_1, & k_1 \\ m_3, & n_3, & k_3 \end{vmatrix}, \quad -Q_m = \begin{vmatrix} n, & k, & l \\ n_1, & k_1, & l_1 \\ n_3, & k_3, & l_3 \end{vmatrix},$$

$$Q_n = \begin{vmatrix} k, & l, & m \\ k_1, & l_1, & m_1 \\ k_3, & l_3, & m_3 \end{vmatrix}, \quad -Q_k = \begin{vmatrix} l, & m, & n \\ l_1, & m_1, & n_1 \\ l_3, & m_3, & n_3 \end{vmatrix},$$

$$-R_l=\begin{vmatrix} m, & n, & k \\ m_1, & n_1, & k_1 \\ m_2, & n_2, & k_2 \end{vmatrix}, \qquad R_m=\begin{vmatrix} n, & k, & l \\ n_1, & k_1, & l_1 \\ n_2, & k_2, & l_2 \end{vmatrix},$$

$$-R_n=\begin{vmatrix} k, & l, & m \\ k_1, & l_1, & m_1 \\ k_2, & l_2, & m_2 \end{vmatrix}, \qquad R_k=\begin{vmatrix} l, & m, & n \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix}.$$

From these, we have

$$P_l^2+P_m^2+P_n^2+P_k^2=\Sigma\begin{vmatrix} m, & n, & k \\ m_2, & n_2, & k_2 \\ m_3, & n_3, & k_3 \end{vmatrix}^2$$

$$=\begin{vmatrix} \Sigma l^2, & \Sigma ll_2, & \Sigma ll_3 \\ \Sigma ll_2, & \Sigma l_2^2, & \Sigma l_2l_3 \\ \Sigma ll_3, & \Sigma l_2l_3, & \Sigma l_3^2 \end{vmatrix}$$

$$=\begin{vmatrix} 1, & 0, & 0 \\ 0, & 1, & \cos\alpha \\ 0, & \cos\alpha, & 1 \end{vmatrix}=\sin^2\alpha;$$

and similarly

$$Q_l^2+Q_m^2+Q_n^2+Q_k^2=\sin^2\beta,$$

$$R_l^2+R_m^2+R_n^2+R_k^2=\sin^2\gamma.$$

Again,

$$-(P_lQ_l+P_mQ_m+P_nQ_n+P_kQ_k)=\Sigma\begin{vmatrix} m, & n, & k \\ m_3, & n_3, & k_3 \\ m_2, & n_2, & k_2 \end{vmatrix}\begin{vmatrix} m, & n, & k \\ m_3, & n_3, & k_3 \\ m_1, & n_1, & k_1 \end{vmatrix}.$$

the right-hand side is (§ 16) equal to

$$\begin{vmatrix} \Sigma l^2, & \Sigma ll_3, & \Sigma ll_2 \\ \Sigma ll_3, & \Sigma l_3^2, & \Sigma l_3l_2 \\ \Sigma ll_1, & \Sigma l_3l_1, & \Sigma l_2l_1 \end{vmatrix}$$

$$=\begin{vmatrix} 1, & 0, & 0 \\ 0, & 1, & \cos\alpha \\ 0, & \cos\beta, & \cos\gamma \end{vmatrix}=\cos\gamma-\cos\alpha\cos\beta;$$

and therefore

$$P_lQ_l+P_mQ_m+P_nQ_n+P_kQ_k=\cos\alpha\cos\beta-\cos\gamma.$$

Similarly

$$Q_lR_l+Q_mR_m+Q_nR_n+Q_kR_k=\cos\beta\cos\gamma-\cos\alpha,$$

$$R_lP_l+R_mP_m+R_nP_n+R_kP_k=\cos\gamma\cos\alpha-\cos\beta.$$

Also

$$lP_l+mP_m+nP_n+kP_k=0,$$

$$lQ_l+mQ_m+nQ_n+kQ_k=0,$$

$$lR_l+mR_m+nR_n+kR_k=0.$$

Substituting the values of L, M, N, K, we have

$$\begin{aligned} D^2 &= D^2(L^2+M^2+N^2+K^2) \\ &= D^2\cos^2\omega+\sin^2\alpha\cos^2\theta+\sin^2\beta\cos^2\phi+\sin^2\gamma\cos^2\psi \\ &\quad +2(\cos\alpha\cos\beta-\cos\gamma)\cos\theta\cos\phi \\ &\quad +2(\cos\beta\cos\gamma-\cos\alpha)\cos\phi\cos\psi \\ &\quad +2(\cos\gamma\cos\alpha-\cos\beta)\cos\psi\cos\theta. \end{aligned}$$

giving two equal and opposite values for $\sin\omega$. These may be taken as $+\omega$ and $-\omega$: or, if the two possible lines are drawn in the same sense, they can be taken as ω and $\pi-\omega$, and then the angle between the two lines is $\pi-2\omega$.

The two lines are given by

$$\left.\begin{aligned} DL,\ DL' &= P_l\cos\alpha+Q_l\cos\beta+R_l\cos\gamma\pm Dl\cos\omega\\ DM,\ DM' &= P_m\cos\alpha+Q_m\cos\beta+R_m\cos\gamma\pm Dm\cos\omega\\ DN,\ DN' &= P_n\cos\alpha+Q_n\cos\beta+R_n\cos\gamma\pm Dn\cos\omega\\ DK,\ DK' &= P_k\cos\alpha+Q_k\cos\beta+R_k\cos\gamma\pm Dk\cos\omega \end{aligned}\right\},$$

the upper signs throughout giving DL, DM, DN, DK; and the lower signs throughout giving DL', DM', DN', DK'.

From these, we have at once

$$D^2=\Sigma\,(P_l\cos\alpha+Q_l\cos\beta+R_l\cos\gamma)^2+D^2\cos^2\omega,$$

$$D^2\Sigma LL'=\Sigma\,(P_l\cos\alpha+Q_l\cos\beta+R_l\cos\gamma)^2-D^2\cos^2\omega\,,$$

and therefore

$$1-\Sigma LL'=2\cos^2\omega,$$

thus verifying the statement that $\pi-2\omega$ is the angle between the lines drawn in the same sense.

Note. We have

$$\begin{aligned} lP_l+mP_m+nP_n+kP_k&=0,\\ l_2P_l+m_2P_m+n_2P_n+k_2P_k&=0,\\ l_3P_l+m_3P_m+n_3P_n+k_3P_k&=0,\\ P_l^2+P_m^2+P_n^2+P_k^2&=\sin^2\alpha, \end{aligned}$$

hence

$$\frac{1}{\sin\alpha}P_l,\quad \frac{1}{\sin\alpha}P_m,\quad \frac{1}{\sin\alpha}P_n,\quad \frac{1}{\sin\alpha}P_k,$$

are the direction-cosines of a line perpendicular to the three directions l, m, n, k; l_2, m_2, n_2, k_2; l_3, m_3, n_3, k_3. As will be seen later (§ 15), these are the direction-cosines of the normal to the flat through the three directions

Similarly

$$\frac{1}{\sin\beta}Q_l,\quad \frac{1}{\sin\beta}Q_m,\quad \frac{1}{\sin\beta}Q_n,\quad \frac{1}{\sin\beta}Q_k,$$

and

$$\frac{1}{\sin\gamma}R_l,\quad \frac{1}{\sin\gamma}R_m,\quad \frac{1}{\sin\gamma}R_n,\quad \frac{1}{\sin\gamma}R_k,$$

are direction-cosines; the former set, of the normal to the flat through the directions l, m, n, k; l_3, m_3, n_3, k_3; l_1, m_1, n_1, k_1; and the latter set, of the normal to the flat through the directions l, m, n, k; l_1, m_1, n_1, k_1; l_2, m_2, n_2, k_2. And, of course, l, m, n, k, are the direction-cosines of the flat through these three perpendicular directions, for it is the flat through the three original lines.

Ex 3. A line is drawn making an angle θ with l_1, m_1, n_1, k_1; an angle ϕ with l_2, m_2, n_2, k_2; an angle ψ with l_3, m_3, n_3, k_3; and an angle χ with l_4, m_4, n_4, k_4; the four lines l_r, m_r, n_r, k_r, $(r=1, 2, 3, 4)$ being a given set of non-orthogonal lines, no three lying in one plane. Obtain a relation connecting the four angles θ, ϕ, ψ, χ.

Projection of lines.

23. We shall have to deal with projections, not merely of lengths, but also of areas, and of volumes.

As regards projections of lines, we take the customary definition that the projection of a line joining two points P and Q upon another line is $P'Q'$, where P' is the projection of P (that is, the foot of the perpendicular from P) on the second line, and Q' is the projection of Q (that is, the foot of the perpendicular from Q) on the second line. This l[illegible] $'Q'$ manifestly depends only upon the two points P and Q, so far as co[illegible]rns the length intercepted on the second line; and it will manifestly be u[illegible] if, keeping the two points P and Q fixed, we proceed from P to Q [illegible]oken continuous line, provided the various portions of the broker [illegible]cessively projected upon the line $P'Q'$, account being taken of [illegible]he projection of each such portion.

Consider the line OP in the figure on p. 7, and its projection on a different line through O, with direction-cosines l', m', n', k', assuming the direction-cosines of OP to be l, m, n, k. Let $OP = r$; then

$$OA = lr, \quad OB = Ah = mr, \quad OC = h\delta = nr, \quad OD = \delta P = kr.$$

For the projection upon the second line, substitute the broken line OA, Ah, $h\delta$, δP, for the direct line OP. The whole projection, upon the line with direction-cosines l', m', n', k', of this broken line is

$$= l' . OA + m' . Ah + n' . h\delta + k' . \delta P$$
$$= (ll' + mm' + nn' + kk')\, r.$$

If the inclination of the two lines is θ, this projection of OP is $r\cos\theta$; hence

$$\cos\theta = ll' + mm' + nn' + kk',$$

the result already known.

We can immediately deduce an expression for the length p of the perpendicular from a point x', y', z', v', on the line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = \frac{v-d}{k}$$

The external point being P, and a, b, c, d, being A', if N is the foot of the perpendicular from P on the line $A'L$,

$$A'N = A'P \cos LA'P$$

or, if $A'P = r'$, and l, m, n, k, are the actual direction-cosines of $A'L$, we have

$$\cos LA'P = l\frac{x'-a}{r'} + m\frac{y'-b}{r'} + n\frac{z'-c}{r'} + k\frac{v'-d}{r'},$$

and therefore

$$A'N = l(x'-a) + m(y'-b) + n(z'-c) + k(v'-d),$$

a result of course derivable, as in the preceding paragraph, by projecting $A'P$ in the same way as OP was there projected. Hence

$$\begin{aligned} p^2 &= r'^2 - A'N^2 \\ &= (x'-a)^2 + (y'-b)^2 + (z'-c)^2 + (v'-d)^2 \\ &\qquad - \{l(x'-a) + m(y'-b) + n(z'-c) + k(v'-d)\}^2. \end{aligned}$$

Further, if X, Y, Z, V, are the coordinates of N, the foot of the perpendicular from P on the line $A'L$,

$$\begin{aligned} X-a &= \text{projection of } A'N \text{ on the axis of } x \\ &= l\,\{l(x'-a) + m(y'-b) + n(z'-c) + k(v'-d)\} = lu, \text{ say}, \\ Y-b &= mu, \\ Z-c &= nu, \\ V-d &= ku; \end{aligned}$$

and the direction-cosines of the perpendicular NP are proportional to

$$x'-X, \quad y'-Y, \quad z'-Z, \quad v'-V,$$

that is, these cosines are

$$\frac{1}{p}\{x'-a-lu\}, \quad \frac{1}{p}\{y'-b-mu\}, \quad \frac{1}{p}\{z'-c-nu\}, \quad \frac{1}{p}\{v'-d-ku\}.$$

Perpendicular from a point on a line.

24. The length and the direction-cosines of the perpendicular, from any external point x', y', z', v', upon a straight line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = \frac{v-d}{k},$$

can be obtained by the following process which, in the sequel, will frequently be adopted for similar purposes.

Let ξ, η, ζ, υ, be any point on the line, at a distance u from a, b, c, d; then

$$\xi = a + lu, \quad \eta = b + mu, \quad \zeta = c + nu, \quad \upsilon = d + ku$$

The distance D of this point from x', y', z', v', is given by

$$\begin{aligned} D^2 &= (x'-\xi)^2 + (y'-\eta)^2 + (z'-\zeta)^2 + (v'-\upsilon)^2 \\ &= \Sigma\,(x'-a-lu)^2 \\ &= \Sigma\,(x'-a)^2 - 2u\Sigma l\,(x'-a) + u^2. \end{aligned}$$

In order to obtain the perpendicular, p, upon the line, we select the value of u which makes D a minimum, that minimum value being p; thus we have

$$-\{\Sigma l\,(x'-a)\} + u = 0,$$

that is,

$$u = \Sigma l\,(x'-a),$$

and now

$$p^2 = \{\Sigma\,(x'-a)^2\} - \Sigma\,\{l\,(x'-a)\}^2.$$

Let X, Y, Z, V, be the point corresponding to this value of u; then the direction-cosines of the perpendicular, measured from the line to x', y', z', v', are

$$\frac{1}{p}(x'-X),\quad \frac{1}{p}(y'-Y),\quad \frac{1}{p}(z'-Z),\quad \frac{1}{p}(v'-V),$$

that is, as before (§ 23),

$$\frac{1}{p}\{x'-a-l\Sigma l(x'-a)\},\quad \frac{1}{p}\{y'-b-m\Sigma l(x'-a)\},$$

$$\frac{1}{p}\{z'-c-n\Sigma l(x'-a)\},\quad \frac{1}{p}\{v'-d-k\Sigma l(x'-a)\}.$$

If these are denoted by $\lambda, \mu, \nu, \kappa$, we have

$$l\lambda+m\mu+n\nu+k\kappa=0,$$

as is to be expected.

The coordinates of the foot of the perpendicular are

$$a+l\Sigma l(x'-a),\quad b+m\Sigma l(x'-a),\quad c+n\Sigma l(x'-a),\quad d+k\Sigma l(x'-a).$$

Ex 1. The location and magnitude of the perpendicular can be obtained by drawing a line through x', y', z', v', with direction-cosines $\lambda, \mu, \nu, \kappa$, such that

$$l\lambda+m\mu+n\nu+k\kappa=0,$$

and requiring it to meet the given line (§ 26). Thus, or otherwise, obtain the four relations of the form

$$lu-p\lambda=x'-a,$$

and verify that

$$p=-\Sigma\lambda(x'-a).$$

Ex. 2 Prove that, in the figure on p 7,

the lines βh, γg, $\delta f'$, are perpendicular to OX;
. . αh, γf, $\delta g'$, . OY;
. . αg, βf, $\delta h'$, ... OZ;
and . .. . . $\alpha f'$, $\beta g'$, $\gamma h'$, ... OV.

Ex. 3. Prove that the perpendiculars from P on ff', gg', hh', respectively are equal to

$$\frac{Of\,.\,Of'}{OP},\quad \frac{Og\,.\,Og'}{OP},\quad \frac{Oh\,.\,Oh'}{OP}.$$

Ex. 4. Prove that the lines Pf and Pf' are perpendicular to one another: likewise the lines Pg and Pg': and the lines Ph and Ph'.

Ex. 5. Prove that gg' and hh' make equal angles with Pf. that hh' and ff' make equal angles with Pg: and that ff' and gg' make equal angles with Ph.

Orthogonal frames.

25. When we come to the theory of curves in four-fold space, especially in regard to the principal frame at any point of such a curve, we shall need certain elementary properties of a system of four orthogonal lines, such as are the four coordinate axes.

Let four such lines be Ox', Oy', Oz', Ov', with direction-cosines as in the tableau

O	x	y	z	v
x'	l_1	m_1	n_1	k_1
y'	l_2	m_2	n_2	k_2
z'	l_3	m_3	n_3	k_3
v'	l_4	m_4	n_4	k_4

.

Thus there are the relations

$$l_r^2 + m_r^2 + n_r^2 + k_r^2 = 1,$$

for $r = 1, 2, 3, 4$; and the relations

$$l_r l_s + m_r m_s + n_r n_s + k_r k_s = 0,$$

for (different) values of r, s, $= 1, 2, 3, 4$. Also, if

$$\Delta = \begin{vmatrix} l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \\ l_4, & m_4, & n_4, & k_4 \end{vmatrix},$$

we have

$$\Delta^2 = \begin{vmatrix} \Sigma l_1^2, & \Sigma l_1 l_2, & \Sigma l_1 l_3, & \Sigma l_1 l_4 \\ \Sigma l_2 l_1, & \Sigma l_2^2, & \Sigma l_2 l_3, & \Sigma l_2 l_4 \\ \Sigma l_3 l_1, & \Sigma l_3 l_2, & \Sigma l_3^2, & \Sigma l_3 l_4 \\ \Sigma l_4 l_1, & \Sigma l_4 l_2, & \Sigma l_4 l_3, & \Sigma l_4^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix} = 1,$$

so that $\Delta = \pm 1$. We assume the directions of Ox', Oy', Oz', Ov', to be such that, by continuous displacement of the frame without other change, Ox' can be brought into coincidence with Ox, Oy' with Oy, Oz' with Oz, and Ov' with Ov. then possible values are l_1, m_1, n_1, $k_1 = 1, 0, 0, 0$, and so for the others. With these possible values, $\Delta = 1$; so we take $\Delta = 1$ generally, as a consequence of the assumption, just made concerning the orientation of the frame: that is,

$$\Delta = \begin{vmatrix} l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \\ l_4, & m_4, & n_4, & k_4 \end{vmatrix} = 1.$$

The minors of Δ are required. We write

$$\frac{\partial\Delta}{\partial\theta_r}=\Theta_r,$$

for $\theta_r = l_r, m_r, n_r, k_r$, and $\Theta_r = L_r, M_r, N_r, K_r$, respectively, and for $r = 1, 2, 3, 4$. Then we have

$$\theta_r=\Theta_r.$$

Also

$$\nabla=\begin{vmatrix} L_1, & M_1, & N_1, & K_1 \\ L_2, & M_2, & N_2, & K_2 \\ L_3, & M_3, & N_3, & K_3 \\ L_4, & M_4, & N_4, & K_4 \end{vmatrix}=1,$$

and

$$\frac{\partial\nabla}{\partial\Theta_r}=\theta_r.$$

Further,

$$l_1m_2-m_1l_2=\begin{vmatrix} M_2, & N_2, & K_2 \\ M_3, & N_3, & K_3 \\ M_4, & N_4, & K_4 \end{vmatrix}\begin{vmatrix} L_1, & N_1, & K_1 \\ L_3, & N_3, & K_3 \\ L_4, & N_4, & K_4 \end{vmatrix}-\begin{vmatrix} L_2, & N_2, & K_2 \\ L_3, & N_3, & K_3 \\ L_4, & N_4, & K_4 \end{vmatrix}\begin{vmatrix} M_1, & N_1, & K_1 \\ M_3, & N_3, & K_3 \\ M_4, & N_4, & K_4 \end{vmatrix}$$

$$=\nabla(N_3K_4-K_3N_4)=n_3k_4-k_3n_4,$$

and so for the others. any second minor in Δ is equal to its complementary second minor.

Also, the tableau can be re-arranged so as to change columns into rows, and rows into columns. It then gives the direction-cosines of Ox, Oy, Oz, Ov, with respect to Ox', Oy', Oz', Ov'; and we have

$$\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2=1,$$

for $\theta = l, m, n, k$; and

$$\theta_1\phi_1+\theta_2\phi_2+\theta_3\phi_3+\theta_4\phi_4=0,$$

for (different) values of $\theta, \phi, = l, m, n, k$.

All these results are easily established, being well-known properties of orthogonal determinants.

Ex. Discuss the types of relations (i) between a constituent of Δ and its minor, and (ii) between a second minor and its complementary, when $\Delta=-1$.

Conditions that two lines may meet.

26 In general, two lines in quadruple space do not meet; the conditions, which must be satisfied in order that they may meet, are obtained as follows. Let the lines be

$$\frac{x-a_1}{l_1}=\frac{y-b_1}{m_1}=\frac{z-c_1}{n_1}=\frac{v-d_1}{k_1},$$

and

$$\frac{x-a_2}{l_2}=\frac{y-b_2}{m_2}=\frac{z-c_2}{n_2}=\frac{v-d_2}{k_2}.$$

Let a supposed meeting-point X, Y, Z, V, be at a distance r_1 from a_1, b_1, c_1, d_1, along the first line, and at a distance r_2 from a_2, b_2, c_2, d_2, along the second line, then

$$\begin{aligned} X &= a_1 + l_1 r_1 = a_2 + l_2 r_2, \\ Y &= b_1 + m_1 r_1 = b_2 + m_2 r_2, \\ Z &= c_1 + n_1 r_1 = c_2 + n_2 r_2, \\ V &= d_1 + k_1 r_1 = d_2 + k_2 r_2. \end{aligned}$$

Consequently, the conditions

$$\left\| \begin{matrix} a_2 - a_1, & b_2 - b_1, & c_2 - c_1, & d_2 - d_1 \\ l_1 \quad , & m_1 \quad , & n_1 \quad , & k_1 \\ l_2 \quad , & m_2 \quad , & n_2 \quad , & k_2 \end{matrix} \right\| = 0$$

must be satisfied: apparently four in number, but actually implying two independent conditions.

The conditions may be interpreted as expressing the property that the two given lines and the line joining a_1, b_1, c_1, d_1, to a_2, b_2, c_2, d_2, lie in one plane—an obvious property when the two given lines meet.

Ex Verify that X is the common value of the expressions

$$\begin{aligned} &\{l_2 m_1 a_1 - l_1 m_2 a_2 + l_1 l_2 (b_2 - b_1)\} \div (m_1 l_2 - l_1 m_2), \\ &\{l_2 n_1 a_1 - l_1 n_2 a_2 + l_1 l_2 (c_2 - c_1)\} \div (n_1 l_2 - l_1 n_2), \\ &\{l_2 k_1 a_1 - l_1 k_2 a_2 + l_1 l_2 (d_2 - d_1)\} \div (k_1 l_2 - l_1 k_2); \end{aligned}$$

and obtain the corresponding expressions for Y, Z, V.

Shortest distance between two lines that do not meet.

27. When the two lines

$$\frac{x - a_1}{l_1} = \frac{y - b_1}{m_1} = \frac{z - c_1}{n_1} = \frac{v - d_1}{k_1},$$

$$\frac{x - a_2}{l_2} = \frac{y - b_2}{m_2} = \frac{z - c_2}{n_2} = \frac{v - d_2}{k_2},$$

do not meet, the magnitude and the position of the shortest distance between them may be required.

Let this shortest distance meet the first line in X_1, Y_1, Z_1, V_1, at a distance r_1 from a_1, b_1, c_1, d_1; and let it meet the second line in X_2, Y_2, Z_2, V_2, at a distance r_2 from a_2, b_2, c_2, d_2. Then

$$\begin{gathered} X_1 = a_1 + l_1 r_1, \quad Y_1 = b_1 + m_1 r_1, \quad Z_1 = c_1 + n_1 r_1, \quad V_1 = d_1 + k_1 r_1, \\ X_2 = a_2 + l_2 r_2, \quad Y_2 = b_2 + m_2 r_2, \quad Z_2 = c_2 + n_2 r_2, \quad V_2 = d_2 + k_2 r_2; \end{gathered}$$

and, if D denote the shortest distance between the lines,

$$\begin{aligned} D^2 &= \Sigma (X_2 - X_1)^2 \\ &= \Sigma (a_2 - a_1 + l_2 r_2 - l_1 r_1)^2. \end{aligned}$$

As D is the shortest distance between the lines, the last expression for D^2 must provide a minimum value for the duly chosen values of r_1 and r_2; and these are given by the equations

$$\begin{aligned} -\Sigma l_1 (a_2 - a_1 + l_2 r_2 - l_1 r_1) &= 0, \\ \Sigma l_2 (a_2 - a_1 + l_2 r_2 - l_1 r_1) &= 0. \end{aligned}$$

Let α denote the inclination of the two lines to one another: then the values of r_1 and r_2 are given by the equations

$$\begin{aligned} r_1 \quad - r_2 \cos \alpha &= \Sigma l_1 (a_2 - a_1), \\ r_1 \cos \alpha - r_2 \quad &= \Sigma l_2 (a_2 - a_1), \end{aligned}$$

so that

$$\begin{aligned} r_1 \sin^2 \alpha &= \Sigma (l_1 - l_2 \cos \alpha)(a_2 - a_1), \\ r_2 \sin^2 \alpha &= \Sigma (l_1 \cos \alpha - l_2)(a_2 - a_1). \end{aligned}$$

Hence

$$X_2 - X_1$$
$$= a_2 - a_1 + \frac{1}{\sin^2 \alpha} \{ l_1 \; \Sigma (l_2 \cos \alpha - l_1)(a_2 - a_1) + l_2 \; \Sigma (l_1 \cos \alpha - l_2)(a_2 - a_1) \},$$

$$Y_2 - Y_1$$
$$= b_2 - b_1 + \frac{1}{\sin^2 \alpha} \{ m_1 \Sigma (l_2 \cos \alpha - l_1)(a_2 - a_1) + m_2 \Sigma (l_1 \cos \alpha - l_2)(a_2 - a_1) \},$$

$$Z_2 - Z_1$$
$$= c_2 - c_1 + \frac{1}{\sin^2 \alpha} \{ n_1 \Sigma (l_2 \cos \alpha - l_1)(a_2 - a_1) + n_2 \Sigma (l_1 \cos \alpha - l_2)(a_2 - a_1) \},$$

$$V_2 - V_1$$
$$= d_2 - d_1 + \frac{1}{\sin^2 \alpha} \{ k_1 \Sigma (l_2 \cos \alpha - l_1)(a_2 - a_1) + k_2 \Sigma (l_1 \cos \alpha - l_2)(a_2 - a_1) \}$$

Also

$$\begin{aligned} D^2 &= \Sigma (a_2 - a_1 + l_2 r_2 - l_1 r_1)^2 \\ &= \Sigma (a_2 - a_1)^2 + 2 r_2 \Sigma l_2 (a_2 - a_1) - 2 r_1 \Sigma l_1 (a_2 - a_1) + r_2^2 - 2 r_1 r_2 \cos \alpha + r_1^2 \\ &= \Sigma (a_2 - a_1)^2 + 2 r_2 (r_1 \cos \alpha - r_2) - 2 r_1 (r_1 - r_2 \cos \alpha) + r_2^2 - 2 r_1 r_2 \cos \alpha + r_1^2 \\ &= \Sigma (a_2 - a_1)^2 - (r_1^2 - 2 r_1 r_2 \cos \alpha + r_2^2) \\ &= \Sigma (a_2 - a_1)^2 - \frac{\Delta}{\sin^2 \alpha}, \end{aligned}$$

where Δ denotes

$$\{ \Sigma l_1 (a_2 - a_1) \}^2 - 2 \{ \Sigma l_1 (a_2 - a_1) \} \{ \Sigma l_2 (a_2 - a_1) \} \cos \alpha + \{ \Sigma l_2 (a_2 - a_1) \}^2.$$

The geometrical significance of the various terms in the expression, giving the value of D, is immediately derivable from a figure.

The direction-cosines of the shortest distance, say λ, μ, ν, κ, are

$$\lambda=\frac{1}{D}(X_2-X_1),\quad \mu=\frac{1}{D}(Y_2-Y_1),\quad \nu=\frac{1}{D}(Z_2-Z_1),\quad \kappa=\frac{1}{D}(V_2-V_1).$$

The two equations, which determine r_1 and r_2, can be written

$$\Sigma l_1(X_2-X_1)=0,\quad \Sigma l_2(X_2-X_1)=0$$

and therefore

$$\Sigma l_1\lambda=0,\quad \Sigma l_2\lambda=0\,.$$

that is, the shortest distance between the two lines is perpendicular to both lines, as is to be expected.

28. It may be pointed out here—for the inadequacy of the property will recur—that the actual direction-cosines of the shortest distance are not determinable solely from the property that it is perpendicular to the two lines, as expressed by the equations

$$\Sigma l_1\lambda=0,\quad \Sigma l_2\lambda=0.$$

The fact is that, in quadruple space, there is a simple infinitude of directions perpendicular to two given lines; for these two equations, together with $\Sigma\lambda^2=1$, are satisfied by a simple infinitude of values of λ, μ, ν, κ. To make their determination precise, we can proceed as follows. Let a line, with direction-cosines λ, μ, ν, κ, satisfying the two relations which express the fact that it is perpendicular to the two lines, be drawn through a point X_1, Y_1, Z_1, V_1, on the first line so as to meet the second line

$$\frac{x-a_2}{l_2}=\frac{y-b_2}{m_2}=\frac{z-c_2}{n_2}=\frac{v-d_2}{k_2}\,.$$

This new line is

$$\frac{x-X_1}{\lambda}=\frac{y-Y_1}{\mu}=\frac{z-Z_1}{\nu}=\frac{v-V_1}{\kappa}\,.$$

the conditions, that it may meet the second line, are that values D and r_2 can be found such that, if the common point be X_2, Y_2, Z_2, V_2, then

$$\begin{aligned}X_2&=a_2+l_2r_2=X_1+\lambda D,\\ Y_2&=b_2+m_2r_2=Y_1+\mu D,\\ Z_2&=c_2+n_2r_2=Z_1+\nu D,\\ V_2&=d_2+k_2r_2=V_1+\kappa D\end{aligned}$$

Moreover, as X_1, Y_1, Z_1, V_1, is taken on the first line

$$\frac{x-a_1}{l_1}=\frac{y-b_1}{m_1}=\frac{z-c_1}{n_1}=\frac{v-d_1}{k_1}\,,$$

there is a quantity r_1 such that

$$\frac{X_1-a_1}{l_1}=\frac{Y_1-b_1}{m_1}=\frac{Z_1-c_1}{n_1}=\frac{V_1-d_1}{k_1}=r_1$$

We thus have

$$\begin{aligned}
\lambda D &= X_2 - X_1 = a_2 - a_1 + l_2 r_2 - l_1 r_1, \\
\mu D &= Y_2 - Y_1 = b_2 - b_1 + m_2 r_2 - m_1 r_1, \\
\nu D &= Z_2 - Z_1 = c_2 - c_1 + n_2 r_2 - n_1 r_1, \\
\kappa D &= V_2 - V_1 = d_2 - d_1 + k_2 r_2 - k_1 r_1,
\end{aligned}$$

together with $\Sigma\lambda l_1 = 0$, $\Sigma\lambda l_2 = 0$. These are the former equations: the remainder of the necessary analysis has already been given.

Ex. Prove that the shortest distance between AD and fP, in the figure on p. 7, is equal to BC; that the shortest distance between BD and gP is equal to AC; and that the shortest distance between CD and hP is equal to AB.

Prove also that these shortest distances bisect the specified portions of each of the lines between which they are measured: that is, the shortest distance between AD and fP is the line joining the middle point of AD and the middle point of fP, and similarly for the other two shortest distances specified.

CHAPTER III.

PLANES.

Forms of the two equations of a plane.

29. We have already seen (§ 13) that two equations are necessary for the mathematical specification of a plane. When the plane is defined by means of three points leading to a system of straight lines, the two equations have the form

$$\left\|\begin{matrix} x-a_1, & y-b_1, & z-c_1, & v-d_1 \\ a_2-a_1, & b_2-b_1, & c_2-c_1, & d_2-d_1 \\ a_3-a_1, & b_3-b_1, & c_3-c_1, & d_3-d_1 \end{matrix}\right\| = 0.$$

When the plane is defined by means of a point and two lines passing in assigned directions through the point, the two equations have the form

$$\left\|\begin{matrix} x-a_1, & y-b_1, & z-c_1, & v-d_1 \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{matrix}\right\| = 0.$$

A deduced equivalent form of the first pair of equations is obtained by representing the coordinates of a point current in the plane by

$$x=\lambda a_1+\mu a_2+\nu a_3,\quad y=\lambda b_1+\mu b_2+\nu b_3,\quad z=\lambda c_1+\mu c_2+\nu c_3,\quad v=\lambda d_1+\mu d_2+\nu d_3,$$

with the restriction $\lambda+\mu+\nu=1$ upon the otherwise arbitrary parameters λ, μ, ν. An equivalent form of the second pair of equations is

$$x-a_1=l_1\rho+l_2\sigma,\quad y-b_1=m_1\rho+m_2\sigma,\quad z-c_1=n_1\rho+n_2\sigma,\quad v-d_1=k_1\rho+k_2\sigma,$$

where ρ and σ can be interpreted as distances ρ and σ taken from a_1, b_1, c_1, d_1, along the respective directions l_1, m_1, n_1, k_1, and l_2, m_2, n_2, k_2. Also, any direction through the point a_1, b_1, c_1, d_1, and therefore any parallel direction through any other point in the plane (that is, any direction in the plane), are given by

$$L=\lambda l_1+\mu l_2,\quad M=\lambda m_1+\mu m_2,\quad N=\lambda n_1+\mu n_2,\quad K=\lambda k_1+\mu k_2.$$

All these forms are adequate for the mathematical representation of the plane. No one of them is unique, even within its own type of expression. Thus if α_1, β_1, γ_1, δ_1, α_2, β_2, γ_2, δ_2; and α_3, β_3, γ_3, δ_3, be three non-collinear points in the given plane, so that the relations

$$\frac{\alpha_3-\alpha_1}{\alpha_2-\alpha_1}=\frac{\beta_3-\beta_1}{\beta_2-\beta_1}=\frac{\gamma_3-\gamma_1}{\gamma_2-\gamma_1}=\frac{\delta_3-\delta_1}{\delta_2-\delta_1}$$

are not simultaneously satisfied, the two equations can be expressed in the form

$$\left\|\begin{matrix} x-\alpha_1, & y-\beta_1, & z-\gamma_1, & v-\delta_1 \\ \alpha_2-\alpha_1, & \beta_2-\beta_1, & \gamma_2-\gamma_1, & \delta_2-\delta_1 \\ \alpha_3-\alpha_1, & \beta_3-\beta_1, & \gamma_3-\gamma_1, & \delta_3-\delta_1 \end{matrix}\right\| = 0.$$

Again, if l_1', m_1', n_1', k_1'; l_2', m_2', n_2', k_2'; be two other distinct directions in the plane, so that

$$i_1' = \alpha i_1 + \beta i_2, \quad i_2' = \gamma i_1 + \delta i_2,$$

where $i = l, m, n, k$, in succession, and if $\alpha\delta - \beta\gamma$ is not zero, the two equations can be expressed in the form

$$\left\|\begin{matrix} x-a_1, & y-b_1, & z-c_1, & v-d_1 \\ l_1' \;, & m_1' \;, & n_1' \;, & k_1' \\ l_2' \;, & m_2' \;, & n_2' \;, & k_2' \end{matrix}\right\| = 0.$$

Nor do these forms exhaust the apparently distinct possibilities of securing the representation of a plane by means of two equations. For two simultaneous independent linear equations

$$A_1x + B_1y + C_1z + D_1v = E_1,$$
$$A_2x + B_2y + C_2z + D_2v = E_2,$$

can be expressed in either of the preceding forms in an unlimited number of ways; and these are equivalent to

$$A_1'x + B_1'y + C_1'z + D_1'v = E_1',$$
$$A_2'x + B_2'y + C_2'z + D_2'v = E_2',$$

where

$$P_1' = \alpha P_1 + \beta P_2, \quad P_2' = \gamma P_1 + \delta P_2,$$

for $P = A, B, C, D, E$, while $\alpha\delta - \beta\gamma$ is not zero, a transformation that can be effected also in an unlimited number of ways.

Canonical form of the equations.

30. Now it may be desirable to have a canonical form of reference for the two equations of a plane. Such canonical form is required to contain, explicitly or implicitly, the least number of independent constants that are sufficient to express a plane, completely general in its position and in its orientation: and it must be such that any given pair of equations, which represent a plane, can have an equivalent canonical expression A canonica form of such a character can be obtained either from the equations

$$x - a = l_1\rho + l_2\sigma, \quad y - b = m_1\rho + m_2\sigma, \quad z - c = n_1\rho + n_2\sigma, \quad v - d = k_1\rho + k_2\sigma,$$

or from the equations

$$A_1x + B_1y + C_1z + D_1v = E_1,$$
$$A_2x + B_2y + C_2z + D_2v = E_2,$$

by deducing from them, in the respective instances, expressions for z and for v in terms of x and y. Let these be*

$$\left.\begin{aligned} z &= f + px + qy \\ v &= h + rx + sy \end{aligned}\right\}.$$

As this form will occasionally be adopted, it will be convenient to indicate the significance of its constants, in reference to the constants in the other forms.

The two equations represent the same plane as the equations

$$x-a,\quad y-b,\quad z-c,\quad v-d = (\rho,\ \sigma\,\big)\!\!\begin{array}{|cccc|} l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{array},$$

provided

$$\left.\begin{aligned} c &= f + pa + qb \\ d &= h + ra + sb \end{aligned}\right\},$$

$$\left.\begin{aligned} n_1 &= pl_1 + qm_1 \\ n_2 &= pl_2 + qm_2 \end{aligned}\right\},\qquad \left.\begin{aligned} k_1 &= rl_1 + sm_1 \\ k_2 &= rl_2 + sm_2 \end{aligned}\right\}.$$

Then

$$\frac{r}{k_1n_2 - m_1k_2} = \frac{s}{l_1k_2 - k_1l_2} = \frac{qr-ps}{k_1n_2 - n_1k_2}$$

$$= \frac{p}{n_1m_2 - m_1n_2} = \frac{q}{l_1n_2 - n_1l_2} = \frac{1}{l_1m_2 - m_1l_2};$$

while

$$f = \frac{1}{\begin{vmatrix} l_1, & m_1 \\ l_2, & m_2 \end{vmatrix}} \begin{vmatrix} a, & b, & c \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix},\qquad h = \frac{1}{\begin{vmatrix} l_1, & m_1 \\ l_2, & m_2 \end{vmatrix}} \begin{vmatrix} a, & b, & d \\ l_1, & m_1, & k_1 \\ l_2, & m_2, & k_2 \end{vmatrix}.$$

Similarly, the two equations

$$A_1x + B_1y + C_1z + D_1v = E_1,$$

$$A_2x + B_2y + C_2z + D_2v = E_2,$$

will be represented by the same canonical form if

$$\frac{p}{D_1A_2 - A_1D_2} = \frac{r}{A_1C_2 - C_1A_2}$$

$$= \frac{q}{D_1B_2 - B_1D_2} = \frac{s}{B_1C_2 - C_1B_2} = \frac{qr-ps}{B_1A_2 - A_1B_2} = \frac{1}{C_1D_2 - D_1C_2}$$

$$= \frac{f}{E_1D_2 - D_1E_2} = \frac{h}{C_1E_2 - E_1C_2}.$$

(The reason for introducing the combination $qr - ps$ will appear later.)

---

* In an exceptional instance such that $C_1D_2 - D_1C_2 = 0$, the form would still serve, merely by an interchange of the axes of reference.

It now appears that the canonical form of the equations remains unchanged, whatever initial point a, b, c, d, and whatever guiding lines l_1, m_1, n_1, k_1, and l_2, m_2, n_2, k_2, be chosen for the mathematical representation of a given plane

$$\left\Vert \begin{matrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{matrix} \right\Vert .$$

For a change of initial point to a', b', c', d', where

$$a'=a+\lambda l_1+\mu l_2,\quad b'=b+\lambda m_1+\mu m_2,\quad c'=c+\lambda n_1+\mu n_2,\quad d'=d+\lambda k_1+\mu k_2,$$

does not affect the values of p, q, r, s, and leaves f and h unaltered. And a change to other guiding lines

$$l_1'=\alpha l_1+\beta l_2,\quad m_1'=\alpha m_1+\beta m_2,\quad n_1'=\alpha n_1+\beta n_2,\quad k_1'=\alpha k_1+\beta k_2,$$
$$l_2'=\gamma l_1+\delta l_2\,,\quad m_2'=\gamma m_1+\delta m_2\,,\quad n_2'=\gamma n_1+\delta n_2\,,\quad k_2'=\gamma k_1+\delta k_2\,,$$

gives determinants of the type

$$l_1'm_2'-m_1'l_2'=(\alpha\delta-\beta\gamma)(l_1m_2-m_1l_2),$$

so that the new values of p, q, r, s, f, h, become (on the removal of the non-zero factor $\alpha\delta-\beta\gamma$) the same as before.

The same remark applies to the two equations

$$\left.\begin{matrix} A_1x+B_1y+C_1z+D_1v=E_1 \\ A_2x+B_2y+C_2z+D_2v=E_2 \end{matrix}\right\} :$$

when two independent linear combinations of these equations are taken, the resulting canonical form is unaltered.

Ex. 1. The two planes

$$\left\Vert \begin{matrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{matrix} \right\Vert =0,\qquad \left.\begin{matrix} A_1(x-a)+B_1(y-b)+C_1(z-c)+D_1(v-d)=0 \\ A_2(x-a)+B_2(y-b)+C_2(z-c)+D_2(v-d)=0 \end{matrix}\right\},$$

are one and the same; prove that

$$\frac{B_1C_2-C_1B_2}{l_1k_2-k_1l_2}=\frac{C_1A_2-A_1C_2}{m_1k_2-k_1m_2}=\frac{A_1B_2-B_1A_2}{n_1k_2-k_1n_2}$$
$$=\frac{A_1D_2-D_1A_2}{m_1n_2-n_1m_2}=\frac{B_1D_2-D_1B_2}{n_1l_2-l_1n_2}=\frac{C_1D_2-D_1C_2}{l_1m_2-m_1l_2}=\frac{\sin\beta}{\sin\alpha},$$

where

$$\Sigma l_1^2=1,\quad \Sigma l_2^2=1,\quad \Sigma l_1l_2=\cos\alpha,$$
$$\Sigma A_1^2=1,\quad \Sigma A_2^2=1,\quad \Sigma A_1A_2=\cos\beta.$$

Ex. 2. A plane is represented as the intersection of two flats by the equations

$$L_1(x-a)+M_1(y-b)+N_1(z-c)+K_1(v-d)=0,$$
$$L_2(x-a)+M_2(y-b)+N_2(z-c)+K_2(v-d)=0,$$

and $L_1L_2+M_1M_2+N_1N_2+K_1K_2=\cos\alpha$, where α is not equal to $\frac{1}{2}\pi$. Shew that the same plane is given as the intersection of the two flats

$$L_1'(x-a)+M_1'(y-b)+N_1'(z-c)+K_1'(v-d)=0,$$
$$L_2'(x-a)+M_2'(y-b)+N_2'(z-c)+K_2'(v-d)=0,$$

where $L_1'L_2'+M_1'M_2'+N_1'N_2'+K_1'K_2'=0$, if

$$\left.\begin{aligned} L_1'\sin\alpha &= L_1\cos(\alpha-\theta)-L_2\cos\theta \\ M_1'\sin\alpha &= M_1\cos(\alpha-\theta)-M_2\cos\theta \\ N_1'\sin\alpha &= N_1\cos(\alpha-\theta)-N_2\cos\theta \\ K_1'\sin\alpha &= K_1\cos(\alpha-\theta)-K_2\cos\theta \end{aligned}\right\}, \qquad \left.\begin{aligned} L_2'\sin\alpha &= L_1\sin(\alpha-\theta)+L_2\sin\theta \\ M_2'\sin\alpha &= M_1\sin(\alpha-\theta)+M_2\sin\theta \\ N_2'\sin\alpha &= N_1\sin(\alpha-\theta)+N_2\sin\theta \\ K_2'\sin\alpha &= K_1\sin(\alpha-\theta)+K_2\sin\theta \end{aligned}\right\},$$

and θ is any arbitrary quantity.

(Owing to the occurrence of an arbitrary quantity θ, the transformation is possible in an unlimited number of ways.)

Direction-cosines of a line in a plane.

31 When the equations of a plane are given in the form

$$\left\|\begin{matrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ ; & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{matrix}\right\|=0,$$

the direction-cosines of any line lying in the plane can be taken as

$$\left.\begin{aligned} L &= \lambda l_1+\mu l_2 \\ M &= \lambda m_1+\mu m_2 \\ N &= \lambda n_1+\mu n_2 \\ K &= \lambda k_1+\mu k_2 \end{aligned}\right\},$$

which usually prove the most convenient in connection with the equations of the plane.

When the equations of the plane are given in the canonical form

$$\left.\begin{aligned} z-f &= px+qy \\ v-h &= rx+sy \end{aligned}\right\},$$

the conditions that the line

$$\frac{x-a}{L}=\frac{y-b}{M}=\frac{z-c}{N}=\frac{v-d}{K}$$

should lie in the plane are

$$c-f=pa+qb, \quad d-h=ra+sb,$$

which do not affect the direction-cosines, and

$$\left.\begin{aligned} N &= pL+qM \\ K &= rL+sM \end{aligned}\right\}.$$

It will be noted that, for these relations of condition, two of the cosines are selected in particular for the expression of the remaining pair. The association of the deduced equations

$$\begin{aligned}(ps-qr)\,L &= sN-qK,\\ (ps-qr)\,M &= -rN+pK,\end{aligned}$$

enables any pair of the four quantities L, M, N, K, to be used for the expression of the remaining pair.

Ex Prove that, if 2α denote the angle between the lines of reference for the equations of the plane in the earlier form, the direction-cosines of any line can be expressed in terms of a single parameter ω by the relations

$$\left.\begin{aligned}L &= \tfrac{1}{2}(l_2+l_1)\frac{\cos\omega}{\cos\alpha}+\tfrac{1}{2}(l_2-l_1)\frac{\sin\omega}{\sin\alpha}\\ M &= \tfrac{1}{2}(m_2+m_1)\frac{\cos\omega}{\cos\alpha}+\tfrac{1}{2}(m_2-m_1)\frac{\sin\omega}{\sin\alpha}\\ N &= \tfrac{1}{2}(n_2+n_1)\frac{\cos\omega}{\cos\alpha}+\tfrac{1}{2}(n_2-n_1)\frac{\sin\omega}{\sin\alpha}\\ K &= \tfrac{1}{2}(k_2+k_1)\frac{\cos\omega}{\cos\alpha}+\tfrac{1}{2}(k_2-k_1)\frac{\sin\omega}{\sin\alpha}\end{aligned}\right\}$$

Prove also that, when the equations of the plane are given in the canonical form, corresponding expressions for L, M, N, K, are given by

$$\left.\begin{aligned}L\sin 2\eta &= \cos\beta\sin(\eta-\theta)\\ M\sin 2\eta &= \cos\gamma\sin(\eta+\theta)\\ N\sin 2\eta &= \sin\beta\cos\delta\sin(\eta-\theta)+\sin\gamma\cos\epsilon\sin(\eta+\theta)\\ K\sin 2\eta &= \sin\beta\sin\delta\sin(\eta-\theta)+\sin\gamma\sin\epsilon\sin(\eta+\theta)\end{aligned}\right\},$$

where $p=\tan\beta\cos\delta$, $r=\tan\beta\sin\delta$, $q=\tan\gamma\cos\epsilon$, $s=\tan\gamma\sin\epsilon$, and

$$\cos 2\eta=\sin\beta\sin\gamma\cos(\delta-\epsilon),$$

while θ is a parameter.

Perpendicular from a point to a plane.

32. The principal magnitude, in the relation between a plane and a point which does not lie in the plane, is the shortest distance of the point from the plane.

Let X, Y, Z, V, be the foot N of the perpendicular from an external point P with coordinates ξ, η, ζ, v, to the plane

$$\left\|\begin{matrix}x-a, & y-b, & z-c, & v-d\\ l_1\,, & m_1\,, & n_1\,, & k_1\\ l_2\,, & m_2\,, & n_2\,, & k_2\end{matrix}\right\|=0.$$

There are quantities r_1 and r_2 such that

$$X-a=l_1r_1+l_2r_2,\quad Y-b=m_1r_1+m_2r_2,\quad Z-c=n_1r_1+n_2r_2,\quad V-d=k_1r_1+k_2r_2.$$

Then the magnitude D^2, where

$$\begin{aligned} D^2 &= \Sigma (\xi - X)^2 \\ &= \Sigma (\xi - a - l_1 r_1 - l_2 r_2)^2, \end{aligned}$$

must be a minimum for all values of r_1 and r_2; hence

$$\Sigma l_1 (\xi - a - l_1 r_1 - l_2 r_2) = 0,$$
$$\Sigma l_2 (\xi - a - l_1 r_1 - l_2 r_2) = 0.$$

These critical equations can be taken in two forms.

In the first form, they can be written

$$\Sigma l_1 (\xi - X) = 0, \quad \Sigma l_2 (\xi - X) = 0;$$

if L, M, N, K, are the direction-cosines of the perpendicular, then

$$\Sigma l_1 L = 0, \quad \Sigma l_2 L = 0,$$

and therefore

$$\Sigma (\alpha l_1 + \beta l_2) L = 0$$

where α and β are any two parametric quantities. Hence the perpendicular from the external point on the plane is perpendicular to every direction in the plane. (It will be seen, hereafter, that the property of being perpendicular to every direction at a point in the plane is not sufficient to determine uniquely the direction of a line: here, the concern is with the perpendicular from an external point on the plane.)

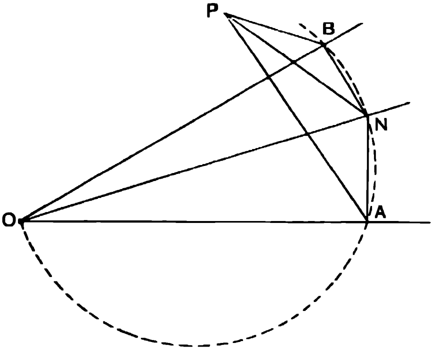

Fig. 2.

In the second form, the critical equations can be written

$$\Sigma l_1 (\xi - a) - r_1 - r_2 \cos \omega = 0,$$
$$\Sigma l_2 (\xi - a) - r_1 \cos \omega - r_2 = 0,$$

where $\cos \omega$ denotes $\Sigma l_1 l_2$, so that ω is the angle AOB, when OA is the line with l_1, m_1, n_1, k_1, for direction-cosines, and OB is the line with l_2, m_2, n_2, k_2, for direction-cosines. Let PA be the perpendicular from the external point P on the line OA, and PB the perpendicular from P on OB: then

$$OA = \Sigma l_1 (\xi - a) = D_1, \quad OB = \Sigma l_2 (\xi - a) = D_2.$$

Thus*

$$r_1 + r_2 \cos \omega = D_1, \quad r_1 \cos \omega + r_2 = D_2;$$

and therefore

$$r_1 \sin^2 \omega = D_1 - D_2 \cos \omega, \quad r_2 \sin^2 \omega = D_2 - D_1 \cos \omega.$$

Let X_1, Y_1, Z_1, V_1, be the coordinates of A, and X_2, Y_2, Z_2, V_2, those of B, so that

$$X_1 - a = l_1 D_1, \quad Y_1 - b = m_1 D_1, \quad Z_1 - c = n_1 D_1, \quad V_1 - d = k_1 D_1,$$
$$X_2 - a = l_2 D_2, \quad Y_2 - b = m_2 D_2, \quad Z_2 - c = n_2 D_2, \quad V_2 - d = k_2 D_2.$$

Now

$$\begin{aligned} \Sigma l_1 (X - X_1) &= \Sigma l_1 \{X - a - (X_1 - a)\} \\ &= \Sigma l_1 (l_1 r_1 + l_2 r_2 - l_1 D_1) = r_1 + r_2 \cos \omega - D_1 = 0, \\ \Sigma l_2 (X - X_2) &= \Sigma l_2 \{X - a - (X_2 - a)\} \\ &= \Sigma l_2 (l_1 r_1 + l_2 r_2 - l_2 D_2) = r_1 \cos \omega + r_2 - D_2 = 0; \end{aligned}$$

and therefore AN is perpendicular to OA, and BN is perpendicular to OB, all the lines lying in the plane AOB. Thus N is the other extremity of a diameter ON of the circle AOB. Also

$$\Sigma (\xi - X)(X - a) = r_1 \Sigma l_1 (\xi - X) + r_2 \Sigma l_2 (\xi - X) = 0,$$

so that PN is perpendicular to ON, as is to be expected.

Next, we have

$$\xi - X = \xi - a - l_1 r_1 - l_2 r_2,$$

with corresponding expressions for $\eta - Y$, $\zeta - Z$, $v - V$; hence

$$\begin{aligned} D^2 &= \Sigma (\xi - X)^2 \\ &= \Sigma (\xi - a)^2 - 2 r_1 \Sigma l_1 (\xi - a) - 2 r_2 \Sigma l_2 (\xi - a) + r_1^2 + 2 r_1 r_2 \cos \omega + r_2^2 \\ &= \Sigma (\xi - a)^2 - \frac{1}{\sin^2 \omega} (D_1^2 - 2 D_1 D_2 \cos \omega + D_2^2), \end{aligned}$$

on substitution. We thus have the length of the perpendicular from the external point P. Moreover, $\Sigma (\xi - a)^2 = OP^2$, and therefore

$$ON^2 = \frac{1}{\sin^2 \omega} (D_1^2 - 2 D_1 D_2 \cos \omega + D_2^2) = \frac{AB^2}{\sin^2 \omega},$$

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* In connection with the discussions of directions in non-orthogonal frames of two dimensions and of three dimensions, which occur in the present chapter and the three succeeding chapters, a comprehensive reference may here be made to E. H. Neville's *Prolegomena to analytical geometry in anisotropic Euclidean space of three dimensions* (Camb. Univ. Press, 1922).

agreeing with the known expression for the diameter of a circle circumscribing a triangle.

Manifestly the five points O, A, N, B, P, lie in a globular region in our quadruple space, the equation of which is

$$(x-a)(x-\xi)+(y-b)(y-\eta)+(z-c)(z-\zeta)+(v-d)(v-\upsilon)=0;$$

and

$$\cos NOA=\frac{OA}{ON}=\frac{D_1\sin\omega}{(D_1^2-2D_1D_2\cos\omega+D_2^2)^{\frac{1}{2}}},$$

$$\cos NOB=\frac{OB}{ON}=\frac{D_2\sin\omega}{(D_1^2-2D_1D_2\cos\omega+D_2^2)^{\frac{1}{2}}}.$$

33. As regards the preceding investigation, the same remark holds as held concerning the construction of the shortest distance between two given lines (§ 28): an assumption, that the perpendicular from P is perpendicular to every direction in the plane, is not sufficient to specify the direction of the line. When a direction L, M, N, K, is thus chosen perpendicular to every line in the plane, we have

$$\Sigma l_1 L=0,\quad \Sigma l_2 L=0,$$

and a line through $\xi, \eta, \zeta, \upsilon$, in that chosen direction is

$$\frac{x-\xi}{L}=\frac{y-\eta}{M}=\frac{z-\zeta}{N}=\frac{v-\upsilon}{K}.$$

There is a simple infinitude of such lines; they manifestly lie in the two flats

$$\Sigma l_1(x-\xi)=0,\quad \Sigma l_2(x-\xi)=0,$$

that is, they lie in another plane. Of this simple infinitude, there is a single line which actually meets the given plane, the sole condition of meeting being

$$\begin{vmatrix} \xi-a, & \eta-b, & \zeta-c, & \upsilon-d \\ L\ , & M\ , & N\ , & K \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{vmatrix}=0.$$

Accordingly, quantities ρ, σ, τ, exist in connection with this single line, such that

$$\begin{aligned} \xi-a&=L\rho+l_1\sigma+l_2\tau,\\ \eta-b&=M\rho+m_1\sigma+m_2\tau,\\ \zeta-c&=N\rho+n_1\sigma+n_2\tau,\\ \upsilon-d&=K\rho+k_1\sigma+k_2\tau. \end{aligned}$$

Multiply by L, M, N, K: we have

$$\rho=\Sigma L(\xi-a)=D.$$

Multiply by l_1, m_1, n_1, k_1: we have

$$\sigma + \tau \cos\omega = \Sigma l_1(\xi - a) = D_1.$$

Multiply by l_2, m_2, n_2, k_2: we have

$$\sigma\cos\omega + \tau = \Sigma l_2(\xi - a) = D_2.$$

Here D, D_1, D_2, are the same quantities as before: so

$$\sigma = r_1, \quad \tau = r_2.$$

Finally, when the four equations are squared and the results are added, we have

$$\Sigma(\xi - a)^2 = \rho^2 + \sigma^2 + 2\sigma\tau\cos\omega + \tau^2;$$

and therefore

$$\begin{aligned} D^2 &= \rho^2 \\ &= \Sigma(\xi - a)^2 - (r_1^2 + 2r_1r_2\cos\omega + r_2^2) \\ &= \Sigma(\xi - a)^2 - \frac{1}{\sin^2\omega}(D_1^2 - 2D_1D_2\cos\omega + D_2^2), \end{aligned}$$

being the former result.

34. We have seen (§ 29) that a plane remains unaltered, when changes are effected in the guiding lines, it is therefore important to establish (or verify) the fact that the expressions, connected with the magnitude and the position of the perpendicular, are invariant under such changes.

Let any two directions l_1', m_1', n_1', k_1', and l_2', m_2', n_2', k_2', be taken in the plane: then there are constants $\gamma, \epsilon, \delta, \eta$, such that

$$l_1' = \gamma l_1 + \epsilon l_2, \quad m_1' = \gamma m_1 + \epsilon m_2, \quad n_1' = \gamma n_1 + \epsilon n_2, \quad k_1' = \gamma k_1 + \epsilon k_2,$$

$$l_2' = \delta l_1 + \eta l_2, \quad m_2' = \delta m_1 + \eta m_2, \quad n_2' = \delta n_1 + \eta n_2, \quad k_2' = \delta k_1 + \eta k_2,$$

while the quantity $\mu, = \gamma\eta - \epsilon\delta$, does not vanish. Let D_1', D_2', ω', D', be the quantities in this representation of the plane, which correspond respectively to D_1, D_2, ω, D, in the former representation. Then

$$D_1' = \Sigma l_1'(\xi - a) = \Sigma(\gamma l_1 + \epsilon l_2)(\xi - a) = \gamma D_1 + \epsilon D_2,$$

$$D_2' = \Sigma l_2'(\xi - a) = \Sigma(\delta l_1 + \eta l_2)(\xi - a) = \delta D_1 + \eta D_2.$$

Also we have

$$1 = \Sigma l_1'^2 = \gamma^2 + 2\epsilon\gamma\cos\omega + \epsilon^2,$$

$$1 = \Sigma l_2'^2 = \delta^2 + 2\delta\eta\cos\omega + \eta^2,$$

$$\cos\omega' = \Sigma l_1' l_2' = \gamma\delta + \epsilon\eta + (\gamma\eta + \epsilon\delta)\cos\omega.$$

Moreover,

$$\mu l_1 = \eta l_1' - \epsilon l_2', \quad \mu l_2 = -\delta l_1' + \gamma l_2',$$

with corresponding expressions for $\mu m_1, \mu n_1, \mu k_1$, and $\mu m_2, \mu n_2, \mu k_2$; hence

$$\mu^2 = \Sigma\mu^2 l_1^2 = \eta^2 - 2\eta\epsilon\cos\omega' + \epsilon^2,$$

$$\mu^2 = \Sigma\mu^2 l_2^2 = \delta^2 - 2\gamma\delta\cos\omega' + \gamma^2,$$

$$\mu^2\cos\omega = \Sigma(\mu l_1 \cdot \mu l_2) = -\eta\delta - \epsilon\gamma + (\eta\gamma + \epsilon\delta)\cos\omega'.$$

Also

$$\sin^2 \omega' = \Sigma\,(l_1' m_2' - m_1' l_2')^2$$
$$= \mu^2 \Sigma\,(l_1 m_2 - m_1 l_2)^2 = \mu^2 \sin^2 \omega.$$

With these results, we have

$$\begin{aligned} &D_1'^2 + D_2'^2 - 2D_1' D_2' \cos \omega' \\ &\quad = (\gamma^2 + \delta^2 - 2\gamma\delta \cos \omega')\, D_1^2 \\ &\qquad + (\epsilon^2 + \eta^2 - 2\epsilon\eta \cos \omega')\, D_2^2 \\ &\qquad\qquad + 2\,\{\epsilon\gamma + \delta\eta - (\gamma\eta + \epsilon\delta) \cos \omega'\}\, D_1 D_2 \\ &\quad = \mu^2 (D_1^2 + D_2^2 - 2D_1 D_2 \cos \omega), \end{aligned}$$

and therefore

$$\frac{D_1'^2 + D_2'^2 - 2D_1' D_2' \cos \omega'}{\sin^2 \omega'} = \frac{D_1^2 + D_2^2 - 2D_1 D_2 \cos \omega}{\sin^2 \omega}.$$

Hence

$$D'^2 = \Sigma\,(\xi - a)^2 - \frac{D_1'^2 + D_2'^2 - 2D_1' D_2' \cos \omega'}{\sin^2 \omega'}$$
$$= \Sigma\,(\xi - a)^2 - \frac{D_1^2 + D_2^2 - 2D_1 D_2 \cos \omega}{\sin^2 \omega} = D^2.$$

consequently the expression for the length of the perpendicular is an invariant under any change of the guiding lines.

Again, in the former representation, the foot of the perpendicular is given by the equation

$$(X - a) \sin^2 \omega = (l_1 - l_2 \cos \omega)\, D_1 + (l_2 - l_1 \cos \omega)\, D_2,$$

and similar equations for Y, Z, V. If its coordinates are X', Y', Z', V', in the changed representation, we have

$$(X' - a) \sin^2 \omega' = (l_1' - l_2' \cos \omega')\, D_1' + (l_2' - l_1' \cos \omega')\, D_2',$$

and similar equations for Y', Z', V'. Substitute for l_1' and l_2' in the equation giving X': the coefficient of l_1 on the right-hand side is

$$\begin{aligned} &= (\gamma - \delta \cos \omega')\, D_1' + (\delta - \gamma \cos \omega')\, D_2' \\ &= (\gamma - \delta \cos \omega')(\gamma D_1 + \epsilon D_2) + (\delta - \gamma \cos \omega')(\delta D_1 + \eta D_2) \\ &= D_1 (\gamma^2 - 2\gamma\delta \cos \omega' + \delta^2) + D_2\,\{\epsilon\gamma + \eta\delta - (\epsilon\delta + \eta\gamma) \cos \omega'\} \\ &= \mu^2 (D_1 - D_2 \cos \omega); \end{aligned}$$

and similarly the coefficient of l_2 in the same expression is

$$= \mu^2 (D_2 - D_1 \cos \omega);$$

so that

$$(X' - a) \sin^2 \omega' = \mu^2\,\{l_1 (D_1 - D_2 \cos \omega) + l_2 (D_2 - D_1 \cos \omega)\}.$$

Hence

$$X' = X:$$

and, similarly, $Y' = Y$, $Z' = Z$, $V' = V$. Thus the expressions for the coordinates of the foot of the perpendicular are invariant under any change in the guiding lines.

Perpendiculars to a plane are not parallel to one another.

35. The plane containing all the lines through ξ, η, ζ, υ, which are perpendicular to every line in the given plane, is determined by the equations

$$l_1(x-\xi)+m_1(y-\eta)+n_1(z-\zeta)+k_1(v-\upsilon)=0,$$
$$l_2(x-\xi)+m_2(y-\eta)+n_2(z-\zeta)+k_2(v-\upsilon)=0.$$

Every direction λ, μ, ν, κ, in this new plane is such that

$$\Sigma l_1\lambda=0,\quad \Sigma l_2\lambda=0,$$

and therefore

$$\Sigma\lambda\,(\alpha l_1+\beta l_2)=0.$$

Thus every direction in either plane is perpendicular to every direction in the other: a property which will be found of significance when (§§ 97–100) the orthogonality of two planes is under discussion.

One other property may be noted, because it is distinct from the corresponding property in three-dimensional geometry: it is that *the perpendiculars from external points to a plane are not parallel to one another*. (The property of parallelism does belong to perpendiculars from external points to a flat, in quadruple space: and the comparison in three-dimensional space is with the perpendiculars from external points on a line.)

With the preceding notation, the direction-cosines of the perpendicular from ξ, η, ζ, υ, to the plane

$$\left\| \begin{matrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{matrix} \right\| = 0,$$

are given by the four equations

$$L^2\ +M^2\ +N^2\ +K^2\ =1,$$
$$l_1L+m_1M+n_1N+k_1K=0,$$
$$l_2L+m_2M+n_2N+k_2K=0,$$
$$\begin{vmatrix} L\ , & M\ , & N\ , & K \\ \xi-a, & \eta-b, & \zeta-c, & \upsilon-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{vmatrix}=0.$$

Hence L, M, N, K, depend upon ξ, η, ζ, υ; and the perpendiculars in question are only parallel to one another for different points ξ, η, ζ, υ, when these points lie upon a straight line passing through a point a, b, c, d, in the plane, that is, when they lie upon a straight line which meets the plane.

Ex. 1. Obtain the equations of the plane through the three points α, β, γ, in the figure on p 7

Prove that the coordinates of the foot of the perpendicular from the point δ on the plane $\alpha\beta\gamma$ are

$$X=\frac{a^3(b^2+c^2)}{a^2b^2+b^2c^2+c^2a^2},\quad Y=\frac{b^3(c^2+a^2)}{a^2b^2+b^2c^2+c^2a^2},\quad Z=\frac{c^3(a^2+b^2)}{a^2b^2+b^2c^2+c^2a^2},\quad V=d.$$

Ex. 2. Given a straight line L and a plane P; prove that the locus of the foot of the perpendicular, drawn from points on L to the plane P, is a straight line.

When the line is

$$\frac{x-\alpha}{\lambda}=\frac{y-\beta}{\mu}=\frac{z-\gamma}{\nu}=\frac{v-\delta}{\kappa},$$

and the plane is

$$\left\| \begin{matrix} x-a, & y-b, & z-c, & v-d \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{matrix} \right\| =0,$$

prove that all these perpendiculars lie in the quadric surface

$$\begin{vmatrix} x-\alpha, & y-\beta, & z-\gamma, & v-\delta \\ a-\alpha, & b-\beta, & c-\gamma, & d-\delta \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{vmatrix} : \Sigma l_1(x-\alpha) \quad \Sigma l_2(x-\alpha)$$

$$= \begin{vmatrix} x-a, & y-b, & z-c, & v-d \\ \lambda, & \mu, & \nu, & \kappa \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{vmatrix} : \Sigma l_1\lambda \quad \Sigma l_2\lambda.$$

36. When the equations of the plane are given in the canonical form

$$z=f+px+qy, \quad v=h+rx+sy,$$

the length of the perpendicular from ξ, η, ζ, v, is similarly derived.

Let the foot of the perpendicular be X, Y, Z, V, so that

$$Z=f+pX+qY, \quad V=h+rX+sY;$$

then

$$D^2=(\xi-X)^2+(\eta-Y)^2+(\zeta-Z)^2+(v-V)^2$$

$$=(\xi-X)^2+(\eta-Y)^2+(\zeta-pX-qY-f)^2+(v-rX-sY-h)^2$$

must be a minimum for all values of X and Y. Hence

$$\xi-X+p(\zeta-Z)+r(v-V)=0,$$

$$\eta-Y+q(\zeta-Z)+s(v-V)=0.$$

In these equations substitute for Z and V in terms of X and Y: and write

$$\Delta=1+p^2+q^2+r^2+s^2+(ps-qr)^2,$$

$$T=\zeta-p\xi-q\eta-f, \quad W=v-r\xi-s\eta-h.$$

When the equations are resolved for $X-\xi$ and $Y-\eta$, we find

$$\Delta(X-\xi)=\{p+s(ps-qr)\}T+\{r-q(ps-qr)\}W,$$

$$\Delta(Y-\eta)=\{q-r(ps-qr)\}T+\{s+p(ps-qr)\}W,$$

and therefore

$$\Delta(Z-\zeta)=\quad(1+r^2+s^2)T-(pr+qs)W,$$

$$\Delta(V-v)=-(pr+qs)T+(1+p^2+q^2)W.$$

Now

$$D^2=(\xi-X)^2+(\eta-Y)^2+(\zeta-Z)^2+(v-V)^2,$$

when the values of $X-\xi$, $Y-\eta$, $Z-\zeta$, $V-v$, are substituted, and reduction is effected, we obtain

$$\Delta D^2=(1+r^2+s^2)T^2-2(pr+qs)TW+(1+p^2+q^2)W^2,$$

thus giving the length of the perpendicular.

If L, M, N, K, are the direction-cosines of the perpendicular, drawn from ξ, η, ζ, v, towards X, Y, Z, V, we have

$$LD\Delta=\Delta(X-\xi),\quad MD\Delta=\Delta(Y-\eta),\quad ND\Delta=\Delta(Z-\zeta),\quad KD\Delta=\Delta(V-v),$$

so that

$$\frac{L}{\{p+s(ps-qr)\}T+\{r-q(ps-qr)\}W}$$

$$=\frac{M}{\{q-r(ps-qr)\}T+\{s+p(ps-qr)\}W}$$

$$=\frac{N}{(1+r^2+s^2)T-(pr+qs)W}$$

$$=\frac{K}{-(pr+qs)T+(1+p^2+q^2)W}$$

$$=\frac{1}{\Delta^{\frac{1}{2}}\{(1+r^2+s^2)T^2-2(pr+qs)TW+(1+p^2+q^2)W^2\}^{\frac{1}{2}}}.$$

We shall return to the expression for D^2, after the discussion of the elementary properties of flats, and we shall obtain an interpretation similar to that obtained (§ 32) for D^2 when the equation was given in the earlier form.

Relations between a plane and a line.

37. We have seen that a line is represented by three linear equations, but not by fewer than three such equations: and that a plane is represented by two linear equations, but not by fewer than two such equations. When therefore the common intersections, if any, of a line and a plane are sought, they will be provided in general by five simultaneous equations, linear in the four variables x, y, z, v. As a rule, such a system will not possess a set of simultaneous roots: and therefore we infer that, in quadruple space, an arbitrary line does not meet an arbitrary plane.

If, however, the five equations are not independent of one another, two alternatives arise. The five equations may be equivalent to four independent equations, each linear in the four variables; in that event, the four independent equations possess a single set of simultaneous roots, and then the line and the plane intersect in a single point. Or the five equations may be equivalent to three independent equations, each linear in the four variables, in that event, the three independent equations possess a singly infinite set of simultaneous roots, each set determining a point lying on the line and the plane: that is, the line lies in the plane. The five equations cannot be equivalent to fewer than three independent equations, each linear in the four variables, because the three equations of a line are irreducible in number. We proceed to the two possible alternatives, in succession.

(i) Let the plane be given by the two equations

$$\left\| \begin{matrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{matrix} \right\| = 0,$$

so that every point lying in the plane is given by

$$x-a=l_1\rho+l_2\sigma,\quad y-b=m_1\rho+m_2\sigma,\quad z-c=n_1\rho+n_2\sigma,\quad v-d=k_1\rho+k_2\sigma,$$

for appropriate values of ρ and σ. Let the line be given by the three equations

$$\frac{x-\alpha}{\lambda}=\frac{y-\beta}{\mu}=\frac{z-\gamma}{\nu}=\frac{v-\delta}{\kappa},$$

so that every point lying on the line is given by

$$x-\alpha=\lambda r,\quad y-\beta=\mu r,\quad z-\gamma=\nu r,\quad v-\delta=\kappa r,$$

for appropriate values of r. If any point is common to the plane and the line, values of ρ, σ, r, must exist such that

$$\begin{aligned} \alpha-a &= l_1\rho+l_2\sigma-\lambda r, \\ \beta-b &= m_1\rho+m_2\sigma-\mu r, \\ \gamma-c &= n_1\rho+n_2\sigma-\nu r, \\ \delta-d &= k_1\rho+k_2\sigma-\kappa r, \end{aligned}$$

and, in order that these equations may co-exist, we must have

$$\left| \begin{matrix} \alpha-a, & l_1\ , & l_2\ , & \lambda \\ \beta-b, & m_1, & m_2, & \mu \\ \gamma-c, & n_1\ , & n_2\ , & \nu \\ \delta-d, & k_1\ , & k_2\ , & \kappa \end{matrix} \right| = 0,$$

which accordingly is the one condition to be satisfied. It is easily seen to be sufficient as well as necessary.

When the condition is satisfied, the coordinates of the point of intersection are determinate, as follows. Let

$$\Sigma l_1 l_2 = \cos\eta, \quad \Sigma l_1 \lambda = \cos\theta, \quad \Sigma l_2 \lambda = \cos\phi.$$

Multiplying the equations by l_1, m_1, n_1, k_1, and adding: then by l_2, m_2, n_2, k_2, and adding: and finally by λ, μ, ν, κ, and adding: we obtain the respective equations

$$\begin{aligned} \rho + \sigma\cos\eta - r\cos\theta &= \Sigma l_1(\alpha - a), \\ \rho\cos\eta + \sigma \qquad - r\cos\phi &= \Sigma l_2(\alpha - a), \\ \rho\cos\theta + \sigma\cos\phi - r \qquad &= \Sigma\lambda(\alpha - a); \end{aligned}$$

and therefore

$$\Delta r = (\cos\theta - \cos\phi\cos\eta)\Sigma l_1(\alpha - a) + (\cos\phi - \cos\theta\cos\eta)\Sigma l_2(\alpha - a) - \sin^2\eta\,\Sigma\lambda(\alpha - a),$$

where

$$\Delta = 1 - \cos^2\eta - \cos^2\theta - \cos^2\phi + 2\cos\eta\cos\theta\cos\phi.$$

With this particular value of r, the coordinates of the point of meeting are

$$\alpha + \lambda r, \quad \beta + \mu r, \quad \gamma + \nu r, \quad \delta + \kappa r.$$

From the condition that the line and the plane intersect, it follows that $\alpha, \beta, \gamma, \delta$, lies on the flat

$$\begin{vmatrix} X - a, & l_1, & l_2, & \lambda \\ Y - b, & m_1, & m_2, & \mu, \\ Z - c, & n_1, & n_2, & \nu \\ V - d, & k_1, & k_2, & \kappa \end{vmatrix} = 0,$$

a flat in which each of the three directions l_1, m_1, n_1, k_1, l_2, m_2, n_2, k_2; and $\lambda, \mu, \nu, \kappa$; lies. Hence the locus of a point from which a straight line can be drawn in a given direction to meet a given plane is a flat containing the plane and the direction.

It might happen that the condition of intersection is satisfied but that the point of intersection is at an infinite distance. In that event, the line would be parallel to the plane. And then we should have

$$\Delta = 0,$$

that is,

$$\eta \pm \theta \pm \phi = 0,$$

which geometrically is only possible if the three directions l_1, m_1, n_1, k_1; l_2, m_2, n_2, k_2; and $\lambda, \mu, \nu, \kappa$; are complanar. The two conditions, one for intersection, and the other for intersection at an infinite distance, are

$$\left\Vert \begin{matrix} \lambda, & \mu, & \nu, & \kappa \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{matrix} \right\Vert = 0:$$

and these are the analytical expressions of complanarity of direction.

Ex. If the plane be given by the equations

$$\left.\begin{array}{l} A_1x+B_1y+C_1z+D_1v=E_1 \\ A_2x+B_2y+C_2z+D_2v=E_2 \end{array}\right\},$$

the condition that the line

$$\frac{x-\alpha}{\lambda}=\frac{y-\beta}{\mu}=\frac{z-\gamma}{\nu}=\frac{v-\delta}{\kappa}$$

should meet the plane is

$$\begin{vmatrix} \Sigma A_1\alpha-E_1, & \Sigma A_1\lambda \\ \Sigma A_2\alpha-E_2, & \Sigma A_2\lambda \end{vmatrix}=0,$$

and the conditions that the line should be parallel to the plane are

$$\Sigma A_1\lambda=0, \quad \Sigma A_2\lambda=0.$$

When the equations of the plane occur in the canonical form

$$z-f=px+qy, \quad v-h=rx+sy,$$

the corresponding condition for intersection is

$$\begin{vmatrix} p\alpha+q\beta-\gamma+f, & p\lambda+q\mu-\nu \\ r\alpha+s\beta-\delta+h, & r\lambda+s\mu-\kappa \end{vmatrix}=0,$$

and the corresponding conditions for parallelism are

$$p\lambda+q\mu-\nu=0, \quad r\lambda+s\mu-\kappa=0$$

38. (ii) The conditions that the line

$$\frac{x-\alpha}{\lambda}=\frac{y-\beta}{\mu}=\frac{z-\gamma}{\nu}=\frac{v-\delta}{\kappa}$$

should lie completely in the plane

$$\left\|\begin{matrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{matrix}\right\|=0$$

are that the equations

$$\left\|\begin{matrix} \alpha-a+\lambda r, & \beta-b+\mu r, & \gamma-c+\nu r, & \delta-d+\kappa r \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{matrix}\right\|=0$$

should be satisfied for all values of r that is, they are

$$\left\|\begin{matrix} \alpha-a, & \beta-b, & \gamma-c, & \delta-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{matrix}\right\|=0,$$

and

$$\left\|\begin{matrix} \lambda\ , & \mu\ , & \nu\ , & \kappa \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{matrix}\right\|=0.$$

The conditions in the first set—equivalent to two—express a requirement that a point $\alpha, \beta, \gamma, \delta$, on the line shall lie in the plane. The conditions in the second set—also equivalent to two—express a requirement that the direction of the line,—determined by $\lambda, \mu, \nu, \kappa$,—shall be complanar with the two directions l_1, m_1, n_1, k_1; and l_2, m_2, n_2, k_2; which determine the orientation of the plane.

Ex. 1. Obtain the conditions, connected with the plane

$$\left.\begin{array}{l} A_1x+B_1y+C_1z+D_1v=E_1 \\ A_2x+B_2y+C_2z+D_2v=E_2 \end{array}\right\}$$

and the line

$$\left.\begin{array}{l} A_3x+B_3y+C_3z+D_3v=E_3 \\ A_4x+B_4y+C_4z+D_4v=E_4 \\ A_5x+B_5y+C_5z+D_5v=E_5 \end{array}\right\},$$

for the following possibilities:

(i) that the line and the plane meet,

(ii) that the line and the plane are parallel,

(iii) that the line lies in the plane.

Ex. 2. It is easy to verify - what has, in fact, been assumed—that, if a line is parallel to a plane, a line drawn parallel to that line and passing through a point in the plane lies wholly in the plane.

The conditions that the line

$$\frac{x-\alpha}{\lambda}=\frac{y-\beta}{\mu}=\frac{z-\gamma}{\nu}=\frac{v-\delta}{\kappa}$$

shall be parallel to the plane

$$\left\|\begin{array}{cccc} x-a', & y-b', & z-c', & v-d' \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{array}\right\|,$$

where a', b', c', d', are the coordinates of any point in the plane, are

$$\left\|\begin{array}{cccc} \lambda, & \mu, & \nu, & \kappa \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{array}\right\|=0.$$

The equations of a line, through a', b', c', d', and parallel to the given line, are

$$\frac{x-a'}{\lambda}=\frac{y-b'}{\mu}=\frac{z-c'}{\nu}=\frac{v-d'}{\kappa}.$$

This line lies wholly in the given plane, on account of the conditions which have just been stated.

Shortest distance between a line and a plane.

39. It has appeared that an arbitrary line and an arbitrary plane do not meet; an enquiry remains concerning the shortest distance between them.

We denote by X', Y', Z', V', any point on the line

$$\frac{x-\alpha}{\lambda}=\frac{y-\beta}{\mu}=\frac{z-\gamma}{\nu}=\frac{v-\delta}{\kappa},$$

so that

$$X' = \alpha + \lambda r', \quad Y' = \beta + \mu r', \quad Z' = \gamma + \nu r', \quad V' = \delta + \kappa r';$$

and by X, Y, Z, V, any point in the plane

$$\left\| \begin{matrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{matrix} \right\| = 0,$$

so that

$$X-a = l_1 r_1 + l_2 r_2, \quad Y-b = m_1 r_1 + m_2 r_2, \quad Z-c = n_1 r_1 + n_2 r_2, \quad V-d = k_1 r_1 + k_2 r_2.$$

The distance between the two selected points is denoted by D, so that

$$\begin{aligned} D^2 &= (X-X')^2 + (Y-Y')^2 + (Z-Z')^2 + (V-V')^2 \\ &= \Sigma (a - \alpha + l_1 r_1 + l_2 r_2 - \lambda r')^2. \end{aligned}$$

For our purpose, D is to be a minimum for all values of r_1, r_2, r'. Hence we have

$$\left. \begin{aligned} \Sigma\lambda\ (a - \alpha + l_1 r_1 + l_2 r_2 - \lambda r') &= 0 \\ \Sigma l_1\ (a - \alpha + l_1 r_1 + l_2 r_2 - \lambda r') &= 0 \\ \Sigma l_2\ (a - \alpha + l_1 r_1 + l_2 r_2 - \lambda r') &= 0 \end{aligned} \right\} .$$

In the first place, these can be written

$$\Sigma\lambda (X - X') = 0, \quad \Sigma l_1 (X - X') = 0, \quad \Sigma l_2 (X - X') = 0\,,$$

or, if L, M, N, K, are the direction-cosines of the shortest distance, so that $X - X' = LD$, $Y - Y' = MD$, $Z - Z' = ND$, $V - V' = KD$, we have

$$\Sigma\lambda L = 0, \quad \Sigma l_1 L = 0, \quad \Sigma l_2 L = 0,$$

and therefore

$$\Sigma (\alpha l_1 + \beta l_2)\, L = 0.$$

Hence the shortest distance is at once, (i) perpendicular to the line, and (ii) perpendicular to every direction in the plane.

Again, as in § 37, let

$$\Sigma l_1 \lambda = \cos\theta, \quad \Sigma l_2 \lambda = \cos\phi, \quad \Sigma l_1 l_2 = \cos\eta,$$

so that θ, ϕ, η, may be regarded as known quantities, when the equations are given in their assumed form. Then the equations for D become

$$\begin{aligned} r' \qquad - r_1 \cos\theta - r_2 \cos\phi &= \Sigma\lambda\, (a - \alpha), \\ r' \cos\theta - r_1 \qquad - r_2 \cos\eta &= \Sigma l_1 (a - \alpha), \\ r' \cos\phi - r_1 \cos\eta - r_2 \qquad &= \Sigma l_2 (a - \alpha), \end{aligned}$$

three equations adequate for the determination of r', r_1, r_2, provided the determinant of the left-hand side does not vanish. This determinant Δ is

$$\begin{aligned} \Delta = &\begin{vmatrix} 1\ , & \cos\theta, & \cos\phi \\ \cos\theta, & 1\ , & \cos\eta \\ \cos\phi, & \cos\eta, & 1 \end{vmatrix} \\ &= 1 - \cos^2\theta - \cos^2\phi - \cos^2\eta + 2\cos\theta\cos\phi\cos\eta \\ &= 4\sin\tfrac{1}{2}(\theta + \phi + \eta)\sin\tfrac{1}{2}(\phi + \eta - \theta)\sin\tfrac{1}{2}(\eta + \theta - \phi)\sin\tfrac{1}{2}(\theta + \phi - \eta)\,; \end{aligned}$$

and so the condition, for the determination of r', r_1, r_2, is that no one of the quantities $\theta \pm \phi \pm \eta$ shall vanish, nor $\theta + \phi + \eta$ be equal to 2π.

Next, from the equations

$$\Sigma L\lambda = 0, \quad \Sigma L l_1 = 0, \quad \Sigma L l_2 = 0,$$

we have

$$\frac{L}{\begin{vmatrix} \mu, & \nu, & \kappa \\ m_1, & n_1, & k_1 \\ m_2, & n_2, & k_2 \end{vmatrix}} = \frac{M}{-\begin{vmatrix} \nu, & \kappa, & \lambda \\ n_1, & k_1, & l_1 \\ n_2, & k_2, & l_2 \end{vmatrix}} = \frac{N}{\begin{vmatrix} \kappa, & \lambda, & \mu \\ k_1, & l_1, & m_1 \\ k_2, & l_2, & m_2 \end{vmatrix}} = \frac{K}{-\begin{vmatrix} \lambda, & \mu, & \nu \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix}} = \frac{1}{\Theta},$$

where

$$\Theta^2 = \Sigma \begin{vmatrix} \mu, & \nu, & \kappa \\ m_1, & n_1, & k_1 \\ m_2, & n_2, & k_2 \end{vmatrix}^2$$

$$= \begin{vmatrix} \Sigma\lambda^2, & \Sigma l_1\lambda, & \Sigma l_2\lambda \\ \Sigma l_1\lambda, & \Sigma l_1^2, & \Sigma l_1 l_2 \\ \Sigma l_2\lambda, & \Sigma l_1 l_2, & \Sigma l_2^2 \end{vmatrix} = \begin{vmatrix} 1, & \cos\theta, & \cos\phi \\ \cos\theta, & 1, & \cos\eta \\ \cos\phi, & \cos\eta, & 1 \end{vmatrix} = \Delta.$$

Hence

$$\begin{aligned} LD &= X - X' = a - \alpha + l_1 r_1 + l_2 r_2 - \lambda r', \\ MD &= Y - Y' = b - \beta + m_1 r_1 + m_2 r_2 - \mu r', \\ ND &= Z - Z' = c - \gamma + n_1 r_1 + n_2 r_2 - \nu r', \\ KD &= V - V' = d - \delta + k_1 r_1 + k_2 r_2 - \kappa r'. \end{aligned}$$

Now $\Sigma L\lambda = 0$, $\Sigma L l_1 = 0$, $\Sigma L l_2 = 0$, and therefore

$$\begin{aligned} D &= L \,.\, LD + M \,.\, MD + N \,.\, ND + K \,.\, KD \\ &= L(a-\alpha) + M(b-\beta) + N(c-\gamma) + K(d-\delta) \\ &= \frac{1}{\Delta^{\frac{1}{2}}} \begin{vmatrix} a-\alpha, & b-\beta, & c-\gamma, & d-\delta \\ \lambda, & \mu, & \nu, & \kappa \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{vmatrix}, \end{aligned}$$

where

$$\Delta = 1 - \cos^2\theta - \cos^2\phi - \cos^2\eta + 2\cos\theta\cos\phi\cos\eta.$$

We thus have an expression for the length of the shortest distance between the line and the plane.

There is an immediate verification of the condition that the line and the plane should meet; for, in that event, $D = 0$, and thus the condition of § 37 reappears.

Note. Later (§ 69) it will be proved that

$$\Delta = \sin^2 \eta \sin^2 \Omega,$$

where Ω is the inclination of the line to the plane.

The consideration of the inclination of a plane to any other homaloidal amplitude is deferred until Chapter V.

Distance between a plane and a line parallel to the plane.

40. But if the line is parallel to the plane, the foregoing expression for D becomes indefinite. The determinant Δ vanishes, because then either $\phi + \theta = \eta$ or $\phi \sim \theta = \eta$; and the determinant

$$\begin{vmatrix} a-\alpha, & b-\beta, & c-\gamma, & d-\delta \\ \lambda\ , & \mu\ , & \nu\ , & \kappa \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{vmatrix}$$

vanishes because (§ 29)

$$\left\Vert \begin{matrix} \lambda, & \mu, & \nu, & \kappa \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{matrix} \right\Vert = 0.$$

We proceed to shew that the perpendicular distance from any point on the line is independent of the position of the point.

As the line is parallel to the plane, the last set of conditions shews that quantities p and q must exist such that

$$\lambda = pl_1 + ql_2, \quad \mu = pm_1 + qm_2, \quad \nu = pn_1 + qn_2, \quad \kappa = pk_1 + qk_2.$$

Multiplying by l_1, m_1, n_1, k_1, and adding · and then multiplying by l_2, m_2, n_2, k_2, and adding · we have

$$\cos \theta = \Sigma \lambda l_1 = p + q \cos \omega,$$

$$\cos \phi = \Sigma \lambda l_2 = p \cos \omega + q,$$

so that

$$p \sin^2 \omega = \cos \theta - \cos \phi \cos \omega, \quad q \sin^2 \omega = \cos \phi - \cos \theta \cos \omega$$

Now take any two points $\alpha, \beta, \gamma, \delta$, and $\xi, \eta, \zeta, \upsilon$, on the line, so that

$$\frac{\xi - \alpha}{\lambda} = \frac{\eta - \beta}{\mu} = \frac{\zeta - \gamma}{\nu} = \frac{\upsilon - \delta}{\kappa}, = \rho,$$

say, let D be the perpendicular from $\xi, \eta, \zeta, \upsilon$, and $\bar{D}$ the perpendicular from $\alpha, \beta, \gamma, \delta$. Then

$$D^2 = \Sigma (\xi - a)^2 - \frac{1}{\sin^2 \omega} (D_1^2 - 2D_1 D_2 \cos \omega + D_2^2),$$

$$\bar{D}^2 = \Sigma (\alpha - a)^2 - \frac{1}{\sin^2 \omega} (\bar{D}_1^2 - 2\bar{D}_1 \bar{D}_2 \cos \omega + \bar{D}_2^2),$$

where

$$D_1 = \Sigma l_1 (\xi - a), \quad D_2 = \Sigma l_2 (\xi - a),$$
$$\bar{D}_1 = \Sigma l_1 (\alpha - a), \quad \bar{D}_2 = \Sigma l_2 (\alpha - a).$$

Hence

$$D_1 = \Sigma l_1 (\alpha + \lambda\rho - a) = \bar{D}_1 + \rho \cos\theta,$$
$$D_2 = \Sigma l_2 (\alpha + \lambda\rho - a) = \bar{D}_2 + \rho \cos\phi,$$

and therefore

$$\begin{aligned} D_1^2 - 2D_1D_2\cos\omega + D_2^2 &= \bar{D}_1^2 - 2\bar{D}_1\bar{D}_2\cos\omega + \bar{D}_2^2 \\ &\quad + 2\rho\,\{\bar{D}_1\cos\theta - (\bar{D}_1\cos\phi + \bar{D}_2\cos\theta)\cos\omega + \bar{D}_2\cos\phi\} \\ &\quad + \rho^2(\cos^2\theta - 2\cos\theta\cos\phi\cos\omega + \cos^2\phi) \\ &= \bar{D}_1^2 - 2\bar{D}_1\bar{D}_2\cos\omega + \bar{D}_2^2 + \rho^2\sin^2\omega \\ &\quad + 2\rho\,\{\bar{D}_1\cos\theta - (\bar{D}_1\cos\phi + \bar{D}_2\cos\theta)\cos\omega + \bar{D}_2\cos\phi\}. \end{aligned}$$

Again,

$$\begin{aligned} \Sigma(\xi - a)^2 &= \Sigma(\alpha + \lambda\rho - a)^2 \\ &= \Sigma(\alpha - a)^2 + \rho^2 + 2\rho\Sigma\lambda(\alpha - a); \end{aligned}$$

and

$$\begin{aligned} \Sigma\lambda(\alpha - a) &= p\Sigma l_1(\alpha - a) + q\Sigma l_2(\alpha - a) \\ &= p\bar{D}_1 + q\bar{D}_2 \\ &= \frac{1}{\sin^2\omega}\{\bar{D}_1(\cos\theta - \cos\phi\cos\omega) + \bar{D}_2(\cos\phi - \cos\theta\cos\omega)\}. \end{aligned}$$

Accordingly,

$$\begin{aligned} D^2 &= \Sigma(\xi - a)^2 - \frac{1}{\sin^2\omega}(D_1^2 - 2D_1D_2\cos\omega + D_2^2) \\ &= \Sigma(\alpha - a)^2 - \frac{1}{\sin^2\omega}(\bar{D}_1^2 - 2\bar{D}_1\bar{D}_2\cos\omega + \bar{D}_2^2) \\ &= \bar{D}^2: \end{aligned}$$

or the perpendicular distances of all points on the line from the plane are the same, their common value being $\bar{D}$.

Shortest distance between a line and a plane, when the equations are in canonical form.

41. When the equations of the plane occur in the canonical form, the calculations follow a somewhat different course The line is

$$\frac{x - \alpha}{\lambda} = \frac{y - \beta}{\mu} = \frac{z - \gamma}{\nu} = \frac{v - \delta}{\kappa}.$$

On the line, the foot of the shortest distance D, between the line and the plane, is taken to be $X', Y', Z', V', = \alpha + \lambda\rho, \beta + \mu\rho, \gamma + \nu\rho, \delta + \kappa\rho$. The plane is

$$z = px + qy + f, \quad v = rx + sy + h;$$

in the plane, the foot of that shortest distance is taken to be X, Y, Z, V and we write

$$p\alpha + q\beta + f - \gamma = T, \quad r\alpha + s\beta + h - \delta = W.$$

Then

$$\begin{aligned}D^2 &= (X - X')^2 + (Y - Y')^2 + (Z - Z')^2 + (V - V')^2\\ &= (X - \alpha - \lambda\rho)^2 + (Y - \beta - \mu\rho)^2\\ &\qquad + \{p(X-\alpha) + q(Y-\beta) + T - \nu\rho\}^2 + \{r(X-\alpha) + s(Y-\beta) + W - \kappa\rho\}^2,\end{aligned}$$

and D^2 has to be a minimum for all values of X, Y, ρ. Consequently

$$\begin{aligned}\lambda(X - X') + \mu(Y - Y') + \nu(Z - Z') + \kappa(V - V') &= 0,\\ X - X' \qquad\qquad + p(Z - Z') + r(V - V') &= 0,\\ Y - Y' + q(Z - Z') + s(V - V') &= 0.\end{aligned}$$

(i) If L, M, N, K, are the direction-cosines of the shortest distance,

$$X - X' = LD, \quad Y - Y' = MD, \quad Z - Z' = ND, \quad V - V' = KD;$$

hence

$$\frac{L}{\begin{vmatrix}\mu, & \nu, & \kappa\\ 0, & p, & r\\ 1, & q, & s\end{vmatrix}} = -\frac{M}{\begin{vmatrix}\nu, & \kappa, & \lambda\\ p, & r, & 1\\ q, & s, & 0\end{vmatrix}} = \frac{N}{\begin{vmatrix}\kappa, & \lambda, & \mu\\ r, & 1, & 0\\ s, & 0, & 1\end{vmatrix}} = -\frac{K}{\begin{vmatrix}\lambda, & \mu, & \nu\\ 1, & 0, & p\\ 0, & 1, & q\end{vmatrix}} = \frac{1}{\Delta^{\frac{1}{2}}},$$

where

$$\Delta = \Sigma\begin{vmatrix}\mu, & \nu, & \kappa\\ 0, & p, & r\\ 1, & q, & s\end{vmatrix}^2 = \begin{vmatrix}\lambda^2 + \mu^2 + \nu^2 + \kappa^2, & \lambda + \nu p + \kappa r, & \mu + \nu q + \kappa s\\ \lambda \quad + \nu p + \kappa r, & 1 + p^2 + r^2, & pq + rs\\ \mu \quad + \nu q + \kappa s, & pq + rs & 1 + q^2 + s^2\end{vmatrix}.$$

Thus the direction-cosines L, M, N, K, are known.

(ii) Again,

$$\begin{aligned}LD &= X - X' = X - \alpha \qquad\qquad - \lambda\rho,\\ MD &= Y - Y' = \qquad\qquad Y - \beta - \mu\rho,\\ ND &= Z - Z' = p(X - \alpha) + q(Y - \beta) - \nu\rho + T,\\ KD &= V - V' = r(X - \alpha) + s(Y - \beta) - \kappa\rho + W,\end{aligned}$$

and

$$L + Np + Kr = 0, \quad M + Nq + Ks = 0, \quad L\lambda + M\mu + N\nu + K\kappa = 0.$$

Hence

$$\begin{aligned}D &= L.LD + M.MD + N.ND + K.KD\\ &= NT + KW\\ &= \frac{1}{\Delta^{\frac{1}{2}}}\{(\kappa - \lambda r - \mu s)T - (\nu - \lambda p - \mu q)W\},\end{aligned}$$

and thus the shortest distance D is known.

(iii) Further, the critical conditions for a minimum of D^2 can be written

$$\begin{aligned}
(\lambda+\nu p+\kappa r)(X-\alpha)+(\mu+\nu q+\kappa s)(Y-\beta)-(\lambda^2+\mu^2+\nu^2+\kappa^2)\rho&=-\nu T-\kappa W,\\
(1+p^2+r^2)(X-\alpha)+(pq+rs)(Y-\beta)-(\lambda+\nu p+\kappa r)\rho&=-pT-rW,\\
(pq+rs)(X-\alpha)+(1+q^2+s^2)(Y-\beta)-(\mu+\nu q+\kappa s)\rho&=-qT-sW.
\end{aligned}$$

Therefore

$$\Delta(X-\alpha)=-T\begin{vmatrix} p, & rs\ , & \lambda+\kappa r \\ q, & 1+s^2, & \mu+\kappa s \\ \nu, & \mu+\kappa s, & \lambda^2+\mu^2+\kappa^2 \end{vmatrix}-W\begin{vmatrix} r, & pq\ , & \lambda+\nu p \\ s, & 1+q^2, & \mu+\nu q \\ \kappa, & \mu+\nu q, & \lambda^2+\mu^2+\nu^2 \end{vmatrix},$$

$$\Delta(Y-\beta)=T\begin{vmatrix} p, & 1+r^2, & \lambda+\kappa r \\ q, & rs\ , & \mu+\kappa s \\ \nu, & \lambda+\kappa r, & \lambda^2+\mu^2+\kappa^2 \end{vmatrix}+W\begin{vmatrix} r, & 1+p^2, & \lambda+\nu p \\ s, & pq\ , & \mu+\nu q \\ \kappa, & \lambda+\nu p, & \lambda^2+\mu^2+\nu^2 \end{vmatrix}.$$

These equations, together with

$$Z-\gamma=p(X-\alpha)+q(Y-\beta)+T,\quad V-\delta=r(X-\alpha)+s(Y-\beta)+W,$$

determine the plane-extremity of the shortest distance D.

(iv) Finally, from the same equations, we have

$$\Delta\rho=T\begin{vmatrix} \nu, & \lambda+\kappa r, & \mu+\kappa s \\ p, & 1+r^2, & rs \\ q, & rs\ , & 1+s^2 \end{vmatrix}+W\begin{vmatrix} \kappa, & \lambda+\nu p, & \mu+\nu q \\ r, & 1+p^2, & pq \\ s, & pq\ , & 1+q^2 \end{vmatrix}.$$

This equation, together with

$$X'-\alpha=\lambda\rho,\quad Y'-\beta=\mu\rho,\quad Z'-\gamma=\nu\rho,\quad V'-\delta=\kappa\rho,$$

determines the line-extremity of the shortest distance D.

Ex. 1. Prove that

$$\Delta=ht^2-2gtw+fw^2,\quad \Delta\rho=htT-g(tW+wT)+fwW,$$

where $t=p\lambda+q\mu-\nu$, $w=r\lambda+s\mu-\kappa$, while

$$f=1+p^2+q^2,\quad g=pr+qs,\quad h=1+r^2+s^2.$$

Ex 2. Verify that $\Delta=0$ when the line is parallel to the plane, so that the expression for D, being

$$\Delta^{-\frac{1}{2}}\{(\kappa-\lambda r-\mu s)T-(\nu-\lambda p-\mu q)W\},$$

then becomes evanescent In this event, obtain an expression for the distance between the line and the plane.

Relation of two planes: parallel planes.

42. A plane in quadruple space is represented by two equations. When the intersection (whatever it may be) of two planes is required, it will be provided analytically by four equations, each of them linear in the four variables x, y, z, v. In general, such a group of equations will provide a single unique set of particular values of x, y, z, v, which determine a unique point. that is, two arbitrary planes meet in a point.

But deviations from the normal form of these values may occur under special limitations, which must be examined in detail. The notion of parallel lines, as lines having the same direction, is simple. The notion of parallelism must be extended to planes: and the extension can be effected by means of the property (§ 13) under which a plane is composed of lines, as follows:

> *Let two planes be denoted by P and P'. Take any point A in P and any point A' in P'. Through A draw any two lines AB and AC in the first plane; and through A' draw, in the quadruple space, a line $A'B'$ parallel to AB and a line $A'C'$ parallel to AC. If both lines $A'B'$ and $A'C'$ lie in the plane P', we say that the planes P and P' are parallel*

It may happen that, for some particular direction AB in the first plane P, the parallel direction $A'B'$ in the quadruple space would lie in the second plane P', and that the result does not hold for any other direction: such would be the fact if the two planes, instead of meeting merely at a point, intersected in a line and if AB were drawn parallel to that line. In such an event, the two planes are not parallel.

Analytically expressed, the definition carries the parallelism of all corresponding lines in the two parallel planes. Let the two different directions AB and AC be given by l_1, m_1, n_1, k_1, and l_2, m_2, n_2, k_2, which therefore are the direction-cosines of $A'B'$ and $A'C'$ in the parallel plane. Any other direction in the plane P is given by

$$\alpha l_1 + \beta l_2, \quad \alpha m_1 + \beta m_2, \quad \alpha n_1 + \beta n_2, \quad \alpha k_1 + \beta k_2.$$

As l_1, m_1, n_1, k_1, and l_2, m_2, n_2, k_2, are the direction-cosines of $A'B'$ and $A'C'$ which lie in P', that new direction also lies in P': that is, to every direction in P there is a parallel direction in P'.

Meeting of two planes: alternatives.

43 Consider now two planes represented by pairs of equations

$$\left.\begin{aligned} A_1x + B_1y + C_1z + D_1v = E_1 \\ A_2x + B_2y + C_2z + D_2v = E_2 \end{aligned}\right\},$$
$$\left.\begin{aligned} A_3x + B_3y + C_3z + D_3v = E_3 \\ A_4x + B_4y + C_4z + D_4v = E_4 \end{aligned}\right\}$$

The range of variation, which is common to the two planes, is given by combining the four equations. In general, they determine one point and only one point.

If however the determinant of the coefficients on the left-hand side is zero, so that

$$\begin{vmatrix} A_1, & B_1, & C_1, & D_1 \\ A_2, & B_2, & C_2, & D_2 \\ A_3, & B_3, & C_3, & D_3 \\ A_4, & B_4, & C_4, & D_4 \end{vmatrix} = 0,$$

then one at least of the quantities x, y, z, v, satisfying the four equations, is infinite unless all the magnitudes M, represented by the determinants

$$\left\| \begin{matrix} A_1, & B_1, & C_1, & D_1, & E_1 \\ A_2, & B_2, & C_2, & D_2, & E_2 \\ A_3, & B_3, & C_3, & D_3, & E_3 \\ A_4, & B_4, & C_4, & D_4, & E_4 \end{matrix} \right\|,$$

vanish.

As the determinant $|A_1, B_2, C_3, D_4|$ vanishes, quantities α, β, γ, δ, exist such that

$$\begin{aligned} \alpha A_1 + \beta A_2 &= \gamma A_3 + \delta A_4 = A, \\ \alpha B_1 + \beta B_2 &= \gamma B_3 + \delta B_4 = B, \\ \alpha C_1 + \beta C_2 &= \gamma C_3 + \delta C_4 = C, \\ \alpha D_1 + \beta D_2 &= \gamma D_3 + \delta D_4 = D. \end{aligned}$$

Every point in the first plane lies in the flat

$$\alpha(A_1x + B_1y + C_1z + D_1v) + \beta(A_2x + B_2y + C_2z + D_2v) = \alpha E_1 + \beta E_2,$$

that is,

$$Ax + By + Cz + Dv = \alpha E_1 + \beta E_2,$$

and every point in the second plane lies in the flat

$$\gamma(A_3x + B_3y + C_3z + D_3v) + \delta(A_4x + B_4y + C_4z + D_4v) = \gamma E_3 + \delta E_4,$$

that is,

$$Ax + By + Cz + Dv = \gamma E_3 + \delta E_4.$$

(i) These two flats are the same, if

$$\alpha E_1 + \beta E_2 = \gamma E_3 + \delta E_4:$$

that is, if all the quantities M vanish. In that event, the two planes lie in one and the same flat. The four simultaneous equations are then not independent of one another; one of them can certainly be deduced as a linear combination of the other three. Suppose that the fourth can thus be deduced; we then are left with three equations

$$\begin{aligned} A_1x + B_1y + C_1z + D_1v &= E_1, \\ A_2x + B_2y + C_2z + D_2v &= E_2, \\ A_3x + B_3y + C_3z + D_3v &= E_3, \end{aligned}$$

which provide the common intersection of the two planes. Now, unless the two independent relations provided by

$$\left\| \begin{matrix} A_1, & B_1, & C_1, & D_1 \\ A_2, & B_2, & C_2, & D_2 \\ A_3, & B_3, & C_3, & D_3 \end{matrix} \right\| = 0$$

are satisfied, these three equations can be resolved so as to give

$$\frac{x}{\lambda} = \frac{y - \beta'}{\mu} = \frac{z - \gamma'}{\nu} = \frac{v - \delta'}{\kappa},$$

where

$$\lambda A_1 + \mu B_1 + \nu C_1 + \kappa D_1 = 0,$$
$$\lambda A_2 + \mu B_2 + \nu C_2 + \kappa D_2 = 0,$$
$$\lambda A_3 + \mu B_3 + \nu C_3 + \kappa D_3 = 0,$$

and therefore

$$\lambda A_4 + \mu B_4 + \nu C_4 + \kappa D_4 = 0.$$

The first two of these equations shew that the particular direction λ, μ, ν, κ, lies in the first plane. The second two shew that the same direction lies in the second plane. Thus the three equations in the variables provide a line, with direction-cosines λ, μ, ν, κ; the direction thus determined lies (and any parallel direction lies) within both planes. Consequently, the two planes lie in one and the same flat; and they intersect in a line, which possesses a finite range, because β', γ', δ', are finite.

(ii) The two flats

$$Ax + By + Cz + Dv = \alpha E_1 + \beta E_2,$$
$$Ax + By + Cz + Dv = \gamma E_3 + \delta E_4,$$

are not the same if $\alpha E_1 + \beta E_2$ and $\gamma E_3 + \delta E_4$ are unequal, they are parallel flats (§ 58). Not fewer than four of the five quantities M are different from zero the vanishing even of two of them would entail the vanishing of the remainder. Because $| A_1, B_2, C_3, D_4 |$ vanishes, the quantities x, y, z, v, corresponding to the non-vanishing quantities M are infinite. Then as before, unless two of the quantities

$$\left\| \begin{array}{cccc} A_1, & B_1, & C_1, & D_1 \\ A_2, & B_2, & C_2, & D_2 \\ A_3, & B_3, & C_3, & D_3 \end{array} \right\|$$

vanish, the two planes lie in parallel flats and intersect at infinity in a direction parallel to the line

$$\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu} = \frac{v}{\kappa}.$$

But it may happen that the relations

$$\left\| \begin{array}{cccc} A_1, & B_1, & C_1, & D_1 \\ A_2, & B_2, & C_2, & D_2 \\ A_3, & B_3, & C_3, & D_3 \end{array} \right\| = 0$$

are satisfied. In that event, the quantities λ, μ, ν, κ, are zero; and the equations of the line cease to provide a definite result. We now have

$$A_3 = \epsilon A_1 + \eta A_2, \quad B_3 = \epsilon B_1 + \eta B_2, \quad C_3 = \epsilon C_1 + \eta C_2, \quad D_3 = \epsilon D_1 + \eta D_2,$$

in addition to the former relations; and these former relations now give

$$A_4 = \epsilon' A_1 + \eta' A_2, \quad B_4 = \epsilon' B_1 + \eta' B_2, \quad C_4 = \epsilon' C_1 + \eta' C_2, \quad D_4 = \epsilon' D_1 + \eta' D_2.$$

If then the first plane is represented by the equations

$$V = E_1, \quad W = E_2,$$

the second plane is represented by the equations

$$\epsilon V + \eta W = E_3, \quad \epsilon' V + \eta' W = E_4,$$

that is, by the equations

$$V = E_1', \quad W = E_2'.$$

Now if l, m, n, k, are the direction-cosines of any line lying in $V = E_1$, $W = E_2$, we have

$$A_1 l + B_1 m + C_1 n + D_1 k = 0,$$
$$A_2 l + B_2 m + C_2 n + D_2 k = 0,$$

consequently

$$A_3 l + B_3 m + C_3 n + D_3 k = 0,$$
$$A_4 l + B_4 m + C_4 n + D_4 k = 0;$$

that is, in the second plane there lies a direction parallel to any assumed direction lying in the first plane, whatever direction be so chosen The two planes are parallel.

The preceding investigation thus yields the following alternative results.

When two planes are given in quadruple space, either

(i) they intersect in a point; or

(ii) they lie in one and the same flat, and intersect in a line lying within a finite range from the origin; or

(iii) they lie in parallel flats, and intersect at infinity in a definite direction; or

(iv) they are parallel planes.

Ex 1 Prove that the planes OAg' and ODf', in the figure on p. 7, meet only in the point O, and that the planes PAg and $P\delta f$ meet only in the point P.

Ex. 2 Prove that the planes $gh\delta$ and $\alpha\beta\gamma$, in the figure on p. 7, intersect at infinity in a direction parallel to the line BC; that the planes ABC and $g'h'D$ intersect at infinity in the same direction; and that the planes

$$ABC', \quad fgh, \quad f'g'h', \quad \alpha\beta\gamma,$$

are parallel to one another.

Ex 3. Prove that the planes

$$\left.\begin{aligned} \frac{x}{a} + \frac{y}{b} &= 1 \\ \frac{z}{c} + \frac{w}{d} &= 1 \end{aligned}\right\}, \quad \left.\begin{aligned} \frac{x}{a} + \frac{z}{c} &= 1 \\ \frac{y}{b} + \frac{w}{d} &= 1 \end{aligned}\right\}, \quad \left.\begin{aligned} \frac{x}{a} + \frac{w}{d} &= 1 \\ \frac{y}{b} + \frac{z}{c} &= 1 \end{aligned}\right\},$$

intersect, by pairs, in straight lines. and find the respective lines.

Ex. 4. Two planes are given, with their pairs of equations in the forms

$$\left.\begin{array}{l} px+qy-z=f \\ rx+sy-v=h \end{array}\right\}, \quad \left.\begin{array}{l} p'x+q'y-z=f' \\ r'x+s'y-v=h' \end{array}\right\}.$$

Shew that, in general, their point of intersection is

$$\Theta x = \begin{vmatrix} f-f', & q-q' \\ h-h', & s-s' \end{vmatrix}, \qquad \Theta y = \begin{vmatrix} p-p', & f-f' \\ r-r', & h-h' \end{vmatrix},$$

$$\Theta z = \begin{vmatrix} h-h', & r-r', & s-s' \\ f, & p, & q \\ f', & p', & q' \end{vmatrix}, \qquad \Theta v = \begin{vmatrix} f-f', & p-p', & q-q' \\ h', & r', & s' \\ h, & r, & s \end{vmatrix},$$

where

$$\Theta = (r'-r)(q'-q)-(s'-s)(p'-p)$$

Prove that, if

$$\frac{h-h'}{f-f'} = \frac{r-r'}{p-p'} = \frac{s-s'}{q-q'},$$

the planes intersect in a line through the point

$$\frac{f'-f}{p-p'}, \quad 0, \quad \frac{pf'-fp'}{p-p'}, \quad \frac{rh'-hr'}{r-r'},$$

having its direction-cosines proportional to

$$-(q-q'), \quad p-p', \quad pq'-qp', \quad \frac{rs'-sr'}{s'-s}(q-q')$$

What are the conditions (*a*) that the planes should lie in two parallel flats, and (*b*) that the planes should be parallel?

Ex. 5 When the equations of two planes are given in the form

$$\left\|\begin{array}{cccc} x-a, & y-b, & z-c, & v-d \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{array}\right\| = 0,$$

$$\left\|\begin{array}{cccc} x-a', & y-b', & z-c', & v-d' \\ l_1', & m_1', & n_1', & k_1' \\ l_2', & m_2', & n_2', & k_2' \end{array}\right\| = 0,$$

shew that their point of intersection is given, as to its coordinates, by equations of the type

$$\begin{vmatrix} (a-a')X, & l_1a', & l_2a', & l_1'a, & l_2'a \\ a-a', & l_1, & l_2, & l_1', & l_2' \\ b-b', & m_1, & m_2, & m_1', & m_2' \\ c-c', & n_1, & n_2, & n_1', & n_2' \\ d-d', & k_1, & k_2, & k_1', & k_2' \end{vmatrix} = 0.$$

Obtain the conditions, that the planes should be parallel, in the form

$$\left\|\begin{array}{cccc} l_1, & m_1, & n_1, & k_1 \\ l_1', & m_1', & n_1', & k_1' \\ l_2', & m_2', & n_2', & k_2' \end{array}\right\| = 0, \qquad \left\|\begin{array}{cccc} l_2, & m_2, & n_2, & k_2 \\ l_1', & m_1', & n_1', & k_1' \\ l_2', & m_2', & n_2', & k_2' \end{array}\right\| = 0$$

Ex 6. (The definition of the parallelism of two planes has been made to depend upon the parallelism of straight lines, that is, of lines having the same direction; and it is not made to depend upon a negative condition of not meeting within a finite range of any selected origin.)

Prove that, with the adopted definition of parallelism, two planes, which lie in the same flat and have their line of intersection at infinity, are parallel to one another.

Ex 7. *Prove that, when two planes meet in a point only, it is possible to draw, through any point not lying in either of them, one (and only one) plane which intersects each of them in a line.*

Take the sole common point for origin, and let the planes be

$$\left.\begin{aligned} z &= px + qy \\ v &= rx + sy \end{aligned}\right\}, \quad \left.\begin{aligned} z &= p'x + q'y \\ v &= r'x + s'y \end{aligned}\right\},$$

where $(p-p')(s-s')-(q-q')(r-r')$ does not vanish because the origin is the only common meeting-place of the planes. Let the third plane (if any) be

$$z = Px + Qy + F, \quad v = Rx + Sy + H,$$

as it passes through an arbitrary point a, b, c, d, we have

$$c = Pa + Qb + F, \quad d = Ra + Sb + H.$$

As the new plane intersects the first plane in a line, we have

$$H(P-p) = F(R-r), \quad H(Q-q) = F(S-s),$$

and as it intersects the second plane in a line, we have

$$H(P-p') = F(R-r'), \quad H(Q-q') = F(S-s').$$

From these relations, we have

$$H(p-p') = F(r-r'), \quad H(q-q') = F(s-s'),$$

and $(p-p')(s-s')-(q-q')(r-r')$ does not vanish. Hence

$$F = 0, \quad H = 0,$$

and therefore the third plane (if any) is

$$z = Px + Qy, \quad v = Rx + Sy,$$

with the conditions

$$c = Pa + Qb, \quad d = Ra + Sb.$$

Because this plane intersects the first in a line, it follows that

$$(P-p)(S-s) = (R-r)(Q-q),$$

and because it intersects the second in a line, that

$$(P-p')(S-s') = (R-r')(Q-q').$$

Now

$$(P-p)a + (Q-q)b = c - pa - qb = \gamma,$$

$$(R-r)a + (S-s)b = d - ra - sb = \delta,$$

where γ and δ do not vanish together, because a, b, c, d, does not lie in the first plane; and therefore

$$(S-s)\gamma = (Q-q)\delta.$$

Similarly, if

$$c - p'a - q'b = \gamma', \quad d - r'a - s'b = \delta',$$

where γ' and δ' do not vanish together, because a, b, c, d, does not lie in the second plane, we have

$$(P-p')\,a+(Q-q')\,b=\gamma',$$

$$(R-r')\,a+(S-s')\,b=\delta',$$

and therefore

$$(S-s')\,\gamma'=(Q-q')\,\delta'$$

Accordingly, we have the four linear equations in P, Q, R, S,

$$\left.\begin{array}{llll} aP+bQ & & & =c \\ & aR+bS & & =d \\ \delta Q & & -\gamma S & =\delta q-\gamma s \\ \delta' Q & & -\gamma' S & =\delta' q'-\gamma' s' \end{array}\right\},$$

which give unique finite values for P, Q, R, S, unless the determinant of the coefficients on the left-hand side which is equal to $a^2(\gamma\delta'-\delta\gamma')$ should vanish. But as the point a, b, c, d, is arbitrary, the quantity $\gamma\delta'-\delta\gamma'$ does not vanish, hence the result, as enunciated.

The existence of some one plane, with the assigned property, can be established as follows Let the two planes be

$$\left.\begin{array}{l} X=\Sigma A_1x-E_1=0 \\ Y=\Sigma A_2x-E_2=0 \end{array}\right\}, \quad \left.\begin{array}{l} Z=\Sigma A_3x-E_3=0 \\ V=\Sigma A_4x-E_4=0 \end{array}\right\},$$

and let

$$\alpha=\Sigma A_1a-E_1,\quad \beta=\Sigma A_2a-E_2,\quad \gamma=\Sigma A_3a-E_3,\quad \delta=\Sigma A_4a-E_4,$$

where both α and β cannot vanish, and both γ and δ cannot vanish A third plane, drawn through the point common to the two planes, is

$$\beta X-\alpha Y=0,\quad \delta Z-\gamma V=0$$

This plane intersects the first plane in the line

$$X=0,\quad Y=0,\quad \delta Z-\gamma V=0,$$

it intersects the second plane in the line

$$Z=0,\quad V=0,\quad \beta X-\alpha Y=0;$$

and it passes through the point a, b, c, d. Hence it possesses the assigned property.

Ex. 8 In a multiple homaloidal space, there are two homaloidal amplitudes A and B, of dimensions m and n respectively; and they have a common range of r dimensions. The least extensive homaloidal amplitude containing A and B is of s dimensions; prove that

$$r+s=m+n$$

Note. The two results, (i) and (ii) on p 70, are particular examples of this theorem

CHAPTER IV.

FLATS.

Directions in a flat.

44. It has been seen that a single equation, linear in the four variables, suffices for the expression of a flat. The form of the equation depends upon the data which determine the flat.

Thus the flat may be required to pass through a point and to contain three assigned directions which are not complanar. In that event, the form of the equation is

$$\begin{vmatrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \\ l_3\ , & m_3\ , & n_3\ , & k_3 \end{vmatrix} = 0,$$

where the three directions, determined by the sets of cosines l_1, m_1, n_1, k_1, l_2, m_2, n_2, k_2; and l_3, m_3, n_3, k_3, are not complanar, so that relations

$$\rho\theta_1 + \sigma\theta_2 + \tau\theta_3 = 0,$$

(for $\theta = l, m, n, k$), are not simultaneously satisfied.

Let l, m, n, k, denote any other direction in the flat, so that the line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = \frac{v-d}{k}$$

lies in the flat; then

$$\begin{vmatrix} l\ , & m\ , & n\ , & k \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \end{vmatrix} = 0.$$

Consequently, quantities λ, μ, ν, exist, such that

$$\left.\begin{aligned} l &= \lambda l_1 + \mu l_2 + \nu l_3 \\ m &= \lambda m_1 + \mu m_2 + \nu m_3 \\ n &= \lambda n_1 + \mu n_2 + \nu n_3 \\ k &= \lambda k_1 + \mu k_2 + \nu k_3 \end{aligned}\right\};$$

and when λ, μ, ν, are regarded as parameters, these expressions give the direction-cosines of any line that lies in the flat. Also, let α be the inclination of the lines l_2, m_2, n_2, k_2, and l_3, m_3, n_3, k_3; let β be the inclination of l_3, m_3, n_3, k_3, and l_1, m_1, n_1, k_1, and let γ be the inclination of l_1, m_1, n_1, k_1, and l_2, m_2, n_2, k_2. As $l^2+m^2+n^2+k^2=1$, the parameters λ, μ, ν, are connected by the relation

$$\lambda^2+\mu^2+\nu^2+2\mu\nu\cos\alpha+2\nu\lambda\cos\beta+2\lambda\mu\cos\gamma=1.$$

Moreover, if X, Y, Z, V, denote any point in the flat, at a distance R from a, b, c, d, along the direction l, m, n, k, we have

$$\left.\begin{aligned} X-a=\ lR&=R\lambda l_1+R\mu l_2+R\nu l_3\\ &=\rho l_1\ \ +\sigma l_2\ \ +\tau l_3\\ Y-b=mR&=\rho m_1\ +\sigma m_2\ +\tau m_3\\ Z-c=nR&=\rho n_1\ \ +\sigma n_2\ \ +\tau n_3\\ V-d=kR&=\rho k_1\ \ +\sigma k_2\ \ +\tau k_3 \end{aligned}\right\},$$

which accordingly give the coordinates of any point in the flat, in terms of three parameters ρ, σ, τ. Manifestly the point X, Y, Z, V, can be reached by moving from the point a, b, c, d, a distance ρ along the direction l_1, m_1, n_1, k_1, then from the extremity of that distance ρ a new distance σ along the direction l_2, m_2, n_2, k_2, and then from the extremity of that distance σ a new distance τ along the direction l_3, m_3, n_3, k_3. Thus ρ, σ, τ, are the coordinates of the point, when it is referred to an oblique frame within the flat, and the formulæ relating to oblique axes in three dimensions are applicable. Clearly

$$\rho^2+\sigma^2+\tau^2+2\sigma\tau\cos\alpha+2\tau\rho\cos\beta+2\rho\sigma\cos\gamma=R^2.$$

Normal to a flat.

45. The preceding form of equation can also be written

$$Lx+My+Nz+Kv=P,$$

where

$$La+Mb+Nc+Kd=P,$$

while

$$\begin{aligned} Ll_1+Mm_1+Nn_1+Kk_1&=0,\\ Ll_2+Mm_2+Nn_2+Kk_2&=0,\\ Ll_3+Mm_3+Nn_3+Kk_3&=0. \end{aligned}$$

If then L, M, N, K, are the direction-cosines of a line, this line is perpendicular to each of the three non-complanar guiding directions through a, b, c, d, which determine the flat. Further, denoting any direction in the flat by l, m, n, k, as before (§ 44), we have

$$\begin{aligned} Ll+Mm+Nn+Kk&=\lambda\Sigma Ll_1+\mu\Sigma Ll_2+\nu\Sigma Ll_3\\ &=0: \end{aligned}$$

or the direction L, M, N, K, is perpendicular to every direction that lies in the flat. Moreover,

$$\frac{L}{\begin{vmatrix} m_1, & n_1, & k_1\\ m_2, & n_2, & k_2\\ m_3, & n_3, & k_3 \end{vmatrix}}=\frac{M}{-\begin{vmatrix} n_1, & k_1, & l_1\\ n_2, & k_2, & l_2\\ n_3, & k_3, & l_3 \end{vmatrix}}=\frac{N}{\begin{vmatrix} k_1, & l_1, & m_1\\ k_2, & l_2, & m_2\\ k_3, & l_3, & m_3 \end{vmatrix}}=\frac{K}{-\begin{vmatrix} l_1, & m_1, & n_1\\ l_2, & m_2, & n_2\\ l_3, & m_3, & n_3 \end{vmatrix}}=\frac{1}{\Delta^{\frac{1}{2}}},$$

where

$$\Delta = \Sigma \begin{vmatrix} m_1, & n_1, & k_1 \\ m_2, & n_2, & k_2 \\ m_3, & n_3, & k_3 \end{vmatrix}^2$$

$$= \begin{vmatrix} \Sigma l_1^2, & \Sigma l_1 l_2, & \Sigma l_1 l_3 \\ \Sigma l_1 l_2, & \Sigma l_2^2, & \Sigma l_2 l_3 \\ \Sigma l_1 l_3, & \Sigma l_2 l_3, & \Sigma l_3^2 \end{vmatrix} = \begin{vmatrix} 1\ , & \cos\gamma, & \cos\beta \\ \cos\gamma, & 1\ , & \cos\alpha \\ \cos\beta, & \cos\alpha, & 1 \end{vmatrix}$$

$$= 1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma.$$

Thus the direction L, M, N, K, is determinate and unique; it is perpendicular to every direction that lies in the flat; and therefore the direction is called *normal to the flat*. In particular, a line through any point a', b', c', d', in the flat, drawn in this direction and therefore given by the equations

$$\frac{x-a'}{L} = \frac{y-b'}{M} = \frac{z-c'}{N} = \frac{v-d'}{K},$$

is called the normal to the flat at the point a', b', c', d'.

Further, the quantity

$$La + Mb + Nc + Kd$$

is the projection upon this direction L, M, N, K, through the origin, of the line joining the origin to the point a, b, c, d, in the flat; that is, it is the length of the perpendicular upon the flat drawn from the origin O. This quantity is equal to P; hence, in the equation of the flat when it has the form

$$Lx + My + Nz + Kv = P,$$

L, M, N, K, are the direction-cosines of the normal to the flat at any point, and P is the length of the perpendicular upon the flat from the origin.

Modes of determining a flat.

46. The foregoing determination of a flat is made by the assignment of a point and of three directions, all of which are to be contained in the flat. There are other modes of determination.

A flat can be determined by four points, which are not complanar. If they are a, b, c, d; a_1, b_1, c_1, d_1; a_2, b_2, c_2, d_2; and a_3, b_3, c_3, d_3; its equation can be taken in the form

$$\begin{vmatrix} x-a, & y-b, & z-c, & v-d \\ a_1-a, & b_1-b, & c_1-c, & d_1-d \\ a_2-a, & b_2-b, & c_2-c, & d_2-d \\ a_3-a, & b_3-b, & c_3-c, & d_3-d \end{vmatrix} = 0,$$

or in any equivalent form (§ 14) such as (e.g.) arises by any permutation of the symbols of the four points.

A flat can be determined by a requirement, that it shall contain a plane

$$\begin{vmatrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{vmatrix} = 0$$

and a point a', b', c', d', not lying in the plane: its equation then is

$$\begin{vmatrix} x-a, & y-b, & z-c, & v-d \\ a'-a, & b'-b, & c'-c, & d'-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{vmatrix} = 0.$$

When the plane is given by equations

$$z-px-qy=f, \quad v-rx-sy=h,$$

the equation of the containing flat is

$$\begin{vmatrix} z-px-qy-f, & v-rx-sy-h \\ c'-pa'-qb'-f, & d'-ra'-sb'-h \end{vmatrix} = 0.$$

A flat can be determined by the requirement, that it shall contain two lines, which do not meet but are not parallel. When the lines are

$$\frac{x-a}{\lambda} = \frac{y-b}{\mu} = \frac{z-c}{\nu} = \frac{v-d}{\kappa},$$

$$\frac{x-a'}{\lambda'} = \frac{y-b'}{\mu'} = \frac{z-c'}{\nu'} = \frac{v-d'}{\kappa'},$$

the equation of the flat can be taken to be

$$\begin{vmatrix} x-a, & y-b, & z-c, & v-d \\ a'-a, & b'-b, & c'-c, & d'-d \\ \lambda\ , & \mu\ , & \nu\ , & \kappa \\ \lambda'\ , & \mu'\ , & \nu'\ , & \kappa' \end{vmatrix} = 0.$$

Ex. 1. Find the equation of a flat

(i) through a given line and two points not on the line

(ii) through a given plane and a given line which meet.

Point out restrictions, if any, on the positions of the amplitudes determining the flat.

Ex. 2. Prove that the three planes

$$\left\| \begin{matrix} x-a, & y-b, & z-c, & v-d \\ l_r\ , & m_r\ , & n_r\ , & k_r \\ l_s\ , & m_s\ , & n_s\ , & k_s \end{matrix} \right\| = 0,$$

for r, $s=1$, 2, 3, lie in the same flat, if l_r, m_r, n_r, k_r, for $r=1$, 2, 3, represent three non-complanar directions.

Ex 3. Given that the two planes

$$\left.\begin{array}{l} z=px+qy+f \\ v=rx+sy+h \end{array}\right\}, \quad \left.\begin{array}{l} z=p'x+q'y+f' \\ v=r'x+s'y+h' \end{array}\right\},$$

meet in a line, obtain the equation of the flat (§ 43) in which the two planes lie.

Ex 4 Find the equation of the flat through the two lines

$$\left.\begin{array}{l} L_1x+M_1y+N_1z+K_1v=P_1 \\ L_2x+M_2y+N_2z+K_2v=P_2 \\ L_3x+M_3y+N_3z+K_3v=P_3 \end{array}\right\}, \quad \left.\begin{array}{l} L_1'x+M_1'y+N_1'z+K_1'v=P_1' \\ L_2'x+M_2'y+N_2'z+K_2'v=P_2' \\ L_3'x+M_3'y+N_3'z+K_3'v=P_3' \end{array}\right\},$$

on the assumption that the two lines do not meet.

Ex. 5. Shew that, in general, a flat cannot be made to pass through more than four arbitrary points in quadruple space.

Verify that the eight points A, β, γ, δ, g, h, f', P, in the figure on p 7, lie in one flat.

Ex. 6. Trace the flat

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{v}{d}=2,$$

by means of guiding points in the same figure.

Ex. 7. Obtain the respective equations of the flats $O\alpha\beta\gamma$, and $PABC$; $ABCD$, and $\alpha\beta\gamma\delta$; in the same figure.

Ex. 8. Given a flat

$$Lx+My+Nz+Kv=P,$$

prove that it is possible to choose three guiding lines for the expression of the flat, such that these lines are perpendicular to one another.

Find three such lines, when the equation is given in the form

$$\left|\begin{array}{cccc} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \\ l_3\ , & m_3\ , & n_3\ , & k_3 \end{array}\right| =0,$$

if $\Sigma l_2l_3=\cos\theta$, $\Sigma l_3l_1=\cos\phi$, $\Sigma l_1l_2=\cos\psi$, where θ, ϕ, ψ, differ from $\frac{1}{2}\pi$.

Flat through a plane and a direction.

47. Consider, in particular, the mode of determination of a flat whereby it is required to contain a plane and the direction of a line not parallel to the plane: or, what is the same thing, to contain a plane, and a line parallel to that direction and meeting the plane.

Let the plane be given by

$$\left\|\begin{array}{cccc} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{array}\right\| =0.$$

The flat must contain the point a, b, c, d. If the associated line, specifying the additional direction to be contained by the flat, should not pass through a, b, c, d, through that point draw a parallel line, having equations

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}=\frac{v-d}{k}.$$

Then the equation of the flat containing the plane and the new line (that is, containing the assigned plane and the assigned direction not parallel to the plane) obviously is

$$\begin{vmatrix} x-a, & y-b, & z-c, & v-d \\ l\ , & m\ , & n\ , & k \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{vmatrix}=0.$$

Let L, M, N, K, as before denote the direction-cosines of a line which is normal to the flat: let

$$\Sigma l_1 l_2=\cos\omega,\quad \Sigma l l_1=\cos\theta,\quad \Sigma l l_2=\cos\phi,$$

so that ω, θ, ϕ, may be regarded as known quantities · and write

$$\Theta=\Sigma\begin{vmatrix} m, & n, & k \\ m_1, & n_1, & k_1 \\ m_2, & n_2, & k_2 \end{vmatrix}^2$$

$$=\begin{vmatrix} 1\ , & \cos\theta, & \cos\phi \\ \cos\theta, & 1\ , & \cos\omega \\ \cos\phi, & \cos\omega, & 1 \end{vmatrix}$$

$$=1-\cos^2\omega-\cos^2\theta-\cos^2\phi+2\cos\omega\cos\theta\cos\phi.$$

Then

$$\left.\begin{aligned} \Theta^{\frac{1}{2}}L &= \phantom{-lh_{12}\ } mh_{12}-ng_{12}+ka_{12} \\ \Theta^{\frac{1}{2}}M &= -lh_{12}\phantom{-mh_{12}}+nf_{12}+kb_{12} \\ \Theta^{\frac{1}{2}}N &= lg_{12}-mf_{12}\phantom{+nf_{12}}+kc_{12} \\ \Theta^{\frac{1}{2}}K &= -la_{12}-mb_{12}-nc_{12} \end{aligned}\right\},$$

where

$$\left.\begin{aligned} a_{12}&=m_1n_2-n_1m_2, & f_{12}&=l_1k_2-k_1l_2 \\ b_{12}&=n_1l_2-l_1n_2, & g_{12}&=m_1k_2-k_1m_2 \\ c_{12}&=l_1m_2-m_1l_2, & h_{12}&=n_1k_2-k_1n_2 \end{aligned}\right\}.$$

48. It is easy to see that if, instead of the guiding lines l_1, m_1, n_1, k_1, and l_2, m_2, n_2, k_2, of the plane, two other guiding lines l_1', m_1', n_1', k_1', and l_2', m_2', n_2', k_2', are chosen for the specification of the plane, the quantities L, M, N, K, are unaltered. For with these alternative lines, we have relations of the form

$$\imath_1'=\gamma\imath_1+\epsilon\imath_2,\quad \imath_2'=\delta\imath_1+\eta\imath_2,$$

(for $\imath=l, m, n, k$), where

$$\mu=\gamma\eta-\delta\epsilon \gtrless 0,$$

$$\gamma^2+\epsilon^2+2\gamma\epsilon\cos\omega=1,\quad \delta^2+\eta^2+2\delta\eta\cos\omega=1,$$

while

$$\mu i_1 = -\eta i_1' + \epsilon i_2', \quad \mu i_2 = -\delta i_1' + \gamma i_2',$$

$$\eta^2 + \epsilon^2 - 2\eta\epsilon \cos\omega' = 1, \quad \delta^2 + \gamma^2 - 2\delta\gamma \cos\omega' = 1,$$

and

$$\cos\omega = \Sigma l_1 l_2, \quad \cos\omega' = \Sigma l_1' l_2'.$$

Then

$$a'_{12}, b'_{12}, c'_{12}, f'_{12}, g'_{12}, h'_{12} = \mu a_{12}, \mu b_{12}, \mu c_{12}, \mu f_{12}, \mu g_{12}, \mu h_{12},$$

and

$$\Theta' = \mu^2 \Theta.$$

Hence

$$\Theta'^{\frac{1}{2}} L' = m h'_{12} - n g'_{12} + k a'_{12} = \mu (m h_{12} - n g_{12} + k a_{12}) = \mu \Theta^{\frac{1}{2}} L,$$

and similarly

$$\Theta'^{\frac{1}{2}} M' = \mu \Theta^{\frac{1}{2}} M, \quad \Theta'^{\frac{1}{2}} N' = \mu \Theta^{\frac{1}{2}} N, \quad \Theta'^{\frac{1}{2}} K' = \mu \Theta^{\frac{1}{2}} K.$$

Thus

$$L' : M' : N' : K' = L : M : N . K;$$

and therefore the magnitudes of the direction-cosines of the normal to the flat are independent of the choice of guiding lines for the specification of the plane.

Length and position of a perpendicular from a point to a flat.

49. When a point does not lie in a flat, we require its shortest distance D from the flat. But it also is necessary to have some convention as to the direction along the line of the distance that is to be taken positive and some accordant convention as to the algebraic measure (positive or negative) as distinct from the sheer geometric magnitude.

Let the point from which the shortest distance is drawn be $\xi, \eta, \zeta, \upsilon$, and let the flat be

$$Lx + My + Nz + Kv = P,$$

where P is definitely positive. (The conventions, when P is zero, will be considered separately) Also, let X, Y, Z, V, be the point in the flat which is the other extremity of the shortest distance from $\xi, \eta, \zeta, \upsilon$. We define that direction of the shortest distance to be the positive direction of measurement, when it is estimated from $\xi, \eta, \zeta, \upsilon$, towards X, Y, Z, V. We take the algebraic measure of the shortest distance to be positive, when $\xi, \eta, \zeta, \upsilon$, and the origin lie on the same side of the flat: we take it to be negative, when $\xi, \eta, \zeta, \upsilon$, and the origin lie on opposite sides of the flat: consequently, the algebraic measure of the shortest distance from the origin to the flat is thus taken, by the assumed convention, to be positive.

When the origin lies in the flat, so that the quantity P in the preceding equation would be zero, a further convention must be adopted. In this event, we shall assume that, when some particular coordinate of the point—we select υ, for the purpose of reference—is positive, the algebraic measure of the shortest distance is negative and that, when υ is negative, the algebraic measure is positive. (This would accord with the preceding convention of the algebraic measure if we assume, arbitrarily, that the flat is displaced slightly along the positive direction of the axis OV, without change of orientation.) Lastly, if υ be zero, we adopt the customary convention of three-dimensional geometry.

Returning to the determination of the shortest distance from $\xi, \eta, \zeta, \upsilon$, to the flat, the quantity D^2, where

$$D^2 = (X - \xi)^2 + (Y - \eta)^2 + (Z - \zeta)^2 + (V - \upsilon)^2,$$

is to be a minimum for all admissible values of X, Y, Z, V, that is, it must be a minimum subject to the condition

$$LX + MY + NZ + KV - P = 0.$$

Hence

$$X - \xi = \lambda L, \quad Y - \eta = \lambda M, \quad Z - \zeta = \lambda N, \quad V - \upsilon = \lambda K,$$

when initially λ is an indeterminate multiplier. Thus

$$\frac{X - \xi}{L} = \frac{Y - \eta}{M} = \frac{Z - \zeta}{N} = \frac{V - \upsilon}{K};$$

and each of these fractions, equal to λ, is also equal to

$$\pm \frac{D}{(L^2 + M^2 + N^2 + K^2)^{\frac{1}{2}}},$$

where D is taken as the purely positive geometric magnitude of the shortest distance and a positive sign is affixed to $(L^2 + M^2 + N^2 + K^2)^{\frac{1}{2}}$, leaving the doubtful sign to be settled under the adopted conventions. Also

$$\begin{aligned} P &= LX + MY + NZ + KV \\ &= L\xi + M\eta + N\zeta + K\upsilon + \{L(X - \xi) + M(Y - \eta) + N(Z - \zeta) + K(V - \upsilon)\} \\ &= L\xi + M\eta + N\zeta + K\upsilon \pm (L^2 + M^2 + N^2 + K^2)^{\frac{1}{2}} D, \end{aligned}$$

with the same positive sign for $(L^2 + M^2 + N^2 + K^2)^{\frac{1}{2}}$.

This doubtful sign has to be determined. We have

$$(L^2 + M^2 + N^2 + K^2)^{\frac{1}{2}} D = \pm (P - L\xi - M\eta - N\zeta - K\upsilon).$$

Now D is positive: $(L^2 + M^2 + N^2 + K^2)^{\frac{1}{2}}$ is positive: when $\xi, \eta, \zeta, \upsilon$, coincides with the origin, that is, is on the same side of the flat as the origin, $P - L\xi - M\eta - N\zeta - K\upsilon$ is positive: consequently the positive sign must be taken. Hence the shortest distance from $\xi, \eta, \zeta, \upsilon$, to the flat is given, as to its algebraic measure, by

$$\frac{P - L\xi - M\eta - N\zeta - K\upsilon}{(L^2 + M^2 + N^2 + K^2)^{\frac{1}{2}}}.$$

Again, the positive direction along the shortest distance is given by the direction from $\xi, \eta, \zeta, \upsilon$, towards X, Y, Z, V; that is, it is given by

$$X - \xi, \quad Y - \eta, \quad Z - \zeta, \quad V - \upsilon.$$

Hence L, M, N, K, taken from the equation of the flat

$$Lx+My+Nz+Kv=P,$$

are proportional to the actual direction-cosines of the shortest distance. Thus, as is to be expected, the shortest distance is normal to the flat.

With these conventions, we say that the perpendicular from ξ, η, ζ, υ, to the flat $Lx+My+Nz+Kv=P$, where P is positive, is

$$\frac{P-L\xi-M\eta-N\zeta-K\upsilon}{(L^2+M^2+N^2+K^2)^{\frac{1}{2}}},$$

the positive sign being taken for the radical; and the coordinates of the foot of the perpendicular are

$$\left.\begin{aligned} X&=\xi+L\frac{P-L\xi-M\eta-N\zeta-K\upsilon}{L^2+M^2+N^2+K^2}\\ Y&=\eta+M\frac{P-L\xi-M\eta-N\zeta-K\upsilon}{L^2+M^2+N^2+K^2}\\ Z&=\zeta+N\frac{P-L\xi-M\eta-N\zeta-K\upsilon}{L^2+M^2+N^2+K^2}\\ V&=\upsilon+K\frac{P-L\xi-M\eta-N\zeta-K\upsilon}{L^2+M^2+N^2+K^2}\end{aligned}\right\}.$$

Clearly the perpendiculars to a flat from any two points ξ, η, ζ, υ; ξ', η', ζ', υ'; are parallel. Their algebraical ratio, being

$$\frac{P-L\xi-M\eta-N\zeta-K\upsilon}{P-L\xi'-M\eta'-N\zeta'-K\upsilon'},$$

is independent of any assumption concerning radical signs.

50. Had we assumed initially that the shortest distance lies along a direction normal to the flat, we should have taken

$$\frac{X-\xi}{L}=\frac{Y-\eta}{M}=\frac{Z-\zeta}{N}=\frac{V-\upsilon}{K}=\rho.$$

The relation between D and ρ is

$$\begin{aligned} D^2&=(X-\xi)^2+(Y-\eta)^2+(Z-\zeta)^2+(V-\upsilon)^2\\ &=\rho^2(L^2+M^2+N^2+K^2).\end{aligned}$$

As the point X, Y, Z, V, lies in the flat $Lx+My+Nz+Kv=P$, we have

$$L\xi+M\eta+N\zeta+K\upsilon+(L^2+M^2+N^2+K^2)\rho=P,$$

so that

$$\rho=\frac{P-L\xi-M\eta-N\zeta-K\upsilon}{L^2+M^2+N^2+K^2}.$$

We thus obtain the former expressions for X, Y, Z, V.

The doubtful sign in estimating the algebraic measure of the perpendicular distance is settled by the convention adopted for the measure; and, again, the length of the perpendicular is obtained in the form

$$\frac{P-L\xi-M\eta-N\zeta-K\upsilon}{(L^2+M^2+N^2+K^2)^{\frac{1}{2}}}.$$

51. When the equation of the flat is given in the form

$$\begin{vmatrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \\ l_3\ , & m_3\ , & n_3\ , & k_3 \end{vmatrix} = 0,$$

any point in the flat, and therefore the foot of the perpendicular from $\xi, \eta, \zeta, \upsilon$, is given by equations

$$\begin{aligned} X &= a + l_1\rho + l_2\sigma + l_3\tau, \\ Y &= b + m_1\rho + m_2\sigma + m_3\tau, \\ Z &= c + n_1\rho + n_2\sigma + n_3\tau, \\ V &= d + k_1\rho + k_2\sigma + k_3\tau, \end{aligned}$$

where ρ, σ, τ, are three parameters. The shortest distance D is given by

$$\begin{aligned} D^2 &= (X-\xi)^2 + (Y-\eta)^2 + (Z-\zeta)^2 + (V-\upsilon)^2 \\ &= \Sigma\,(a + l_1\rho + l_2\sigma + l_3\tau - \xi)^2, \end{aligned}$$

and this has to be a minimum for all values of ρ, σ, τ. Thus

$$\begin{aligned} \Sigma l_1 (a + l_1\rho + l_2\sigma + l_3\tau - \xi) &= 0, \\ \Sigma l_2 (a + l_1\rho + l_2\sigma + l_3\tau - \xi) &= 0, \\ \Sigma l_3 (a + l_1\rho + l_2\sigma + l_3\tau - \xi) &= 0. \end{aligned}$$

Taken in one manner, these equations can be written

$$\Sigma l_1 (X-\xi) = 0, \quad \Sigma l_2 (X-\xi) = 0, \quad \Sigma l_3 (X-\xi) = 0,$$

and therefore

$$\frac{X-\xi}{\begin{vmatrix} m_1, & n_1, & k_1 \\ m_2, & n_2, & k_2 \\ m_3, & n_3, & k_3 \end{vmatrix}} = \frac{Y-\eta}{-\begin{vmatrix} n_1, & k_1, & l_1 \\ n_2, & k_2, & l_2 \\ n_3, & k_3, & l_3 \end{vmatrix}} = \frac{Z-\zeta}{\begin{vmatrix} k_1, & l_1, & m_1 \\ k_2, & l_2, & m_2 \\ k_3, & l_3, & m_3 \end{vmatrix}} = \frac{V-\upsilon}{-\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}}$$

$$= \pm \frac{D}{\Theta^{\frac{1}{2}}},$$

where

$$\Theta = \Sigma \begin{vmatrix} m_1, & n_1, & k_1 \\ m_2, & n_2, & k_2 \\ m_3, & n_3, & k_3 \end{vmatrix}^2 = 1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma,$$

with the former notation. The positive sign is given to $\Theta^{\frac{1}{2}}$: and, as before, the doubtful sign must be settled.

Hence

$$a + l_1\rho + l_2\sigma + l_3\tau = X = \xi \pm \frac{D}{\Theta^{\frac{1}{2}}} \begin{vmatrix} m_1, & n_1, & k_1 \\ m_2, & n_2, & k_2 \\ m_3, & n_3, & k_3 \end{vmatrix},$$

and so for the others; hence

$$\begin{vmatrix} a-\xi, & b-\eta, & c-\zeta, & d-v \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \\ l_3\ , & m_3\ , & n_3\ , & k_3 \end{vmatrix} = \pm D\Theta^{\frac{1}{2}}.$$

Now D and $\Theta^{\frac{1}{2}}$ are positive; and the perpendicular has its sign the same as when ξ, η, ζ, v, coincides with the origin, according to our convention. Hence the perpendicular on the flat is, in algebraic measure, given by

$$\frac{1}{\Theta^{\frac{1}{2}}}\begin{vmatrix} a-\xi, & b-\eta, & c-\zeta, & d-v \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \\ l_3\ , & m_3\ , & n_3\ , & k_3 \end{vmatrix}.$$

Its direction-cosines are

$$\Theta^{-\frac{1}{2}}\begin{vmatrix} m_1, & n_1, & k_1 \\ m_2, & n_2, & k_2 \\ m_3, & n_3, & k_3 \end{vmatrix}, \quad -\Theta^{-\frac{1}{2}}\begin{vmatrix} n_1, & k_1, & l_1 \\ n_2, & k_2, & l_2 \\ n_3, & k_3, & l_3 \end{vmatrix},$$

$$\Theta^{-\frac{1}{2}}\begin{vmatrix} k_1, & l_1, & m_1 \\ k_2, & l_2, & m_2 \\ k_3, & l_3, & m_3 \end{vmatrix}, \quad -\Theta^{-\frac{1}{2}}\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}.$$

From the critical equations rendering D^2 a minimum, when these are taken in another manner, we have

$$\begin{aligned} \Sigma l_1(\xi-a) &= \rho \qquad\quad + \sigma\cos\gamma + \tau\cos\beta, \\ \Sigma l_2(\xi-a) &= \rho\cos\gamma + \sigma \qquad\quad + \tau\cos\alpha, \\ \Sigma l_3(\xi-a) &= \rho\cos\beta + \sigma\cos\alpha + \tau \qquad . \end{aligned}$$

Also

$$X - a = \rho l_1 + \sigma l_2 + \tau l_3;$$

hence

$$\begin{vmatrix} X-a\ , & l_1\ , & l_2\ , & l_3 \\ \Sigma l_1(\xi-a), & 1\ , & \cos\gamma, & \cos\beta \\ \Sigma l_2(\xi-a), & \cos\gamma, & 1\ , & \cos\alpha \\ \Sigma l_3(\xi-a), & \cos\beta, & \cos\alpha, & 1 \end{vmatrix} = 0.$$

Similarly, for $Y-b$, $Z-c$, $V-d$: being equations which give the coordinates of the foot of the perpendicular from ξ, η, ζ, v, on the flat.

Ex. 1. Prove that the flats $ABCD$, $ff'gg'hh'$, $\alpha\beta\gamma\delta$, in the figure on p 7, divide the line OP into four equal parts

Ex. 2. Denoting the perpendicular from α upon the flat $fg'h'A$ by p_1, the perpendicular from β upon the flat $gh'f'B$ by p_2, the perpendicular from γ upon the flat $hf'g'C$ by p_3, and the perpendicular from δ upon the flat $fghD$ by p_4, (all in the same figure), prove that

$$\frac{1}{p_1^2}+\frac{1}{p_2^2}+\frac{1}{p_3^2}+\frac{1}{p_4^2}=7\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right).$$

Ex. 3 Find the length of the perpendicular from the point a_s, b_s, c_s, d_s, on the flat through the four points a_r, b_r, c_r, d_r, (for $r=1, 2, 3, 4$)

Obtain also the coordinates of the foot of this perpendicular.

Ex. 4 A flat is required to contain the plane

$$\left.\begin{aligned} L_1x+M_1y+N_1z+K_1v=1 \\ L_2x+M_2y+N_2z+K_2v=1 \end{aligned}\right\},$$

and also the linear direction λ, μ, ν, κ. Prove that the distance of the flat from the origin is

$$\frac{\Sigma L_2\lambda \sim \Sigma L_1\lambda}{\{\Sigma L_1^2(\Sigma L_2\lambda)^2 - 2(\Sigma L_1L_2)(\Sigma L_1\lambda)(\Sigma L_2\lambda) + \Sigma L_2^2(\Sigma L_1\lambda)^2\}^{\frac{1}{2}}}$$

Perpendicular from a point on the plane of cleavage of two flats.

52. We now are in a position to obtain the magnitude and the position of the shortest distance from an external point ξ, η, ζ, υ, to a plane, when the equations of the plane represent it as the intersection of two flats

$$L_1x + M_1y + N_1z + K_1v = P_1,$$
$$L_2x + M_2y + N_2z + K_2v = P_2,$$

where L_1, M_1, N_1, K_1, and L_2, M_2, N_2, K_2, are the direction-cosines of the respective normals to the two flats And it will prove to be possible, as in § 32, to derive a geometrical construction for that shortest distance, in association with the perpendiculars from ξ, η, ζ, υ, on the two flats.

Let X, Y, Z, V, be the foot of the shortest distance on the plane. Then the quantity D^2, where

$$D^2 = (X-\xi)^2 + (Y-\eta)^2 + (Z-\zeta)^2 + (V-\upsilon)^2,$$

must be a minimum for all admissible values of X, Y, Z, V: that is, it must be a minimum subject to the two conditions

$$L_1X + M_1Y + N_1Z + K_1V - P_1 = 0,$$
$$L_2X + M_2Y + N_2Z + K_2V - P_2 = 0.$$

According to the usual critical tests, we have

$$\begin{aligned} X-\xi &= \lambda L_1 + \mu L_2, \\ Y-\eta &= \lambda M_1 + \mu M_2, \\ Z-\zeta &= \lambda N_1 + \mu N_2, \\ V-\upsilon &= \lambda K_1 + \mu K_2, \end{aligned}$$

where initially λ and μ are two indeterminate multipliers.

In the first place, we have

$$\left\| \begin{matrix} X-\xi, & Y-\eta, & Z-\zeta, & V-\upsilon \\ L_1, & M_1, & N_1, & K_1 \\ L_2, & M_2, & N_2, & K_2 \end{matrix} \right\| = 0.$$

and therefore the shortest distance lies in the plane

$$\begin{vmatrix} x-\xi, & y-\eta, & z-\zeta, & v-\upsilon \\ L_1\,, & M_1\,, & N_1\,, & K_1 \\ L_2\,, & M_2\,, & N_2\,, & K_2 \end{vmatrix} = 0.$$

This plane passes through the point ξ, η, ζ, υ, and it contains the directions determined by L_1, M_1, N_1, K_1, and L_2, M_2, N_2, K_2: that is, it is the plane determined by the two perpendiculars from ξ, η, ζ, υ, upon the two flats, respectively, the equations of which compose the equations of the plane.

Let α denote the inclination of the normals to the two flats, so that

$$\cos\alpha = L_1L_2 + M_1M_2 + N_1N_2 + K_1K_2.$$

We write

$$Q_1 = P_1 - L_1\xi - M_1\eta - N_1\zeta - K_1\upsilon,$$

$$Q_2 = P_2 - L_2\xi - M_2\eta - N_2\zeta - K_2\upsilon,$$

so that Q_1 and Q_2 are the perpendiculars upon the two flats from the point ξ, η, ζ, υ. Substituting for P_1 and P_2 in terms of X, Y, Z, V, we have

$$\begin{aligned} Q_1 &= L_1(X-\xi) + M_1(Y-\eta) + N_1(Z-\zeta) + K_1(V-\upsilon) \\ &= \lambda + \mu\cos\alpha, \\ Q_2 &= L_2(X-\xi) + M_2(Y-\eta) + N_2(Z-\zeta) + K_2(V-\upsilon) \\ &= \lambda\cos\alpha + \mu. \end{aligned}$$

Also, when L, M, N, K, are the direction-cosines of the shortest distance,

$$\begin{aligned} LD &= X-\xi = \lambda L_1 + \mu L_2\,, \\ MD &= Y-\eta = \lambda M_1 + \mu M_2, \\ ND &= Z-\zeta = \lambda N_1 + \mu N_2\,, \\ KD &= V-\upsilon = \lambda K_1 + \mu K_2; \end{aligned}$$

so that the direction L, M, N, K, lies in the plane through ξ, η, ζ, υ, determined by L_1, M_1, N_1, K_1, and L_2, M_2, N_2, K_2: that is, by the two normals. Hence

$$\begin{aligned} D^2 &= \Sigma(\lambda L_1 + \mu L_2)^2 \\ &= \lambda^2 + 2\lambda\mu\cos\alpha + \mu^2 \\ &= \frac{1}{\sin^2\alpha}(Q_1^2 - 2Q_1Q_2\cos\alpha + Q_2^2), \end{aligned}$$

which gives the length of the shortest distance: while

$$\begin{aligned} LD\sin^2\alpha &= L_1\,(Q_1 - Q_2\cos\alpha) + L_2\,(-Q_1\cos\alpha + Q_2), \\ MD\sin^2\alpha &= M_1(Q_1 - Q_2\cos\alpha) + M_2(-Q_1\cos\alpha + Q_2), \\ ND\sin^2\alpha &= N_1(Q_1 - Q_2\cos\alpha) + N_2(-Q_1\cos\alpha + Q_2), \\ KD\sin^2\alpha &= K_1(Q_1 - Q_2\cos\alpha) + K_2(-Q_1\cos\alpha + Q_2), \end{aligned}$$

which give the direction-cosines of that shortest distance.

Let F_1 be the foot of the perpendicular from ξ, η, ζ, υ, the point A, on the flat $\Sigma L_1 x = P_1$; let F_2 be the foot of the perpendicular from A on the flat $\Sigma L_2 x = P_2$; and let F be the foot of the perpendicular drawn to the plane. Then AF_1, AF, AF_2, have been proved to be complanar. Also FF_1, a line in the first flat, is perpendicular to AF_1, and FF_2, a line in the second flat, is perpendicular to AF_2. Hence the four points A, F_1, F, F_2, lie on a circle of which AF is the diameter.

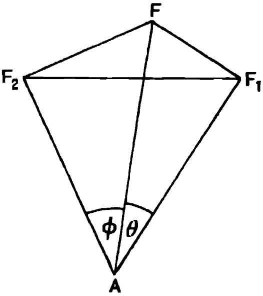

Fig. 3.

Further, we have

$$D\cos\theta = AF_1 = Q_1 = \lambda + \mu\cos\alpha, \quad D\cos\phi = AF_2 = Q_2 = \lambda\cos\alpha + \mu,$$

where $F_1AF = \theta$, $F_2AF = \phi$, and $\theta + \phi = \alpha$, also

$$F_1F_2 = D\sin\alpha.$$

53. It is to be noted that, while the plane is given by the two equations $\Sigma L_1 x - P_1 = 0$ and $\Sigma L_2 x - P_2 = 0$ which are not unique as a representation because they can be superseded by any linear combined equivalent pair, the length and the position of AF are the same for every such equivalent pair. This invariance, a property to be expected in connection with the shortest distance from the plane, is established algebraically as follows.

Let the plane be represented by the equivalent pair of equations

$$L_1'x + M_1'y + N_1'z + K_1'v = P_1',$$
$$L_2'x + M_2'y + N_2'z + K_2'v = P_2',$$

where

$$L_1' = \gamma L_1 + \epsilon L_2, \quad M_1' = \gamma M_1 + \epsilon M_2, \quad N_1' = \gamma N_1 + \epsilon N_2, \quad K_1' = \gamma K_1 + \epsilon K_2,$$
$$L_2' = \delta L_1 + \eta L_2, \quad M_2' = \delta M_1 + \eta M_2, \quad N_2' = \delta N_1 + \eta N_2, \quad K_2' = \delta K_1 + \eta K_2,$$
$$P_1' = \gamma P_1 + \epsilon P_2, \quad P_2' = \delta P_1 + \eta P_2;$$

the constants γ, ϵ, δ, η, are subject to the negative condition that ω, where

$$\omega = \gamma\eta - \delta\epsilon,$$

does not vanish, and we take L_1', M_1', N_1', K_1', and L_2', M_2', N_2', K_2', as direction-cosines, so that

$$\Sigma L_1'^2 = \gamma^2 + 2\gamma\epsilon\cos\alpha + \epsilon^2 = 1, \quad \Sigma L_2'^2 = \delta^2 + 2\delta\eta\cos\alpha + \eta^2 = 1.$$

Also, let Q_1' and Q_2' be the perpendiculars upon the new flats, and let α' be their inclination; then

$$\cos\alpha' = \Sigma L_1'L_2' = \gamma\delta + \epsilon\eta + (\gamma\eta + \delta\epsilon)\cos\alpha,$$
$$\sin^2\alpha' = \Sigma(L_1'M_2' - M_1'L_2')^2 = (\gamma\eta - \delta\epsilon)^2\,\Sigma(L_1M_2 - M_1L_2)^2 = \omega^2\sin^2\alpha,$$
$$Q_1' = P_1' - \Sigma L_1'\xi$$
$$= \gamma P_1 + \epsilon P_2 - \Sigma(\gamma L_1 + \epsilon L_2)\xi = \gamma Q_1 + \epsilon Q_2,$$
$$Q_2' = P_2' - \Sigma L_2'\xi$$
$$= \delta P_1 + \eta P_2 - \Sigma(\delta L_1 + \eta L_2)\xi = \delta Q_1 + \eta Q_2.$$

And, inversely, we have

$$\omega L_1 = \eta L_1' - \epsilon L_2', \quad \omega L_2 = -\delta L_1' + \gamma L_2',$$

with corresponding expressions for M_1, N_1, K_1, and M_2, N_2, K_2; hence

$$\begin{aligned} \omega^2 &= \Sigma\omega^2 L_1^2 = \Sigma(\eta L_1' - \epsilon L_2')^2 = \eta^2 - 2\epsilon\eta\cos\alpha' + \epsilon^2, \\ \omega^2 &= \Sigma\omega^2 L_2^2 = \Sigma(-\delta L_1' + \gamma L_2')^2 = \delta^2 - 2\gamma\delta\cos\alpha' + \gamma^2, \\ \omega^2\cos\alpha &= \Sigma\omega L_1 . \omega L_2 = \Sigma(\eta L_1' - \epsilon L_2')(-\delta L_1' + \gamma L_2') \\ &= -\eta\delta - \gamma\epsilon + (\gamma\eta + \delta\epsilon)\cos\alpha'. \end{aligned}$$

In the new expressions connected with the shortest distance, let D' denote its magnitude; let X', Y', Z', V', denote its foot, lying in the plane; and let λ', μ', be the new indeterminate multipliers, corresponding to the former λ, μ. Then

$$\begin{gathered} X' - \xi = \lambda' L_1' + \mu' L_2', \quad Y' - \eta = \lambda' M_1' + \mu' M_2', \\ Z' - \zeta = \lambda' N_1' + \mu' N_2', \quad V' - v = \lambda' K_1' + \mu' K_2', \\ Q_1' = \lambda' + \mu'\cos\alpha', \quad Q_2' = \lambda'\cos\alpha' + \mu'. \end{gathered}$$

From the last two, we have

$$\begin{aligned} \lambda' + \mu'\cos\alpha' &= Q_1' = \gamma Q_1 + \epsilon Q_2 = \gamma(\lambda + \mu\cos\alpha) + \epsilon(\lambda\cos\alpha + \mu), \\ \lambda'\cos\alpha' + \mu' &= Q_2' = \delta Q_1 + \eta Q_2 = \delta(\lambda + \mu\cos\alpha) + \eta(\lambda\cos\alpha + \mu); \end{aligned}$$

when these are resolved, and the foregoing relations are used, they yield

$$\begin{aligned} \omega\lambda\sin^2\alpha &= (\lambda' + \mu'\cos\alpha')(\delta\cos\alpha + \eta) - (\lambda'\cos\alpha' + \mu')(\gamma\cos\alpha + \epsilon) \\ &= \omega\sin^2\alpha(\gamma\lambda' + \delta\mu'), \end{aligned}$$

on reduction, and similarly

$$\omega\mu\sin^2\alpha = \omega\sin^2\alpha(\epsilon\lambda' + \eta\mu'),$$

that is,

$$\lambda = \gamma\lambda' + \delta\mu', \quad \mu = \epsilon\lambda' + \eta\mu'.$$

Hence

$$\lambda' L_1' + \mu' L_2' = \lambda'(\gamma L_1 + \epsilon L_2) + \mu'(\delta L_1 + \eta L_2) = \lambda L_1 + \mu L_2,$$

and therefore

$$X' = X.$$

Similarly $Y' = Y$, $Z' = Z$, $V' = V$: that is, the expressions for the foot of the perpendicular give an invariable position for all transformations of the equations of the plane.

Similarly

$$\begin{aligned} &Q_1'^2 - 2Q_1'Q_2'\cos\alpha' + Q_2'^2 \\ &\qquad = Q_1^2(\gamma^2 - 2\gamma\delta\cos\alpha' + \delta^2) + Q_2^2(\eta^2 - 2\epsilon\eta\cos\alpha' + \epsilon^2) \\ &\qquad\quad - 2Q_1Q_2\{(\gamma\eta + \delta\epsilon)\cos\alpha' - (\delta\eta + \gamma\epsilon)\} \\ &\qquad = (Q_1^2 - 2Q_1Q_2\cos\alpha + Q_2^2)\,\omega^2\sin^2\alpha, \end{aligned}$$

so that, as $\sin^2 \alpha' = \omega^2 \sin^2 \alpha$, there are the equalities

$$D'^2 = \frac{Q_1'^2 - 2Q_1'Q_2' \cos \alpha' + Q_2'^2}{\sin^2 \alpha'} = \frac{Q_1^2 - 2Q_1Q_2 \cos \alpha + Q_2^2}{\sin^2 \alpha} = D^2,$$

which shew that the expression for the length of the shortest distance also is invariantive.

Ex. When the plane is given by the equations

$$lx + my + nz + kv = 1, \quad l'x + m'y + n'z + k'v = 1,$$

shew that the perpendicular from ξ, η, ζ, v, on the plane is equal to

$$\left\{\frac{(1 - \Sigma l\xi)^2 - 2(1 - \Sigma l\xi)(1 - \Sigma l'\xi)\Sigma ll' + (1 - \Sigma l'\xi)^2}{(\Sigma l^2)(\Sigma l'^2) - (\Sigma ll')^2}\right\}^{\frac{1}{2}},$$

and find the coordinates of the foot of the perpendicular.

Intersection of a flat and a line.

54. As the equations of a line are three in number, its meeting (if any) with a flat will be determined by four equations, each linear in the variables: that is, a line generally intersects a flat in a point.

Let the flat be

$$Lx + My + Nz + Kv = P;$$

and let the line have

$$\frac{x - \alpha}{\lambda} = \frac{y - \beta}{\mu} = \frac{z - \gamma}{\nu} = \frac{v - \delta}{\kappa}$$

for its equations. At the common point, if any, let r be the common value of these four fractions; then r is determined by

$$L(\alpha + \lambda r) + M(\beta + \mu r) + N(\gamma + \nu r) + K(\delta + \kappa r) = P,$$

so that

$$r = \frac{P - L\alpha - M\beta - N\gamma - K\delta}{L\lambda + M\mu + N\nu + K\kappa};$$

and the coordinates of the point of intersection are

$$X = \alpha + \lambda r, \quad Y = \beta + \mu r, \quad Z = \gamma + \nu r, \quad V = \delta + \kappa r.$$

The result gives an infinite value for r, if

$$L\lambda + M\mu + N\nu + K\kappa = 0,$$

provided $P - L\alpha - M\beta - N\gamma - K\delta$ does not vanish. In that event, the direction, determined by λ, μ, ν, κ, is perpendicular to the direction, determined by L, M, N, K, which is the normal to the flat: that is, the direction is contained within the flat. But the point α, β, γ, δ, does not lie in the flat, because $P - L\alpha - M\beta - N\gamma - K\delta$ is not zero: thus the line meets the flat at an infinite distance, or we can say that the line is parallel to the flat.

The result gives an indeterminate value for r, if

$$L\lambda + M\mu + N\nu + K\kappa = 0,$$

and also

$$P - L\alpha - M\beta - N\gamma - K\delta = 0.$$

In that event, the equation

$$L(\alpha + \lambda R) + M(\beta + \mu R) + N(\gamma + \nu R) + K(\delta + \kappa R) = P,$$

is satisfied for all values of R: that is, the line lies in the flat.

Ex. 1. When the line does not lie in the flat but is parallel to it, prove that the shortest distance between the line and the flat is

$$\frac{P - L\alpha - M\beta - N\gamma - K\delta}{(L^2 + M^2 + N^2 + K^2)^{\frac{1}{2}}}.$$

Ex. 2. Prove that, for the figure on p. 7, the lines $O\beta$ and $O\gamma$ are parallel to the flat $Pagh$, and that the distance $O\delta$ is bisected by that flat.

Ex. 3 Prove that the line

$$\left.\begin{matrix} L_1x + M_1y + N_1z + K_1v = P_1 \\ L_2x + M_2y + N_2z + K_2v = P_2 \\ L_3x + M_3y + N_3z + K_3v = P_3 \end{matrix}\right\}$$

lies in the flat

$$Lx + My + Nz + Kv = P,$$

if the conditions

$$\left\| \begin{matrix} L_1, & M_1, & N_1, & K_1, & P_1 \\ L_2, & M_2, & N_2, & K_2, & P_2 \\ L_3, & M_3, & N_3, & K_3, & P_3 \\ L, & M, & N, & K, & P \end{matrix} \right\| = 0$$

are satisfied.

Intersection of two flats

55. We have seen that a plane can be represented by the equations of two flats, taken simultaneously. Hence a plane can be regarded as a plane of cleavage of two different flats.

When the flats are

$$L_1x + M_1y + N_1z + K_1v = P_1,$$
$$L_2x + M_2y + N_2z + K_2v = P_2,$$

any direction l, m, n, k, in their plane of cleavage is such that

$$L_1l + M_1m + N_1n + K_1k = 0,$$
$$L_2l + M_2m + N_2n + K_2k = 0.$$

Hence the normal to each flat is perpendicular to every such direction: that is, to every direction in the plane of cleavage.

Ex. Obtain the equations of the flats $Pagh$ and $P\delta g'h'$, in the figure on p. 7; and shew that the intersection of these flats is the plane PBC.

Find the plane in which the flats $Pagh$ and $P\delta f'g'$ intersect, inserting points (other than P) in the diagram sufficient to determine the plane.

Manifestly, when two planes exist in one and the same flat, they intersect in a line, a property already (§ 43) established. For if the two planes are given by the two pairs of equations

$$\left.\begin{array}{l}\Sigma Lx\;=P\\ \Sigma L_1x=P_1\end{array}\right\},\quad \left.\begin{array}{l}\Sigma Lx\;=P\\ \Sigma L_2x=P_2\end{array}\right\},$$

where $\Sigma Lx=P$ is the flat in which they both lie, then the common intersection of the planes is

$$\Sigma Lx=P,\quad \Sigma L_1x=P_1,\quad \Sigma L_2x=P_2,$$

that is, it is a line

COROLLARY. Two planes in any three-dimensional space meet in a line For they can be taken as existing in a common flat $v=0$, being given by

$$Ax+By+Cz+Dv=E,\ v=0;\quad A'x+B'y+C'z+D'v=E',\ v=0;$$

their line of intersection in the flat $v=0$ is given by

$$Ax+By+Cz=E,\quad A'x+B'y+C'z=E'.$$

Intersection of a flat and a plane, or of three flats.

56. When the intersection of a flat by a plane is required, it can be given by three linear equations, made up of the single equation of the flat and the two equations of the plane. Thus the intersection will, in general, be a line.

If the equations of the plane are the combined equations of two flats, the intersection will be given by

$$\begin{array}{l}Lx\;+My\;+Nz\;+Kv\;=P,\\ \left.\begin{array}{l}L_1x+M_1y+N_1z+K_1v=P_1\\ L_2x+M_2y+N_2z+K_2v=P_2\end{array}\right\},\end{array}$$

taken simultaneously. Accordingly (§ 17) the intersection is a line, always on the implicit assumption that there are three equations, that is, that the three equations are unconnected by an identical relation.

If the equations of the plane are

$$\left\|\begin{array}{cccc}x-a, & y-b, & z-c, & v-d\\ l_1\;, & m_1\;, & n_1\;, & k_1\\ l_2\;, & m_2\;, & n_2\;, & k_2\end{array}\right\|=0,$$

every point in the plane is given by means of the relations

$$x-a=l_1r_1+l_2r_2,\quad y-b=m_1r_1+m_2r_2,\quad z-c=n_1r_1+n_2r_2,\quad v-d=k_1r_1+k_2r_2$$

Hence all points common to the plane and the flat are given by those values of the parameters r_1 and r_2 which satisfy the relation

$$P-\Sigma La+r_1\Sigma Ll_1+r_2\Sigma Ll_2=0.$$

(This relation is merely the expression of a property in projections. Let A be the point a, b, c, d; let AN_1 be a distance r_1 along the direction l_1, m_1, n_1, k_1, and AN_2 be a distance r_2 along the direction l_2, m_2, n_2, k_2, so that the point x, y, z, v, is the fourth corner of the parallelogram N_1AN_2P in the plane. The property in question is that the projection of the broken line AN_1P upon the normal to the flat from A is equal to the projection of the straight line AP upon that normal.) Hence the points common to the plane and the flat are given by

$$\begin{aligned} x - a &= l_1 r_1 + l_2 r_2 \\ &= \frac{l_1}{\Sigma l_1 L}(-P + \Sigma La) + \left(l_2 - l_1 \frac{\Sigma l_2 L}{\Sigma l_1 L}\right) r_2, \end{aligned}$$

and similarly for y, z, v; that is, the common points constitute the line

$$\frac{x - a + \frac{l_1}{\Sigma l_1 L}(P - \Sigma La)}{l_2 \Sigma l_1 L - l_1 \Sigma l_2 L} = \frac{y - b + \frac{m_1}{\Sigma l_1 L}(P - \Sigma La)}{m_2 \Sigma l_1 L - m_1 \Sigma l_2 L}$$

$$= \frac{z - c + \frac{n_1}{\Sigma l_1 L}(P - \Sigma La)}{n_2 \Sigma l_1 L - n_1 \Sigma l_2 L} = \frac{v - d + \frac{k_1}{\Sigma l_1 L}(P - \Sigma La)}{k_2 \Sigma l_1 L - k_1 \Sigma l_2 L}.$$

Ex 1. Shew that, if $F_1=0$, $F_2=0$, $F_3=0$, $F_4=0$ are four flats, and if the planes

$$\left.\begin{matrix} F_1=0 \\ F_2=0 \end{matrix}\right\}, \quad \left.\begin{matrix} F_3=0 \\ F_4=0 \end{matrix}\right\},$$

intersect in a line and not in a point only, then the other two pairs of planes given by

$$\left.\begin{matrix} F_1=0 \\ F_3=0 \end{matrix}\right\}, \quad \left.\begin{matrix} F_2=0 \\ F_4=0 \end{matrix}\right\}; \qquad \left.\begin{matrix} F_1=0 \\ F_4=0 \end{matrix}\right\}, \quad \left.\begin{matrix} F_2=0 \\ F_3=0 \end{matrix}\right\};$$

also meet in the same line

Ex. 2. Prove that the flat $Pagh$ and the plane $fg'h'$, in the figure on p. 7, intersect in a line parallel to BC through the middle point of OD

What is the intersection of the same plane with the flat $Pf'gh$?

Ex. 3. Obtain the line-intersection of the flat $Lx+My+Nz+Kv=P$ and the plane

$$z=px+qy+f, \quad v=rx+sy+h,$$

in the form

$$\frac{x-\theta}{M+Nq+Ks} = \frac{y}{-(Np+Kr+L)} = \frac{z-f-p\theta}{K(sp-qr)-Lq-Mp} = \frac{v-h-r\theta}{-Ls+Mr-N(sp-qr)},$$

where

$$\theta = \frac{P-Nf-Kg}{L+Np+Kr}.$$

Conditions that a plane is contained in a flat.

57. The intersection of the flat and the plane is definitely a line when, in the first instance, the three equations

$$Lx + My + Nz + Kv = P,$$

$$\left.\begin{aligned} L_1x + M_1y + N_1z + K_1v &= P_1 \\ L_2x + M_2y + N_2z + K_2v &= P_2 \end{aligned}\right\},$$

are independent of one another, that is, are unconnected by any linear relation. When the equations are not thus unconnected, they reduce to two equations: they cannot reduce to fewer than two, because a plane (which requires two equations for its expression) is presumed given. When they reduce to two only, there must subsist relations

$$L = \rho L_1 + \sigma L_2, \quad M = \rho M_1 + \sigma M_2, \quad N = \rho N_1 + \sigma N_2,$$
$$K = \rho K_1 + \sigma K_2, \quad P = \rho P_1 + \sigma P_2,$$

that is, we must have the three relations

$$\left\| \begin{matrix} L, & M, & N, & K, & P \\ L_1, & M_1, & N_1, & K_1, & P_1 \\ L_2, & M_2, & N_2, & K_2, & P_2 \end{matrix} \right\| = 0.$$

Moreover, we then have

$$\rho(\Sigma L_1 x - P_1) + \sigma(\Sigma L_2 x - P_2) = \Sigma L x - P:$$

that is, any point, lying in the plane and therefore having its coordinates satisfying the equations $\Sigma L_1 x - P_1 = 0$ and $\Sigma L_2 x - P_2 = 0$, has its coordinates satisfying the equation $\Sigma L x - P = 0$ and therefore lies in the flat. Hence, under the foregoing conditions, the plane represented by the second and third equations lies in the flat represented by the first equation.

It follows that, when the conditions are satisfied, the plane represented by any two of the equations lies in the flat represented by the third equation.

Ex. Discuss the contingency that arises, when only the conditions

$$\left| \begin{matrix} L, & M, & N, & K \\ L_1, & M_1, & N_1, & K_1 \\ L_2, & M_2, & N_2, & K_2 \end{matrix} \right| = 0$$

out of the preceding set are satisfied.

Parallelism of flats.

58. As it proved necessary to apply the property of parallelism to planes by an extension of its property in connection with lines, so it proves convenient to apply the property of parallelism to flats, by a similar extension of the property in connection with lines. We proceed as in § 42, using as a basis the synthetic construction of a flat from straight lines.

Let two flats be denoted by F and F′. Take any point A in F and any point A′ in F′. Through A draw any three arbitrary directions AB, AC, AD, in F, subject to the negative limitation that these three directions are non-complanar. Through A′, in quadruple space, draw three directions A′B′, A′C′, A′D′, respectively parallel to AB, AC, AD. If all these three directions A′B′, A′C′, A′D′, lie in F′, we say that the flats F′ and F are parallel.

It might happen that a direction $A'B'$ could be drawn in F' parallel to a particular direction AB in F: the property would be inadequate to secure the desired parallelism. It also might happen that directions $A'B'$ and $A'C'$ could be drawn in F', respectively parallel to two particular directions AB and AC in F: the property still would be inadequate to secure the desired parallelism. In the former event, it would mean that two parallel directions can be taken in the respective flats: in the latter, it would mean that two parallel planes could be taken in the respective flats.

For the parallelism of flats, it is necessary to have the three concomitant non-complanar parallel directions. When the property is possessed, it is possible to obtain in F' a direction parallel to any assumed arbitrary direction in F, a characteristic result established as follows.

Let l_1, m_1, n_1, k_1; l_2, m_2, n_2, k_2; l_3, m_3, n_3, k_3; determine three non-complanar directions in F. Suppose that those three directions can be drawn in F'. Then any direction in F is given by

$$\begin{aligned} l &= \alpha l_1 + \beta l_2 + \gamma l_3, \\ m &= \alpha m_1 + \beta m_2 + \gamma m_3, \\ n &= \alpha n_1 + \beta n_2 + \gamma n_3, \\ k &= \alpha k_1 + \beta k_2 + \gamma k_3. \end{aligned}$$

These same magnitudes l, m, n, k, also determine a direction in F': that is, a direction in F' can be drawn parallel to any direction whatever assumed in F, which is the property in question.

That the agreement in parallelism of (e.g.) only two directions would be inadequate to secure full parallelism, is easily seen. Thus if, in F', the first two of the specified directions are possible but not a third non-complanar direction, we cannot have

$$\alpha\theta_1 + \beta\theta_2 + \gamma\theta_3 = \alpha\theta_1 + \beta\theta_2 + \gamma'\theta_3',$$

for $\theta = l, m, n, k$, when the ratios $l_3' : m_3' : n_3' : k_3'$ are not equal to the ratios $l_3 : m_3 : n_3 : k_3$; that is, lines could be drawn in F, parallels to which could not be drawn in F'. The two flats would not then possess complete parallelism, though parallel directions would exist for $\gamma = 0$, $\gamma' = 0$, while α and β remain arbitrary.

It thus appears that, when the flat F is

$$\begin{vmatrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \\ l_3\ , & m_3\ , & n_3\ , & k_3 \end{vmatrix} = 0,$$

the parallel flat F' through the point a' is

$$\begin{vmatrix} x-a', & y-b', & z-c', & v-d' \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \end{vmatrix} = 0.$$

Consequently, if the normal to the flat F has L, M, N, K, for its direction-cosines, then L, M, N, K, are also the direction-cosines of the normal to the flat F': in other words, the normals to two parallel flats are themselves parallel.

It is an immediate corollary that the flats

$$Lx + My + Nz + Kv = P_1,$$
$$Lx + My + Nz + Kv = P_2,$$

where P_1 and P_2 are distinct from one another, are parallel to one another. Further we say that, as two flats usually have a plane of cleavage, two parallel flats have their plane of cleavage at infinity.

Ex 1. Obtain the distance between two parallel flats F and F'.

Ex. 2. Prove that the flats $ABCD$, $ff'gg'hh'$, $\alpha\beta\gamma\delta$, in the figure on p. 7, are parallel to one another; likewise the pairs of flats $O\beta\gamma\delta$ and $PBCD$, $O\gamma\delta\alpha$ and $PCDA$, $O\delta\alpha\beta$ and $PDAB$, $O\alpha\beta\gamma$ and $PABC$.

Find where a flat, through f, g, h, parallel to the flat $O\beta\gamma\delta$ is met by the line OP.

Ex. 3. Shew that the necessary and sufficient conditions, in order that the flats

$$\begin{vmatrix} x-a, & y-b, & z-c, & v-d \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \end{vmatrix} = 0, \qquad \begin{vmatrix} x-a' & y-b' & z-c' & v-d' \\ l_1' & m_1' & n_1' & k_1' \\ l_2' & m_2' & n_2' & k_2' \\ l_3' & m_3' & n_3' & k_3' \end{vmatrix} = 0,$$

shall be parallel, are that relations

$$\left.\begin{aligned} l_r &= \alpha_r l_1' + \beta_r l_2' + \gamma_r l_3' \\ m_r &= \alpha_r m_1' + \beta_r m_2' + \gamma_r m_3' \\ n_r &= \alpha_r n_1' + \beta_r n_2' + \gamma_r n_3' \\ k_r &= \alpha_r k_1' + \beta_r k_2' + \gamma_r k_3' \end{aligned}\right\}$$

shall be satisfied, for $r = 1, 2, 3$.

Inclinations of homaloidal amplitudes of various types.

59. We now proceed to consider the inclinations of the various homaloidal amplitudes, line, plane, flat, to one another.

The case, when both the amplitudes are lines, has been already discussed (§§ 19, 20). It is the simplest of all the cases, for each of the amplitudes admits, within itself, a unique direction which is one of its essential characteristics.

When one of the amplitudes is a plane, there is no individual direction, which is contained by the plane and can be regarded as an essential characteristic. Nor is there any individual direction, perpendicular to the plane, and belonging to it alone without specification of some external property: so that, within the aggregate of lines perpendicular to the plane at any point in the plane, there is no one definite line which can be selected as an essentially associated characteristic direction.

When one of the amplitudes is a flat, there is no individual direction which is contained by the flat and can be regarded as an essential characteristic. But there is one unique direction, which is perpendicular to every direction in the flat and which is the normal to the flat: that normal can be taken as a direction, essentially characteristic of the flat.

Hence, after the case of two lines already considered, the cases, next in simplicity, are

(i) the inclination of a line and a flat, and

(ii) the inclination of two flats.

The respective inclinations of a plane to a line, to a plane, to a flat, remain for separate consideration, at a later stage.

Inclination of a line to a flat.

60. In estimating the inclination of a line to a flat, there are two methods of proceeding.

The more obvious method is to frame an estimate, by taking the inclination of the given line to the normal to the flat, because the direction of this normal is unique in relation to the flat. If the flat be

$$\begin{vmatrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \\ l_3\ , & m_3\ , & n_3\ , & k_3 \end{vmatrix} = 0,$$

and if

$$\Sigma l_2 l_3 = \cos\alpha, \quad \Sigma l_3 l_1 = \cos\beta, \quad \Sigma l_1 l_2 = \cos\gamma,$$

the equation of the flat can be written in the form

$$Lx + My + Nz + Kv = P,$$

where L, M, N, K, the direction-cosines of the normal, are

$$\frac{L}{\begin{vmatrix} m_1, & n_1, & k_1 \\ m_2, & n_2, & k_2 \\ m_3, & n_3, & k_3 \end{vmatrix}} = \frac{M}{-\begin{vmatrix} n_1, & k_1, & l_1 \\ n_2, & k_2, & l_2 \\ n_3, & k_3, & l_3 \end{vmatrix}} = \frac{N}{\begin{vmatrix} k_1, & l_1, & m_1 \\ k_2, & l_2, & m_2 \\ k_3, & l_3, & m_3 \end{vmatrix}} = \frac{K}{-\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}} = \Theta^{-\frac{1}{2}},$$

the value of Θ being

$$1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma.$$

Let the inclination of this normal to the given line with direction-cosines l, m, n, k, be $\frac{1}{2}\pi-\vartheta$; then

$$\sin\vartheta=lL+mM+nN+kK,$$

that is,

$$\Theta^{\frac{1}{2}}\sin\vartheta=\begin{vmatrix} l, & m, & n, & k \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \end{vmatrix}$$

61. The foregoing method however has the apparent disadvantage that it brings the given line into relation with the flat, only by estimating its inclination to a line which lies outside the flat, though that outside position has a corporate association with all directions in the flat. By the alternative method, we first take the inclination ϕ of the given line to any direction in the flat: and, to avoid an ambiguity of sign necessarily dependent on the sense in which a direction is measured, we consider the magnitude $\cos^2\phi$.

Now this magnitude can never be negative, though it can be zero: it can never be greater than unity, though it can be unity. It is always possible to select a line, in an unlimited number of ways, in the flat so as to be perpendicular to the given line. For take a direction

$$\begin{aligned} \lambda &= \rho l_1 + \sigma l_2 + \tau l_3, \\ \mu &= \rho m_1 + \sigma m_2 + \tau m_3, \\ \nu &= \rho n_1 + \sigma n_2 + \tau n_3, \\ \kappa &= \rho k_1 + \sigma k_2 + \tau k_3, \end{aligned}$$

where the parameters ρ, σ, τ, are subject to the relation

$$\rho^2+\sigma^2+\tau^2+2\sigma\tau\cos\alpha+2\tau\rho\cos\beta+2\rho\sigma\cos\gamma=1;$$

and let

$$\Sigma ll_1=\cos\alpha', \quad \Sigma ll_2=\cos\beta', \quad \Sigma ll_3=\cos\gamma'.$$

Then ϕ is a right angle, and $\cos^2\phi$ assumes its least possible value, when

$$\Sigma\lambda l=0:$$

that is, when

$$\rho\cos\alpha'+\sigma\cos\beta'+\tau\cos\gamma'=0.$$

Manifestly, there is an infinitude of values of ρ, σ, τ, satisfying the two relations: and therefore it is possible, in an infinite number of ways, to secure that $\cos^2\phi$ acquires its least value, which is zero, though it may not provide a minimum for $\cos\phi$.

We therefore proceed to find the maximum value which $\cos^2\phi$ can acquire.

Denoting the corresponding value of ϕ by θ, we take the maximum value of $\cos^2\phi$ for all values ρ, σ, τ, that are admissible under the one condition

$$\rho^2 + \sigma^2 + \tau^2 + 2\sigma\tau\cos\alpha + 2\tau\rho\cos\beta + 2\rho\sigma\cos\gamma = 1,$$

while, for the maximum value,

$$\begin{aligned}\cos\theta &= l\lambda + m\mu + n\nu + k\kappa\\ &= \rho\cos\alpha' + \sigma\cos\beta' + \tau\cos\gamma'.\end{aligned}$$

The critical equations are

$$\begin{aligned}\cos\alpha' &= \Omega\,(\rho \qquad\quad + \sigma\cos\gamma + \tau\cos\beta),\\ \cos\beta' &= \Omega\,(\rho\cos\gamma + \sigma \qquad\quad + \tau\cos\alpha),\\ \cos\gamma' &= \Omega\,(\rho\cos\beta + \sigma\cos\alpha + \tau \qquad\quad),\end{aligned}$$

where Ω is, initially, an indeterminate magnitude. Multiplying by ρ, σ, τ, and adding, we have $\cos\theta = \Omega$; and thus the foregoing equations are

$$\begin{aligned}\frac{\cos\alpha'}{\cos\theta} &= \rho \qquad\quad + \sigma\cos\gamma + \tau\cos\beta,\\ \frac{\cos\beta'}{\cos\theta} &= \rho\cos\gamma + \sigma \qquad\quad + \tau\cos\alpha,\\ \frac{\cos\gamma'}{\cos\theta} &= \rho\cos\beta + \sigma\cos\alpha + \tau\end{aligned}$$

Also we have

$$\cos\theta = \rho\cos\alpha' + \sigma\cos\beta' + \tau\cos\gamma';$$

hence

$$\begin{vmatrix}\cos\alpha', & 1\ , & \cos\gamma, & \cos\beta\\ \cos\beta', & \cos\gamma, & 1\ , & \cos\alpha\\ \cos\gamma', & \cos\beta, & \cos\alpha, & 1\\ \cos^2\theta, & \cos\alpha', & \cos\beta', & \cos\gamma'\end{vmatrix} = 0,$$

and therefore

$$\Theta\cos^2\theta = \begin{vmatrix}\cos\alpha', & 1\ , & \cos\gamma, & \cos\beta\\ \cos\beta', & \cos\gamma, & 1\ , & \cos\alpha\\ \cos\gamma', & \cos\beta, & \cos\alpha, & 1\\ 0\ , & \cos\alpha', & \cos\beta', & \cos\gamma'\end{vmatrix}.$$

From the expression for $\sin\vartheta$ in § 60, we have

$$\begin{aligned}\Theta\sin^2\vartheta &= \begin{vmatrix}1\ , & \cos\alpha', & \cos\beta', & \cos\gamma'\\ \cos\alpha', & 1\ , & \cos\gamma\ , & \cos\beta\\ \cos\beta', & \cos\gamma\ , & 1\ , & \cos\alpha\\ \cos\gamma', & \cos\beta, & \cos\alpha\ , & 1\end{vmatrix}\\ &= \Theta + \begin{vmatrix}0\ , & \cos\alpha', & \cos\beta', & \cos\gamma'\\ \cos\alpha', & 1\ , & \cos\gamma\ , & \cos\beta\\ \cos\beta', & \cos\gamma\ , & 1\ , & \cos\alpha\\ \cos\gamma', & \cos\beta, & \cos\alpha\ , & 1\end{vmatrix};\end{aligned}$$

and therefore

$$\cos^2\theta = \cos^2 \vartheta.$$

Thus the same angle, $\theta, = \vartheta$, is obtained for the measure by the two methods.

62. It is to be expected, on considering the geometrical configuration, that the direction of the line λ, μ, ν, κ, with which the line l, m, n, k, makes this angle θ, lies in the plane through the given line and a direction normal to the flat: so that, in fact, the direction λ, μ, ν, κ, is the intersection of the flat by this plane. This expectation is verified, as follows:

The equations of the plane, through the normal to the flat and the given line l, m, n, k, are

$$\left\| \begin{matrix} x-a, & y-b, & z-c, & v-d \\ l\ , & m\ , & n\ , & k \\ L\ , & M\ , & N\ , & K \end{matrix} \right\| = 0;$$

and any direction in this plane is given by

$$\lambda' = pl + qL, \quad \mu' = pm + qM, \quad \nu' = pn + qN, \quad \kappa' = pk + qK.$$

If this direction is the intersection of the plane and the flat

$$L(x-a) + M(y-b) + N(z-c) + K(v-d) = 0,$$

then

$$L\lambda' + M\mu' + N\nu' + K\kappa' = 0,$$

so that

$$p\Sigma Ll + q\Sigma L^2 = 0,$$

that is,

$$p\sin\theta + q = 0.$$

Also

$$1 = \Sigma\lambda'^2 = p^2 + 2pq\sin\theta + q^2;$$

hence

$$\frac{p}{1} = \frac{q}{-\sin\theta} = \frac{1}{\cos\theta},$$

and therefore

$$\lambda' = \frac{l - L\sin\theta}{\cos\theta}, \quad \mu' = \frac{m - M\sin\theta}{\cos\theta}, \quad \nu' = \frac{n - N\sin\theta}{\cos\theta}, \quad \kappa' = \frac{k - K\sin\theta}{\cos\theta}.$$

With the preceding values of λ, given by

$$\lambda = \rho l_1 + \sigma l_2 + \tau l_3,$$

and the corresponding expressions for μ, ν, κ, we have

$$\Sigma L(\lambda' - \lambda) = \frac{\Sigma Ll - \Sigma L^2\sin\theta}{\cos\theta} - \rho\Sigma Ll_1 - \sigma\Sigma Ll_2 - \tau\Sigma Ll_3 = 0,$$

because $\Sigma Ll = \sin\theta$, $\Sigma Ll_1 = 0 = \Sigma Ll_2 = \Sigma Ll_3$. Also

$$\Sigma l_1(\lambda' - \lambda) = \frac{\Sigma ll_1 - \Sigma Ll_1\sin\theta}{\cos\theta} - \rho\Sigma l_1^2 - \sigma\Sigma l_1 l_2 - \tau\Sigma l_1 l_3$$

$$= \frac{\cos\alpha'}{\cos\theta} - \rho - \sigma\cos\gamma - \tau\cos\beta = 0,$$

$$\Sigma l_2(\lambda'-\lambda)=\frac{\Sigma l l_2-\Sigma l_2 L\sin\theta}{\cos\theta}-\rho\Sigma l_1 l_2-\sigma\Sigma l_2^2-\tau\Sigma l_2 l_3$$

$$=\frac{\cos\beta'}{\cos\theta}-\rho\cos\gamma-\sigma-\tau\cos\alpha=0,$$

$$\Sigma l_3(\lambda'-\lambda)=\frac{\Sigma l l_3-\Sigma l_3 L\sin\theta}{\cos\theta}-\rho\Sigma l_1 l_3-\sigma\Sigma l_2 l_3-\tau\Sigma l_3^2$$

$$=\frac{\cos\gamma'}{\cos\theta}-\rho\cos\beta-\sigma\cos\alpha-\tau=0.$$

Now the determinant

$$\begin{vmatrix} L, & M, & N, & K \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \end{vmatrix}, =\Theta^{\frac{1}{2}},$$

and does not vanish: hence, as

$$\Sigma L(\lambda'-\lambda)=0,\quad \Sigma l_1(\lambda'-\lambda)=0,\quad \Sigma l_2(\lambda'-\lambda)=0,\quad \Sigma l_3(\lambda'-\lambda)=0,$$

we have

$$\lambda'-\lambda=0,\quad \mu'-\mu=0,\quad \nu'-\nu=0,\quad \kappa'-\kappa=0,$$

thus establishing the property in question.

63. From these results, or by an independent assumption of what is suggested by a geometrical figure, we can infer the inclination of a line to a flat in the following way.

Let the line be

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}=\frac{v-d}{k},$$

and the flat be

$$Lx+My+Nz+Kv=P.$$

The line meets the flat at a distance r from a, b, c, d, where

$$L(a+lr)+M(b+mr)+N(c+nr)+K(d+kr)=P,$$

so that

$$r(Ll+Mm+Nn+Kk)=P-La-Mb-Nc-Kd.$$

The perpendicular, p, from a, b, c, d, on the flat is

$$p=P-La-Mb-Nc-Kd.$$

Hence, if θ be the inclination of the line to the flat, defined (from the pure geometry) by the relation

$$p=r\sin\theta,$$

we obtain the former result

$$\sin\theta=Ll+Mm+Nn+Kk.$$

This process is, however, hardly more than a modification of the first mode (§ 60) of obtaining the inclination.

Ex. In the figure on p. 7, prove that the lines ff', fg', fh', are parallel to the flat $\alpha\beta\gamma\delta'$; and that OP is perpendicular to the flat, only when $OA=OB=OC=OD$.

Projection of a line on a flat.

64. With the discussion of the inclination of a line to a flat, it is natural to associate the projection of the line on the flat. Manifestly the projection can be obtained, by drawing perpendiculars from points of the line upon the flat and taking the locus of the feet of the perpendiculars. Let the flat be

$$Lx + My + Nz + Kv = P,$$

and the line be

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = \frac{v-d}{k}.$$

Any point on this line can be represented by coordinates

$$\xi, \eta, \zeta, v, \quad = a + l\rho, \quad b + m\rho, \quad c + n\rho, \quad d + k\rho,$$

where ρ is parametric. When X, Y, Z, V, denote the coordinates of the foot of the perpendicular of length R drawn to the flat from ξ, η, ζ, v, we have

$$X = \xi + LR, \quad Y = \eta + MR, \quad Z = \zeta + NR, \quad V = v + KR.$$

This point lies in the flat; hence

$$\begin{aligned} P &= LX + MY + NZ + KV \\ &= R + L\xi + M\eta + N\zeta + Kv \\ &= R + \Sigma La + \rho \Sigma Ll. \end{aligned}$$

Thus

$$\begin{aligned} X &= a + l\rho + LR \\ &= a + L(P - \Sigma La) + \rho(l - L\,\Sigma Ll), \\ Y &= b + M(P - \Sigma La) + \rho(m - M\Sigma Ll), \\ Z &= c + N(P - \Sigma La) + \rho(n - N\Sigma Ll), \\ V &= d + K(P - \Sigma La) + \rho(k - K\Sigma Ll). \end{aligned}$$

Therefore the locus of X, Y, Z, V, that is, the projection of the line, is given by

$$\frac{X-\alpha}{l - L\Sigma lL} = \frac{Y-\beta}{m - M\Sigma lL} = \frac{Z-\gamma}{n - N\Sigma lL} = \frac{V-\delta}{k - K\Sigma lL},$$

where

$$\frac{\alpha - a}{L} = \frac{\beta - b}{M} = \frac{\gamma - c}{N} = \frac{\delta - d}{K} = P - \Sigma La,$$

and $\alpha, \beta, \gamma, \delta$, are the coordinates of the foot of the perpendicular from a, b, c, d, on the flat.

With the preceding notation, ϑ is the inclination of the line to the flat (§ 60): thus ϑ should be the angle between the line and its projection. Now

$$(l - L\Sigma lL)^2 + (m - M\Sigma lL)^2 + (n - N\Sigma lL)^2 + (k - K\Sigma lL)^2 = \cos^2\vartheta,$$

so that the *direction-cosines of the projection* are

$$\frac{l - L\Sigma lL}{\cos\vartheta}, \quad \frac{m - M\Sigma lL}{\cos\vartheta}, \quad \frac{n - N\Sigma lL}{\cos\vartheta}, \quad \frac{k - K\Sigma lL}{\cos\vartheta},$$

and ϑ clearly is the inclination of this projection to the original line.

Ex. When the flat is given by the equation

$$\begin{vmatrix} x-a', & y-b', & z-c', & v-d' \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \\ l_3\ , & m_3\ , & n_3\ , & k_3 \end{vmatrix} = 0,$$

and when

$$\cos\theta_{23} = \Sigma l_2 l_3, \quad \cos\theta_{31} = \Sigma l_3 l_1, \quad \cos\theta_{12} = \Sigma l_1 l_2,$$

$$\cos\theta_{01} = \Sigma l l_1\ , \quad \cos\theta_{02} = \Sigma l l_2\ , \quad \cos\theta_{03} = \Sigma l l_3\ ,$$

shew that the direction-cosines, of the projection of the line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = \frac{v-d}{k}$$

on the flat, are proportional to λ, μ, ν, κ, where

$$\begin{vmatrix} \varpi\ , & t_1\ , & t_2\ , & t_3 \\ \cos\theta_{01}, & 1\ , & \cos\theta_{12}, & \cos\theta_{13} \\ \cos\theta_{02}, & \cos\theta_{12}, & 1\ , & \cos\theta_{23} \\ \cos\theta_{03}, & \cos\theta_{13}, & \cos\theta_{23}, & 1 \end{vmatrix} = 0,$$

in which $\varpi = \lambda$, μ, ν, κ, in turn, and $t = l$, m, n, k, for the respective values of ϖ.

Inclination of two flats.

65. We now pass to the consideration of the inclination of two flats

$$L_1 x + M_1 y + N_1 z + K_1 v = P_1, \quad L_2 x + M_2 y + N_2 z + K_2 v = P_2,$$

to one another; and, as for the preceding investigation, there are two methods.

We assume, as an inference from the fact that the normal to a flat is a uniquely determinate direction, that the inclination between the normals to the two flats can be taken as a measure of the inclination of the two flats. On this assumption, and denoting the angle between the directions of the respective normals to be θ, we have

$$\cos\theta = L_1 L_2 + M_1 M_2 + N_1 N_2 + K_1 K_2.$$

By the alternative method adopted in § 61, we take any direction l_1, m_1, n_1, k_1, in the first flat, so that

$$L_1 l_1 + M_1 m_1 + N_1 n_1 + K_1 k_1 = 0, \quad l_1^2 + m_1^2 + n_1^2 + k_1^2 - 1 = 0,$$

and any direction l_2, m_2, n_2, k_2, in the second flat, so that

$$L_2 l_2 + M_2 m_2 + N_2 n_2 + K_2 k_2 = 0, \quad l_2^2 + m_2^2 + n_2^2 + k_2^2 - 1 = 0.$$

Now it is always possible, even after the direction in the first flat has been selected, to choose a direction in the second flat which is perpendicular to that selected direction; for the direction-cosines l_2, m_2, n_2, k_2, need only satisfy the relation

$$l_1 l_2 + m_1 m_2 + n_1 n_2 + k_1 k_2 = 0,$$

in addition to the two relations connected with their existence in the second flat. Accordingly, if

$$\cos\phi = l_1 l_2 + m_1 m_2 + n_1 n_2 + k_1 k_2,$$

the quantity $\cos^2\phi$ can always be made zero: it can never be greater than unity: and accordingly there will be a maximum value for $\cos^2\phi$, which usually will lie between 0 and 1. We denote the value of ϕ, associated with this maximum value of $\cos^2\phi$, by θ. In order that $\cos^2\phi$ may be a maximum, the critical equations are

$$\left.\begin{array}{ll} l_2 = \lambda L_1 + \mu l_1\,, & l_1 = \lambda' L_2 + \mu' l_2 \\ m_2 = \lambda M_1 + \mu m_1, & m_1 = \lambda' M_2 + \mu' m_2 \\ n_2 = \lambda N_1 + \mu n_1\,, & n_1 = \lambda' N_2 + \mu' n_2 \\ k_2 = \lambda K_1 + \mu k_1\,, & k_1 = \lambda' K_2 + \mu' k_2 \end{array}\right\}$$

Multiply the first set throughout by l_1, m_1, n_1, k_1, and add: then

$$\cos\theta = \mu,$$

so that

$$l_2 - l_1\cos\theta = \lambda L_1\,,\quad m_2 - m_1\cos\theta = \lambda M_1,$$

$$n_2 - n_1\cos\theta = \lambda N_1,\quad k_2 - k_1\cos\theta = \lambda K_1,$$

and, when these are squared and added, we have

$$\sin^2\theta = \lambda^2,$$

so that we can take

$$\sin\theta = \epsilon\lambda,$$

where ϵ is ± 1.

Again multiply the second set throughout by l_2, m_2, n_2, k_2, and add · then

$$\cos\theta = \mu',$$

so that

$$l_1 - l_2\cos\theta = \lambda' L_2\,,\quad m_1 - m_2\cos\theta = \lambda' M_2,$$

$$n_1 - n_2\cos\theta = \lambda' N_2,\quad k_1 - k_2\cos\theta = \lambda' K_2;$$

and, when these are squared and added, we have

$$\sin^2\theta = \lambda'^2,$$

so that we can take

$$\sin\theta = \eta\lambda',$$

where η is ± 1.

Hence we have

$$\begin{array}{ll} l_2 - l_1\cos\theta = \epsilon L_1\sin\theta, & l_1 - l_2\cos\theta = \eta L_2\sin\theta, \\ m_2 - m_1\cos\theta = \epsilon M_1\sin\theta, & m_1 - m_2\cos\theta = \eta M_2\sin\theta, \\ n_2 - n_1\cos\theta = \epsilon N_1\sin\theta, & n_1 - n_2\cos\theta = \eta N_2\sin\theta, \\ k_2 - k_1\cos\theta = \epsilon K_1\sin\theta, & k_1 - k_2\cos\theta = \eta K_2\sin\theta. \end{array}$$

Consequently,

$$\begin{aligned}\epsilon\eta \sin^2\theta\,\Sigma L_1 L_2 &= \Sigma\,(l_2 - l_1\cos\theta)(l_1 - l_2\cos\theta)\\ &= \Sigma l_1 l_2 - \cos\theta\,\Sigma l_1^2 - \cos\theta\,\Sigma l_2^2 + \cos^2\theta\,\Sigma l_1 l_2\\ &= -\cos\theta\sin^2\theta,\end{aligned}$$

and therefore

$$\cos\theta = -\epsilon\eta\,\Sigma L_1 L_2.$$

Now the line l_1, m_1, n_1, k_1, can be drawn in either sense along its direction, and likewise for l_2, m_2, n_2, k_2, along its direction: so that two doubtful signs are at our disposal. Accordingly, we choose

$$\epsilon = 1, \quad \eta = -1;$$

and then

$$\cos\theta = L_1L_2 + M_1M_2 + N_1N_2 + K_1K_2,$$

while

$$\begin{aligned} l_2 - l_1\cos\theta &= L_1\sin\theta, & l_1 - l_2\cos\theta &= -L_2\sin\theta,\\ m_2 - m_1\cos\theta &= M_1\sin\theta, & m_1 - m_2\cos\theta &= -M_2\sin\theta,\\ n_2 - n_1\cos\theta &= N_1\sin\theta, & n_1 - n_2\cos\theta &= -N_2\sin\theta,\\ k_2 - k_1\cos\theta &= K_1\sin\theta, & k_1 - k_2\cos\theta &= -K_2\sin\theta.\end{aligned}$$

We thus obtain the same value of θ as on the earlier assumption: that is, the two methods of estimating the inclination of two flats lead to the same result.

Further, the last equations give

$$\left.\begin{aligned} l_1\sin\theta &= L_1\cos\theta - L_2\\ m_1\sin\theta &= M_1\cos\theta - M_2\\ n_1\sin\theta &= N_1\cos\theta - N_2\\ k_1\sin\theta &= K_1\cos\theta - K_2\end{aligned}\right\},\qquad \left.\begin{aligned} l_2\sin\theta &= L_1 - L_2\cos\theta\\ m_2\sin\theta &= M_1 - M_2\cos\theta\\ n_2\sin\theta &= N_1 - N_2\cos\theta\\ k_2\sin\theta &= K_1 - K_2\cos\theta\end{aligned}\right\}.$$

The first of these sets shews that the line l_1, m_1, n_1, k_1, lying in the first flat, also lies in the plane

$$\left\Vert\begin{matrix} x-a, & y-b, & z-c, & v-d\\ L_1, & M_1, & N_1, & K_1\\ L_2, & M_2, & N_2, & K_2\end{matrix}\right\Vert = 0\cdot$$

and the second of these sets shews that the line l_2, m_2, n_2, k_2, lying in the second flat, lies in the same plane. Hence *the two lines*, by reference to which the inclination of the two flats is measured, *are intersections of the respective flats by a plane through the directions of the two normals to the flats.*

66. Inferences can be deduced, as to parallel flats and perpendicular flats.

(i) When the two flats are *parallel*, their inclination is zero, as estimated by the relation between their normals. Conversely, when the inclination is zero, we are led to the known criteria for parallelism. When $\theta = 0$, we have

$$L_1L_2 + M_1M_2 + N_1N_2 + K_1K_2 = 1,$$

and therefore, as

$$L_1^2+M_1^2+N_1^2+K_1^2=1, \quad L_2^2+M_2^2+N_2^2+K_2^2=1,$$

we have

$$(\Sigma L_1^2)(\Sigma L_2^2)-(\Sigma L_1L_2)^2=0.$$

Thus

$$(L_1M_2-M_1L_2)^2+(M_1N_2-N_1M_2)^2+(N_1L_2-L_1N_2)^2$$
$$+(L_1K_2-K_1L_2)^2+(M_1K_2-K_1M_2)^2+(N_1K_2-K_1N_2)^2=0.$$

All the quantities L, M, N, K, are real: so that the last equation can hold only if each of the six squares vanishes, i.e., if

$$\frac{L_2}{L_1}=\frac{M_2}{M_1}=\frac{N_2}{N_1}=\frac{K_2}{K_1},$$

being the analytical conditions sufficient to secure that the two flats $\Sigma L_1x=P_1$, $\Sigma L_2x=P_2$, are parallel.

(ii) When the two flats are *perpendicular*, we have $\theta=\frac{1}{2}\pi$; then

$$L_1L_2+M_1M_2+N_1N_2+K_1K_2=0.$$

The condition that a direction λ, μ, ν, κ, lies in the flat $\Sigma L_1x=P_1$ is

$$L_1\lambda+M_1\mu+N_1\nu+K_1\kappa=0;$$

hence the normal to the second flat lies in the first. Similarly, the normal to the first flat lies in the second. Hence, when two flats are perpendicular to one another, each contains the direction of the normal to the other.

Hence also when two flats are perpendicular, a line, drawn through any point in one of them perpendicular to the other, lies in the first flat.

Ex 1. Prove that any flat, through a normal to a flat, is perpendicular to the flat.

Ex. 2. *Three flats are, each of them, perpendicular to a fourth flat; prove that the line, which the three flats determine, is perpendicular to the fourth flat.*

Let the flats be

$$\Sigma L_1x=P_1, \quad \Sigma L_2x=P_2, \quad \Sigma L_3x=P_3,$$

and let the fourth flat be

$$\Sigma Lx=P,$$

then, as the last is perpendicular to each of the other three,

$$\Sigma L_1L=0, \quad \Sigma L_2L=0, \quad \Sigma L_3L=0.$$

Let the line of intersection of the three flats be

$$\frac{x-\alpha}{\lambda}=\frac{y-\beta}{\mu}=\frac{z-\gamma}{\nu}=\frac{u-\delta}{\kappa},$$

as this line lies in each of the flats, we have

$$\Sigma L_1\lambda=0, \quad \Sigma L_2\lambda=0, \quad \Sigma L_3\lambda=0.$$

Consequently

$$\frac{\lambda}{L}=\frac{\mu}{M}=\frac{\nu}{N}=\frac{\kappa}{K},$$

that is, the line common to the three flats coincides, in direction, with a normal to the fourth flat.

Ex. 3. *Given two perpendicular flats, prove that a direction, which lies in one flat and is at right angles to every direction in their plane of cleavage, is perpendicular to the other flat.*

Let the flats be given, as to directions, by the equations

$$L_1x+M_1y+N_1z+K_1v=0, \quad L_2x+M_2y+N_2z+K_2v=0.$$

If their plane of cleavage be represented, as to directions, by the equations

$$\left\| \begin{matrix} x, & y, & z, & v \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{matrix} \right\| =0,$$

each of the directions l_1, m_1, n_1, k_1, and l_2, m_2, n_2, k_2, lies in both flats; hence

$$L_1l_1+M_1m_1+N_1n_1+K_1k_1=0, \quad L_2l_1+M_2m_1+N_2n_1+K_2k_1=0,$$

$$L_1l_2+M_1m_2+N_1n_2+K_1k_2=0, \quad L_2l_2+M_2m_2+N_2n_2+K_2k_2=0.$$

Let $\lambda, \mu, \nu, \kappa$, be the direction-cosines of a line in the first flat, and perpendicular to the two guiding directions (and therefore to every direction) in the plane of cleavage. Then

$$\left.\begin{aligned} L_1\lambda+M_1\mu+N_1\nu+K_1\kappa&=0 \\ l_1\lambda+m_1\mu+n_1\nu+k_1\kappa&=0 \\ l_2\lambda+m_2\mu+n_2\nu+k_2\kappa&=0 \end{aligned}\right\}.$$

But we have

$$\left.\begin{aligned} L_1L_2+M_1M_2+N_1N_2+K_1K_2&=0 \\ l_1L_2+m_1M_2+n_1N_2+k_1K_2&=0 \\ l_2L_2+m_2M_2+n_2N_2+k_2K_2&=0 \end{aligned}\right\};$$

consequently, as in the preceding example,

$$\frac{\lambda}{L_2}=\frac{\mu}{M_2}=\frac{\nu}{N_2}=\frac{\kappa}{K_2},$$

so that the direction coincides with the normal to the second flat.

Ex. 4. Two flats are perpendicular to a third flat; prove that the normal to the third flat lies in the plane of cleavage of the two flats.

Hence shew that, when three flats are perpendicular in pairs, the normal to any one of them lies in the plane of cleavage of the other two.

CHAPTER V.

INCLINATIONS OF LINES AND FLATS TO PLANES: PROJECTIONS.

Orthogonal planes: perpendicular planes.

67. The orientation of a line in quadruple space is settled simply, because of its determination by the uniqueness of its direction. The orientation of a flat can be regarded as settled, no less simply, because of its determination by the uniqueness of the (common) direction of the normals to the flat.

But the orientation of a plane in quadruple space (or in n-fold space, where $n > 3$) cannot be settled with the same simplicity. There is the initial difficulty that, within the plane, no line exists with any characteristic uniqueness; there is the further difficulty that, at any point in the plane, there is no direction, perpendicular to the plane, with any intrinsic characteristic uniqueness. Indeed, it will appear that the orientation of a plane is made most definite, not by the use of linear directions as for a line and for a flat, but by the use of the coordinate planes of reference.

Meanwhile, one property of directions, through a point in the plane and having an orthogonal bearing towards the plane, may be noted. Let the plane be

$$\left\| \begin{array}{cccc} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{array} \right\| = 0.$$

A line through the point a, b, c, d, is given by the equations

$$\frac{x-a}{\lambda} = \frac{y-b}{\mu} = \frac{z-c}{\nu} = \frac{v-d}{\kappa};$$

it is perpendicular to the guiding lines of the plane if

$$\lambda l_1 + \mu m_1 + \nu n_1 + \kappa k_1 = 0,$$

$$\lambda l_2 + \mu m_2 + \nu n_2 + \kappa k_2 = 0.$$

Then,

$$\lambda (\alpha l_1 + \beta l_2) + \mu (\alpha m_1 + \beta m_2) + \nu (\alpha n_1 + \beta n_2) + \kappa (\alpha k_1 + \beta k_2) = 0,$$

for all values of α and β: that is, the line is perpendicular to every direction in the plane. But the two relations $\Sigma\lambda l_1 = 0$, $\Sigma\lambda l_2 = 0$, together with the permanent relation $\Sigma\lambda^2 = 1$, are satisfied by an unlimited range of values of $\lambda, \mu, \nu, \kappa$: that is, there is an unlimited number of lines through a, b, c, d, perpendicular to every direction in the plane. Manifestly, any point on such a line through a, b, c, d, satisfies both the equations

$$\left.\begin{array}{l} l_1(x-a)+m_1(y-b)+n_1(z-c)+k_1(v-d)=0 \\ l_2(x-a)+m_2(y-b)+n_2(z-c)+k_2(v-d)=0 \end{array}\right\},$$

which constitute another plane through the point. Thus all the perpendicular lines in question lie in a second plane. Moreover, for any direction L, M, N, K, in this new plane, we have

$$l_1L+m_1M+n_1N+k_1K=0,$$

$$l_2L+m_2M+n_2N+k_2K=0,$$

and therefore, as before

$$(\alpha l_1+\beta l_2)L+(\alpha m_1+\beta m_2)M+(\alpha n_1+\beta n_2)N+(\alpha k_1+\beta k_2)K=0;$$

that is, the direction L, M, N, K, being any whatever in the new plane, is perpendicular to the direction $\alpha l_1+\beta l_2$, $\alpha m_1+\beta m_2$, $\alpha n_1+\beta n_2$, $\alpha k_1+\beta k_2$, being any whatever in the original plane.

When two planes possess the property, that the direction of every line in either is at right angles to the direction of every line in the other, the two planes are said to be *orthogonal* (sometimes *completely orthogonal*): and thus all the directions at any point in a plane, which are perpendicular to the plane, lie in the completely orthogonal plane through that point. But though the new plane is uniquely definite, it does not intrinsically and without a further imposed external condition provide any unique characteristic line perpendicular to the original plane.

Further, it is to be noted that this relation, in the quality of orthogonality, is not the only kind of relation in which two planes can stand towards one another if they are to be considered perpendicular. Thus, in the figure on p. 7, the plane ZOV (that is, the plane $x=0$, $y=0$) is completely orthogonal to the plane XOY (that is, the plane $z=0$, $v=0$): every line in either plane is perpendicular to every line in the other. But, in the customary solid geometry, we are accustomed to regard two planes, such as XOY and XOZ in that figure, as perpendicular to one another: the two planes exist in the common flat $OXYZ$ (that is, the ordinary three-dimensional space given by $v=0$); and lines drawn in the respective planes at any point in OX, perpendicular to OX in the respective planes, are perpendicular to one another. On this property, the two planes are described as perpendicular. Yet not every direction in XOY is perpendicular to every direction in XOZ: indeed, in XOY a direction, parallel to OX, and in XOZ, a direction also parallel to OX, are parallel to one another.

Thus it is desirable to discriminate between the two kinds of rectangular relation between planes. When two planes are such, that every direction in either is perpendicular to every direction in the other, they will be called *orthogonal* to one another. When two planes are such that it is possible to select a unique direction in either plane perpendicular to every direction in the other plane, the two planes will be called *perpendicular* to one another. The same discrimination, between orthogonal planes and perpendicular planes, will appear from the analytical tests.

In passing, it may be remarked that a corresponding discrimination in the kinds of perpendicularity of flats is necessary, when these spread through homaloidal space of n dimensions, for $n > 4$. Owing to the uniqueness of the normal when $n = 4$, such discrimination is unnecessary in quadruple space.

68. The relation of perpendicularity between a line and a plane can be exhibited in a slightly different form.

Let the plane be given by the equations

$$L_1(x-a) + M_1(y-b) + N_1(z-c) + K_1(v-d) = 0,$$

$$L_2(x-a) + M_2(y-b) + N_2(z-c) + K_2(v-d) = 0;$$

and let a direction through a, b, c, d, be taken, parallel to the given line, so that the equations of a new line thus drawn are

$$\frac{x-a}{\lambda} = \frac{y-b}{\mu} = \frac{z-c}{\nu} = \frac{v-d}{\kappa}.$$

Let l, m, n, k, be a direction in the plane: then

$$L_1 l + M_1 m + N_1 n + K_1 k = 0,$$

$$L_2 l + M_2 m + N_2 n + K_2 k = 0.$$

When this direction is perpendicular to the given external direction (and therefore to this line), the relation

$$\lambda l + \mu m + \nu n + \kappa k = 0$$

is satisfied. Thus there are three homogeneous equations in l, m, n, k. Usually, they suffice to determine the ratios $l : m : n : k$; that is, usually it is possible to determine a single direction in a plane perpendicular to an assigned external direction.

But it may happen that the ratios $l : m : n : k$ are not thus determinate. The direction l, m, n, k, in the plane must satisfy the two equations $\Sigma L_1 l = 0$ and $\Sigma L_2 l = 0$: but, now, the third equation $\Sigma \lambda l = 0$, though satisfied, does not provide any further datum for the determination of l, m, n, k. The direction $\lambda, \mu, \nu, \kappa$, is perpendicular to every direction l, m, n, k, satisfying the two conditions $\Sigma L_1 l = 0$, $\Sigma L_2 l = 0$, that is, to every direction in the plane. The direction $\lambda, \mu, \nu, \kappa$, is then perpendicular to the plane. The necessary and sufficient conditions are

$$\left\| \begin{array}{cccc} \lambda, & \mu, & \nu, & \kappa \\ L_1, & M_1, & N_1, & K_1 \\ L_2, & M_2, & N_2, & K_2 \end{array} \right\| = 0.$$

Consequently the line in a direction $\lambda, \mu, \nu, \kappa$, through any point a', b', c', d', lies in the plane

$$\begin{Vmatrix} x-a', & y-b', & z-c', & v-d' \\ L_1, & M_1, & N_1, & K_1 \\ L_2, & M_2, & N_2, & K_2 \end{Vmatrix} = 0;$$

and any direction in this plane is perpendicular to every direction in the given plane. Thus the latter plane is *orthogonal* to the first plane: and the relation is reciprocal.

Ex. 1. *When two planes intersect in a line, the line perpendicular to both of them at any point along their intersection is normal to the flat in which both the planes lie.*

Let the origin be taken at any point on the line of intersection of the planes: and let the equation of the flat, in which both the planes lie, be

$$Lx + My + Nz + Kv = 0.$$

Then the equations of the two planes can be taken to be

$$\left.\begin{array}{l} Lx + My + Nz + Kv = 0 \\ L_1x + M_1y + N_1z + K_1v = 0 \end{array}\right\}, \quad \left.\begin{array}{l} Lx + My + Nz + Kv = 0 \\ L_2x + M_2y + N_2z + K_2v = 0 \end{array}\right\},$$

respectively. The plane through the origin orthogonal to the first is

$$\begin{Vmatrix} x, & y, & z, & v \\ L, & M, & N, & K \\ L_1, & M_1, & N_1, & K_1 \end{Vmatrix} = 0;$$

and every line in this plane is perpendicular to the first plane. The plane through the origin orthogonal to the second is

$$\begin{Vmatrix} x, & y, & z, & v \\ L, & M, & N, & K \\ L_2, & M_2, & N_2, & K_2 \end{Vmatrix} = 0;$$

and every line in this plane is perpendicular to the second plane. The two new planes intersect in the line through the origin

$$\frac{x}{L} = \frac{y}{M} = \frac{z}{N} = \frac{v}{K},$$

so that this line is perpendicular to both of the original planes. It manifestly is normal, at the origin, to the flat

$$Lx + My + Nz + Kv = 0$$

in which both the planes lie. Hence the proposition.

Ex. 2. *Planes, orthogonal to a given plane, and drawn through different points along a given line, are parallel to one another and lie in one flat.*

Let the given plane be

$$\left.\begin{array}{l} L_1x + M_1y + N_1z + K_1v = P_1 \\ L_2x + M_2y + N_2z + K_2v = P_2 \end{array}\right\},$$

and let the given line be

$$\frac{x-a}{\lambda} = \frac{y-b}{\mu} = \frac{z-c}{\nu} = \frac{v-d}{\kappa}.$$

A plane, through any point $a+\lambda r$, $b+\mu r$, $c+\nu r$, $d+\kappa r$, on the line, and orthogonal to the given plane, is given by the equations

$$\left\Vert \begin{matrix} x-a-\lambda r, & y-b-\mu r, & z-c-\nu r, & v-d-\kappa r \\ L_1\ \ , & M_1\ \ , & N_1\ \ , & K_1 \\ L_2\ \ , & M_2\ \ , & N_2\ \ , & K_2 \end{matrix} \right\Vert = 0.$$

For different values of r, all these planes are parallel to one another. Any plane, determined by a particular value of r, lies in the flat

$$\left| \begin{matrix} x-a, & y-b, & z-c, & v-d \\ \lambda\ , & \mu\ , & \nu\ , & \kappa \\ L_1\ , & M_1\ , & N_1\ , & K_1 \\ L_2\ , & M_2\ , & N_2\ , & K_2 \end{matrix} \right| = 0.$$

This equation is independent of r. and therefore the family of parallel planes, for all the values of r, lies in this flat.

Inclination of a line to a plane.

69. Among matters relating to orientation of planes, we begin with the inclination of a line to a plane.

Let the plane be

$$\left\Vert \begin{matrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{matrix} \right\Vert = 0$$

In order to deal solely with inclinations, we take, through the point a, b, c, d, in the plane, a direction $\lambda, \mu, \nu, \kappa$, parallel to the direction of the line: thus, without loss of generality as regards inclination, we take the line to be

$$\frac{x-a}{\lambda} = \frac{y-b}{\mu} = \frac{z-c}{\nu} = \frac{v-d}{\kappa}.$$

Any direction in the plane is given by

$$\lambda' = pl_1 + ql_2, \quad \mu' = pm_1 + qm_2, \quad \nu' = pn_1 + qn_2, \quad \kappa' = pk_1 + qk_2,$$

where

$$p^2 + q^2 + 2pq\cos\omega = 1,$$

ω denoting the angle between the selected guiding lines of the plane. Let ϕ denote the angle between this direction and the given line, so that

$$\cos\phi = \lambda\lambda' + \mu\mu' + \nu\nu' + \kappa\kappa'.$$

It is always possible to select one plane-direction $\lambda', \mu', \nu', \kappa'$, which is perpendicular to the given line; for, writing

$$\lambda l_1 + \mu m_1 + \nu n_1 + \kappa k_1 = \cos\alpha', \quad \lambda l_2 + \mu m_2 + \nu n_2 + \kappa k_2 = \cos\beta',$$

the necessary values of p and q are given by

$$p\cos\alpha' + q\cos\beta' = 0,\quad p^2 + q^2 + 2pq\cos\omega = 1.$$

For all other directions in the plane, $\cos^2\phi$ will usually be different from zero —the exceptions arise, of course, when $\alpha' = \frac{1}{2}\pi$ and $\beta' = \frac{1}{2}\pi$.

Now $\cos^2\phi$ cannot be greater than unity. Also, $\cos^2\phi$ cannot be equal to unity; for then we should be obliged to have

$$pl_1 + ql_2 = \lambda' = \lambda,\quad pm_1 + qm_2 = \mu' = \mu,\quad pn_1 + qn_2 = \nu' = \nu,\quad pk_1 + qk_2 = \kappa' = \kappa,$$

where λ, μ, ν, κ, are given quantities, and values p and q cannot usually be obtained to satisfy the four conditions. Thus $\cos^2\phi$, an essentially positive quantity, can be zero and cannot be unity; it therefore must have a maximum value, giving a minimum (stationary) value to ϕ.

When $\cos^2\phi$ is a maximum, we shall denote the associated value of ϕ by θ, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$; and we shall call this angle θ the inclination of the line to the plane. We thus have to make $\cos^2\phi$ a maximum, for admissible values of λ', μ', ν', κ'; that is, for admissible values of p and q, these quantities being subject to the relation

$$p^2 + q^2 + 2pq\cos\omega = 1.$$

Now, generally,

$$\cos\phi = \Sigma\lambda\lambda' = p\cos\alpha' + q\cos\beta'.$$

Thus the critical equations are

$$\cos\alpha' = t\,(p + q\cos\omega),\quad \cos\beta' = t\,(p\cos\omega + q),$$

where t initially is an indeterminate multiplier: and now the particular value of ϕ is θ. Multiplying the critical equations by p and q respectively, and adding, we have

$$\cos\theta = t\,(p^2 + 2pq\cos\omega + q^2) = t;$$

and now the critical equations are

$$\left.\begin{aligned} \frac{\cos\alpha'}{\cos\theta} &= p \qquad + q\cos\omega \\ \frac{\cos\beta'}{\cos\theta} &= p\cos\omega + q \end{aligned}\right\}.$$

Also

$$\cos\theta = p\cos\alpha' + q\cos\beta';$$

consequently

$$\begin{vmatrix} \dfrac{\cos\alpha'}{\cos\theta}, & 1, & \cos\omega \\ \dfrac{\cos\beta'}{\cos\theta}, & \cos\omega, & 1 \end{vmatrix} = 0;$$

therefore

$$\sin^2\omega\cos^2\theta = \cos^2\alpha' - 2\cos\alpha'\cos\beta'\cos\omega + \cos^2\beta',$$

and also

$$\sin^2\omega\sin^2\theta = 1 - \cos^2\alpha' - \cos^2\beta' - \cos^2\omega + 2\cos\alpha'\cos\beta'\cos\omega.$$

Either of these equations determines the inclination of the line to the plane.

For the fuller interpretation of this result, we compare it with the investigation (§ 32) of the perpendicular from an external point on a plane. In Fig. 2 (§ 32, p. 49), let P be any point in the line through O, with the direction λ, μ, ν, κ; let OA, OB, be the guiding lines of the plane: PA, PB, the perpendiculars from P on these guiding lines: and ON the diameter, through O, of the circle AOB drawn in the plane. Then PN is the perpendicular from P on the plane.

Let $OP = D$ · then

$$AOP = \alpha', \quad OA = D_1 = D\cos\alpha', \quad BOP = \beta', \quad OB = D_2 = D\cos\beta'.$$

Also let $NOA = \alpha$, $NOB = \beta$, $AOB = \omega$; thus

$$\cos\alpha = \Sigma l_1\lambda' = p + q\cos\omega,$$

$$\cos\beta = \Sigma l_2\lambda' = p\cos\omega + q.$$

Hence

$$\cos\alpha' = \cos\alpha\cos\theta, \quad \cos\beta' = \cos\beta\cos\theta;$$

and

$$ON = \frac{OA}{\cos\alpha} = \frac{D_1}{\cos\alpha} = \frac{D\cos\alpha'}{\cos\alpha} = D\cos\theta.$$

In the flat $AOPB$, a three-dimensional homaloidal (ordinary) space, the plane PBN is perpendicular to the line OB, because PBO and NBO are right angles; and, similarly, the plane PAN is perpendicular to the line OA Hence the line PN, the intersection of the planes NAP, NBP, lying in the one flat $AOPB$, is perpendicular to the plane AOB. Consequently, the angle NOP is equal to θ.

In order to find the inclination of the line OP to the plane AOB, we use the flat $AOPB$. In that three-dimensional homaloidal space, construct a sphere on OP as diameter: let the section of this sphere by the plane AOB be the circle AOB, and let ON be the diameter of this circle through O. Then NOP is the inclination of the line OP to the plane AOB.

The analytical expression for $\cos^2\theta$ is

$$\frac{1}{D^2\sin^2\omega}(D_1^2 - 2D_1D_2\cos\omega + D_2^2);$$

the quantity D, $= OP$, is independent of the plane: and the rest of the expression is (§ 34) invariantive for all selections of guiding lines for the determination of the plane. Hence $\cos^2\theta$ also is an invariant, as it should be an invariant, for all changes in the choice of guiding lines of the plane.

Associated with every external direction OP through O, there lies within the plane a line of reference ON such that the angle NOP is the inclination of the line to the plane. Its direction-cosines are

$$\lambda' = l_1 p + l_2 q = \frac{1}{\sin^2\omega\cos\theta}\{l_1 (\cos\alpha' - \cos\beta'\cos\omega) + l_2 (\cos\beta' - \cos\alpha'\cos\omega)\},$$

$$\mu' = m_1 p + m_2 q = \frac{1}{\sin^2\omega\cos\theta}\{m_1 (\cos\alpha' - \cos\beta'\cos\omega) + m_2 (\cos\beta' - \cos\alpha'\cos\omega)\},$$

$$\nu' = n_1 p + n_2 q = \frac{1}{\sin^2\omega\cos\theta}\{n_1 (\cos\alpha' - \cos\beta'\cos\omega) + n_2 (\cos\beta' - \cos\alpha'\cos\omega)\},$$

$$\kappa' = k_1 p + k_2 q = \frac{1}{\sin^2\omega\cos\theta}\{k_1 (\cos\alpha' - \cos\beta'\cos\omega) + k_2 (\cos\beta' - \cos\alpha'\cos\omega)\}.$$

Further, we have $ON = D\cos\theta$; and so, for the x-coordinate of N, we have

$$X - a = \lambda' D\cos\theta = \frac{D}{\sin^2\omega}\{(l_1 - l_2\cos\omega)\cos\alpha' + (l_2 - l_1\cos\omega)\cos\beta'\}.$$

Also, if ξ, η, ζ, ν, are the coordinates of P, then $\xi - a = \lambda D$; so that, if l, m, n, k, are the direction-cosines of PN,

$$l \,.\, D\sin\theta = \xi - X = \xi - a - (X - a),$$

and therefore

$$l\sin\theta = \lambda - \frac{1}{\sin^2\omega}\{(l_1 - l_2\cos\omega)\cos\alpha' - (l_2 - l_1\cos\omega)\cos\beta'\}.$$

Now

$$\cos\alpha' = \Sigma\lambda l_1, \quad \cos\beta' = \Sigma\lambda l_2;$$

hence, if we write

$$e_{rs} = r_1 s_1 + r_2 s_2 - (r_1 s_2 + r_2 s_1)\cos\omega,$$

for $r = l, m, n, k$, and $s = l, m, n, k$, where r and s may be the same, we have

$$\left.\begin{aligned}
-l\sin\theta\sin^2\omega &= \lambda(e_{ll} - \sin^2\omega) + \mu e_{lm} + \nu e_{ln} + \kappa e_{lk} \\
-m\sin\theta\sin^2\omega &= \lambda e_{lm} + \mu(e_{mm} - \sin^2\omega) + \nu e_{mn} + \kappa e_{mk} \\
-n\sin\theta\sin^2\omega &= \lambda e_{ln} + \mu e_{mn} + \nu(e_{nn} - \sin^2\omega) + \kappa e_{nk} \\
-k\sin\theta\sin^2\omega &= \lambda e_{lk} + \mu e_{mk} + \nu e_{nk} + \kappa(e_{kk} - \sin^2\omega)
\end{aligned}\right\}$$

which give the direction-cosines of PN.

Projection of a line on a plane.

70. Manifestly, ON is the projection of OP upon the plane AOB: and it helps to provide an estimate of the inclination of OP to the plane. The projection is definite, save for the instances when the line OP is at right angles to OA and OB: as already indicated, the exceptional lines OP then lie in a plane orthogonal to the plane AOB.

But the line OP determines a flat when associated with the plane AOB. The equation of the flat $POAB$ clearly is

$$\begin{vmatrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \\ \lambda\ , & \mu\ , & \nu\ , & \kappa \end{vmatrix} = 0.$$

One plane, containing the line PN, is

$$\left.\begin{aligned}(x-\xi)\,l_1+(y-\eta)\,m_1+(z-\zeta)\,n_1+(v-\upsilon)\,k_1=0\\(x-\xi)\,l_2+(y-\eta)\,m_2+(z-\zeta)\,n_2+(v-\upsilon)\,k_2=0\end{aligned}\right\}.$$

The intersection of this flat and this plane is the line PN. For any point in the flat is

$$\begin{aligned}x&=a+\ l_1\rho+\ l_2\sigma+\lambda\tau,\\y&=b+m_1\rho+m_2\sigma+\mu\tau,\\z&=c+\ n_1\rho+\ n_2\sigma+\nu\tau,\\v&=d+\ k_1\rho+\ k_2\sigma+\kappa\tau.\end{aligned}$$

and if this point lies in the new plane, we have

$$\begin{aligned}\Sigma l_1(a-\xi)+\rho+\sigma\cos\omega+\tau\cos\alpha'&=0,\\\Sigma l_2(a-\xi)+\rho\cos\omega+\sigma+\tau\cos\beta'&=0.\end{aligned}$$

Hence

$$\begin{aligned}\rho+\sigma\cos\omega&=D_1-\tau\cos\alpha'=(D-\tau)\cos\alpha',\\\rho\cos\omega+\sigma&=D_2-\tau\cos\beta'=(D-\tau)\cos\beta';\end{aligned}$$

and therefore

$$\begin{aligned}\rho\sin^2\omega&=(D-\tau)(\cos\alpha'-\cos\beta'\cos\omega),\\\sigma\sin^2\omega&=(D-\tau)(\cos\beta'-\cos\alpha'\cos\omega).\end{aligned}$$

Hence for the point x, y, z, v, common to the flat and the new plane,

$$x-\xi=a-\xi+\frac{D-\tau}{\sin^2\omega}\{l_1(\cos\alpha'-\cos\beta'\cos\omega)+l_2(\cos\beta'-\cos\alpha'\cos\omega)\}+\lambda\tau.$$

But $\xi-a=\lambda D$, and therefore

$$\begin{aligned}x-\xi&=(D-\tau)\left[-\lambda+\frac{1}{\sin^2\omega}\{l_1(\cos\alpha'-\cos\beta'\cos\omega)+l_2(\cos\beta'-\cos\alpha'\cos\omega)\}\right]\\&=-(D-\tau)\,l\sin\theta.\end{aligned}$$

Similarly for $y-\eta$, $z-\zeta$, $v-\upsilon$. Hence *the intersection of the flat $POAB$ and the plane* $\Sigma l_1(x-\xi)=0$, $\Sigma l_2(x-\xi)=0$, *is the line*

$$\frac{x-\xi}{l}=\frac{y-\eta}{m}=\frac{z-\zeta}{n}=\frac{v-\upsilon}{k}:$$

that is to say, it is the line PN.

It is an immediate corollary that the point N, the foot of the perpendicular upon the original plane from the point P in the line OP, is the single point of intersection of the orthogonal planes

$$\begin{Vmatrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{Vmatrix} = 0,$$

and

$$\left.\begin{aligned} l_1(x-\xi)+m_1(y-\eta)+n_1(z-\zeta)+k_1(v-\upsilon)=0 \\ l_2(x-\xi)+m_2(y-\eta)+n_2(z-\zeta)+k_2(v-\upsilon)=0 \end{aligned}\right\}.$$

The result of the main investigation, as regards the inclination, is:

The inclination θ *of a line, with direction-cosines* λ, μ, ν, κ, *to a plane with guiding lines having direction-cosines* l_1, m_1, n_1, k_1, *and* l_2, m_2, n_2, k_2, *is given by the relations*

$$\sin^2\omega\cos^2\theta = \cos^2\alpha' + \cos^2\beta' - 2\cos\alpha'\cos\beta'\cos\omega,$$

$$\sin^2\omega\sin^2\theta = 1 - \cos^2\omega - \cos^2\alpha' - \cos^2\beta' + 2\cos\omega\cos\alpha'\cos\beta',$$

where $\cos\omega = \Sigma l_1 l_2$, $\cos\alpha' = \Sigma\lambda l_1$, $\cos\beta' = \Sigma\lambda l_2$.

These formulæ are manifestly the formulæ of spherical trigonometry, giving the distance of the vertex of a triangle from the opposite side.

Line perpendicular to a plane: line parallel to a plane.

71. Two special cases of this general result in § 70 are to be noted.

(i) When the inclination θ of the line to the plane is equal to $\frac{1}{2}\pi$, so that the line is perpendicular to the plane, we have

$$\cos\alpha' = \cos\alpha\cos\theta = 0, \quad \cos\beta' = \cos\beta\cos\theta = 0,$$

so that

$$\lambda l_1 + \mu m_1 + \nu n_1 + \kappa k_1 = 0,$$

$$\lambda l_2 + \mu m_2 + \nu n_2 + \kappa k_2 = 0:$$

the line λ, μ, ν, κ, is perpendicular to every direction in the plane.

(ii) When the inclination θ of the line to the plane vanishes, so that the direction λ, μ, ν, κ, coincides with the direction λ', μ', ν', κ', in the plane, it follows that the line OP, which has been drawn through a point O in the plane parallel to the postulated line, lies entirely in the plane. Then the given line is parallel to the plane: and, as

$$\lambda = \lambda' = pl_1 + ql_2, \quad \mu = \mu' = pm_1 + qm_2, \quad \nu = \nu' = pn_1 + qn_2, \quad \kappa = \kappa' = pk_1 + qk_2,$$

the conditions are

$$\begin{Vmatrix} \lambda, & \mu\ , & \nu\ , & \kappa \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{Vmatrix} = 0.$$

Ex. 1. When the plane is given as the intersection of two flats

$$L_1x+M_1y+N_1z+K_1v=P_1,$$
$$L_2x+M_2y+N_2z+K_2v=P_2,$$

where $\Sigma L_1^2=1$, $\Sigma L_2^2=1$, prove:

(i) that the inclination ϑ of a line with direction-cosines λ, μ, ν, κ, to the plane is given by

$$\sin^2\eta\sin^2\vartheta=\cos^2\alpha+\cos^2\beta-2\cos\alpha\cos\beta\cos\eta,$$

where $\cos\eta=\Sigma L_1L_2$, $\cos\alpha=\Sigma L_1\lambda$, $\cos\beta=\Sigma L_2\lambda$. and

(ii) that the direction-cosines of the line of reference in the plane with which the given line makes the inclination ϑ are λ', μ', ν', κ', where

$$\left.\begin{aligned}(\lambda-\lambda'\cos\vartheta)\sin^2\eta&=L_1(\cos\alpha-\cos\beta\cos\eta)+L_2(\cos\beta-\cos\alpha\cos\eta)\\(\mu-\mu'\cos\vartheta)\sin^2\eta&=M_1(\cos\alpha-\cos\beta\cos\eta)+M_2(\cos\beta-\cos\alpha\cos\eta)\\(\nu-\nu'\cos\vartheta)\sin^2\eta&=N_1(\cos\alpha-\cos\beta\cos\eta)+N_2(\cos\beta-\cos\alpha\cos\eta)\\(\kappa-\kappa'\cos\vartheta)\sin^2\eta&=K_1(\cos\alpha-\cos\beta\cos\eta)+K_2(\cos\beta-\cos\alpha\cos\eta)\end{aligned}\right\}$$

Ex. 2 When a plane is given in the form

$$z=f+px+qy,\quad v=h+rx+sy,$$

the inclination ψ of the line

$$\frac{x-a}{\lambda}=\frac{y-b}{\mu}=\frac{z-c}{\nu}=\frac{v-d}{\kappa}$$

to the plane is given by

$$\Delta\cos^2\psi=AS^2-2BRS+CR^2,$$

where

$$R=\lambda+\nu p+\kappa r,\quad S=\mu+\nu q+\kappa s,$$
$$A=1+p^2+r^2,\quad B=pq+rs,\quad C=1+q^2+s^2,$$
$$\Delta=1+p^2+q^2+r^2+s^2+(ps-qr)^2=AC-B^2.$$

Prove that the line is perpendicular to the plane, if $\lambda+\nu p+\kappa r=0$, $\mu+\nu q+\kappa s=0$; and is parallel to the plane, if $\nu=p\lambda+q\mu$, $\kappa=r\lambda+s\mu$.

Equations of the projection of a line on a plane.

72. With the investigation of the inclination between a line and a plane, we associate the projection of the line upon the plane. The equations of the plane can be taken in the form

$$\left\|\begin{matrix}x-a, & y-b, & z-c, & v-d\\ l_1, & m_1, & n_1, & k_1\\ l_2, & m_2, & n_2, & k_2\end{matrix}\right\|=0.$$

The form of the result is affected by the relation between the line and the plane as regards meeting. If the line actually intersects the plane, we assume this point to be selected as the point a, b, c, d, in the equations of the plane: while, if the line does not meet the plane, we take its equations to be

$$\frac{x-\alpha}{\lambda}=\frac{y-\beta}{\mu}=\frac{z-\gamma}{\nu}=\frac{v-\delta}{\kappa},$$

where α, β, γ, δ, are not equal to a, b, c, d, respectively. In either event, the direction-cosines of the line are taken to be λ, μ, ν, κ.

Let ξ, η, ζ, v, be any point on the line, so that

$$\xi = \alpha + \lambda\rho, \quad \eta = \beta + \mu\rho, \quad \zeta = \gamma + \nu\rho, \quad v = \delta + \kappa\rho,$$

being the parametric distance along the line. If X, Y, Z, V, are the coordinates of the foot of the perpendicular from ξ, η, ζ, v, on the plane, then (§ 34) we have

$$X = a + \frac{1}{\sin^2\omega}\{(l_1 - l_2\cos\omega)D_1 + (l_2 - l_1\cos\omega)D_2\},$$

with like expressions for Y, Z, V, where

$$D_1 = \Sigma l_1(\xi - a) = \Sigma l_1(\alpha - a) + \rho\cos\alpha',$$

$$D_2 = \Sigma l_2(\xi - a) = \Sigma l_2(\alpha - a) + \rho\cos\beta',$$

while

$$\cos\alpha' = \Sigma l_1\lambda, \quad \cos\beta' = \Sigma l_2\lambda.$$

Accordingly, let there be quantities A, B, C, D, such that

$$\left.\begin{aligned}
(A - a)\sin^2\omega &= (l_1 - l_2\cos\omega)\Sigma l_1(\alpha - a) + (l_2 - l_1\cos\omega)\Sigma l_2(\alpha - a)\\
(B - b)\sin^2\omega &= (m_1 - m_2\cos\omega)\Sigma l_1(\alpha - a) + (m_2 - m_1\cos\omega)\Sigma l_2(\alpha - a)\\
(C - c)\sin^2\omega &= (n_1 - n_2\cos\omega)\Sigma l_1(\alpha - a) + (n_2 - n_1\cos\omega)\Sigma l_2(\alpha - a)\\
(D - d)\sin^2\omega &= (k_1 - k_2\cos\omega)\Sigma l_1(\alpha - a) + (k_2 - k_1\cos\omega)\Sigma l_2(\alpha - a)
\end{aligned}\right\}$$

Also, write

$$\left.\begin{aligned}
L &= (l_1 - l_2\cos\omega)\cos\alpha' + (l_2 - l_1\cos\omega)\cos\beta'\\
M &= (m_1 - m_2\cos\omega)\cos\alpha' + (m_2 - m_1\cos\omega)\cos\beta'\\
N &= (n_1 - n_2\cos\omega)\cos\alpha' + (n_2 - n_1\cos\omega)\cos\beta'\\
K &= (k_1 - k_2\cos\omega)\cos\alpha' + (k_2 - k_1\cos\omega)\cos\beta'
\end{aligned}\right\}$$

Then the foot of the perpendicular from the point $\alpha + \lambda\rho$, $\beta + \mu\rho$, $\gamma + \nu\rho$, $\delta + \kappa\rho$, upon the plane is given by

$$X = A + \frac{\rho}{\sin^2\omega}L, \quad Y = B + \frac{\rho}{\sin^2\omega}M, \quad Z = C + \frac{\rho}{\sin^2\omega}N, \quad V = D + \frac{\rho}{\sin^2\omega}K.$$

Hence the locus of X, Y, Z, V, that is, the projection of the line upon the plane, is given by the equations

$$\frac{X - A}{L} = \frac{Y - B}{M} = \frac{Z - C}{N} = \frac{V - D}{K}.$$

Let

$$\Omega = (\cos^2\alpha' + \cos^2\beta' - 2\cos\alpha'\cos\beta'\cos\omega)^{\frac{1}{2}}\sin\omega\,;$$

then the direction-cosines of the projection of the line are

$$\frac{L}{\Omega}, \quad \frac{M}{\Omega}, \quad \frac{N}{\Omega}, \quad \frac{K}{\Omega}.$$

As is to be expected, these direction-cosines are independent of the contingency of intersection by the line and the plane. Should the line and the plane meet, we can take $\alpha, \beta, \gamma, \delta, = a, b, c, d$; and equations of the projection can be taken

$$\frac{X-a}{L}=\frac{Y-b}{M}=\frac{Z-c}{N}=\frac{V-d}{K},$$

thus giving the projection of the line

$$\frac{x-a}{\lambda}=\frac{y-b}{\mu}=\frac{z-c}{\nu}=\frac{v-d}{\kappa}$$

upon the plane

$$\left\|\begin{matrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{matrix}\right\|=0.$$

The inclination of the given line to its projection is less than its inclination to any other line in the plane, a result that follows from the investigation (§ 69) of the inclination of the line to the plane.

Direction-cosines of the projection of the line.

73. The direction-cosines of the projection are proportional to L, M, N, K. Now

$$L=(l_1-l_2\cos\omega)\,\Sigma l_1\lambda+(l_2-l_1\cos\omega)\,\Sigma l_2\lambda.$$

But it is convenient to have these expressed linearly in terms of $\lambda, \mu, \nu, \kappa$; so with the definitions of § 69, viz.

$$e_{rs}=r_1s_1+r_2s_2-(r_1s_2+r_2s_1)\cos\omega,$$

we have

$$\left.\begin{aligned} L &= e_{ll}\lambda+e_{lm}\mu+e_{ln}\nu+e_{lk}\kappa \\ M &= e_{lm}\lambda+e_{mm}\mu+e_{mn}\nu+e_{mk}\kappa \\ N &= e_{ln}\lambda+e_{mn}\mu+e_{nn}\nu+e_{nk}\kappa \\ K &= e_{lk}\lambda+e_{mk}\mu+e_{nk}\nu+e_{kk}\kappa \end{aligned}\right\}.$$

The direction-cosines of the perpendicular $\xi, \eta, \zeta, \upsilon$, on the plane, denoted by l, m, n, k, were given in § 52; it is easy to verify that

$$Ll+Mm+Nn+Kk=0,$$

as is to be expected. Also

$$L^2+M^2+N^2+K^2=\Omega^2=(\cos^2\alpha'+\cos^2\beta'-2\cos\alpha'\cos\beta'\cos\omega)\sin^2\omega.$$

Finally, the length of the projection upon the plane of the segment ρ of the given line between $\xi, \eta, \zeta, \upsilon$, and a, b, c, d, should be equal to $\rho\cos\theta$. Now

$$\Omega=\cos\theta\sin^2\omega;$$

and therefore

$$X - A = \frac{\rho}{\sin^2\omega} L = \frac{L}{\Omega} \rho \cos\theta,$$

with

$$Y - B = \frac{M}{\Omega} \rho \cos\theta, \quad Z - C = \frac{N}{\Omega} \rho \cos\theta, \quad V - D = \frac{K}{\Omega} \rho \cos\theta.$$

The direction-cosines of the projection are L/Ω, M/Ω, N/Ω, K/Ω; thus the length of the projection of the segment is $\rho \cos\theta$.

Ex. Required the projection of the line upon the plane $v=0$, $z=0$.

Here the plane is

$$\left\|\begin{matrix} x, & y, & z, & v \\ 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \end{matrix}\right\| = 0,$$

so that $l_1, m_1, n_1, k_1, = 1, 0, 0, 0$; $l_2, m_2, n_2, k_2, = 0, 1, 0, 0$; $\omega = \frac{1}{2}\pi$; and $a, b, c, d, = 0, 0, 0, 0$. Thus

$$\cos\alpha' = \lambda, \quad \cos\beta' = \mu, \quad \cos\theta = (\lambda^2 + \mu^2)^{\frac{1}{2}};$$

$$L = \lambda, \quad M = \mu;$$

$$D_1 = \alpha, \quad D_2 = \beta;$$

and therefore the equation of the projected line in the plane $v=0$, $z=0$, is

$$\frac{X-\alpha}{\lambda} = \frac{Y-\beta}{\mu}.$$

The full equations of the projected line in quadruple space are

$$\frac{X-\alpha}{\lambda} = \frac{Y-\beta}{\mu}, \quad Z=0, \quad V=0;$$

and the direction-cosines of the projection are

$$\frac{\lambda}{(\lambda^2+\mu^2)^{\frac{1}{2}}}, \quad \frac{\mu}{(\lambda^2+\mu^2)^{\frac{1}{2}}}, \quad 0, \quad 0.$$

Similarly for the projection upon any other of the planes of reference in the quadruple space.

Theorem as to the projection of two lines into one another.

74. The different course of the analysis, when the equations of a plane in their canonical form are used, is illustrated by the establishment of the theorem that, *when two lines are given, it is possible to draw a plane through either of them, such that this line is the projection of the other line upon the plane**.

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* The corresponding property in three dimensions is as follows: Given two lines

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}, \quad \frac{x-\alpha}{\lambda} = \frac{y-\beta}{\mu} = \frac{z-\gamma}{\nu},$$

the first is the projection of the second on the plane

$$(\Sigma l\lambda)\, \Sigma \{(x-a)(\alpha-a)\} = \{\Sigma l(\alpha-a)\}\{\Sigma\lambda(x-a)\},$$

and the second is the projection of the first on the plane

$$(\Sigma\lambda l)\, \Sigma \{(x-\alpha)(a-\alpha)\} = \{\Sigma\lambda(a-\alpha)\}\{\Sigma l(x-\alpha)\}.$$

We take the first line to be

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}=\frac{v-d}{k}=R,$$

and the second line to be

$$\frac{x-\alpha}{\lambda}=\frac{y-\beta}{\mu}=\frac{z-\gamma}{\nu}=\frac{v-\delta}{\kappa}=S.$$

A plane through the first is given by the equations

$$\left.\begin{aligned} z-c &= p(x-a)+q(y-b) \\ v-d &= r(x-a)+s(y-b) \end{aligned}\right\},$$

provided p, q, r, s, satisfy the relations

$$n=pl+qm, \quad k=rl+sm.$$

Take a point ξ, η, ζ, v, on the second line; and let X, Y, Z, V, be the foot of the perpendicular from this point upon the assumed plane. Then the quantity

$$(\xi-X)^2+(\eta-Y)^2+(\zeta-Z)^2+(v-V)^2$$

must be a minimum, subject to the conditions

$$Z-c=p(X-a)+q(Y-b), \quad V-d=r(X-a)+s(Y-b):$$

that is,

$$(\xi-X)^2+(\eta-Y)^2+\{\zeta-c-p(X-a)-q(Y-b)\}^2+\{v-d-r(X-a)-s(Y-b)\}^2$$

must be a minimum for all admissible values of X and Y. The critical equations are

$$\xi-X+p(\zeta-Z)+r(v-V)=0,$$

$$\eta-Y+q(\zeta-Z)+s(v-V)=0.$$

By the demands of the theorem, X, Y, Z, V, is to lie on the first line, so that

$$X=a+lR, \quad Y=b+mR, \quad Z=c+nR, \quad V=d+kR;$$

then the two critical equations are

$$\xi-a+p(\zeta-c)+r(v-d)=R(l+pn+rk),$$

$$\eta-b+q(\zeta-c)+s(v-d)=R(m+qn+sk);$$

and therefore

$$\frac{\xi-a+p(\zeta-c)+r(v-d)}{\eta-b+q(\zeta-c)+s(v-d)}=\frac{l+pn+rk}{m+qn+sk}.$$

This condition allows the perpendicular from a particular point ξ, η, ζ, v, on the second line, drawn to the assumed plane, to lie upon the first line in that assumed plane. When the whole second line projects into the first line, this condition must be satisfied for all values of ξ, η, ζ, v, such that

$$\frac{\xi-\alpha}{\lambda}=\frac{\eta-\beta}{\mu}=\frac{\zeta-\gamma}{\nu}=\frac{v-\delta}{\kappa}=S,$$

for current parametric values of S. Consequently we have

$$\frac{\alpha - a + p(\gamma - c) + r(\delta - d)}{\beta - b + q(\gamma - c) + s(\delta - d)} = \frac{l + pn + rk}{m + qn + sk},$$

$$\frac{\lambda + p\nu + r\kappa}{\mu + q\nu + s\kappa} = \frac{l + pn + rk}{m + qn + sk},$$

being two equations involving p, q, r, s; taken with the earlier relations

$$n = pl + qm, \quad k = rl + sm,$$

they suffice potentially for the determination of p, q, r, s.

The second of the new conditions, being

$$(\lambda + p\nu + r\kappa)(m + qn + sk) = (l + pn + rk)(\mu + q\nu + s\kappa),$$

is

$$\lambda m - l\mu + p(\nu m - n\mu) + q(\lambda n - l\nu) + r(\kappa m - k\mu) + s(\lambda k - l\kappa) + (ps - qr)(\nu k - n\kappa) = 0.$$

Now, as

$$p = \frac{1}{l}(n - qm), \quad r = \frac{1}{l}(k - sm),$$

we have

$$ps - qr = \frac{1}{l}(sn - qk),$$

so that the condition becomes linear in q and s; and, if

$$\cos\theta = l\lambda + m\mu + n\nu + k\kappa,$$

its form becomes

$$m\cos\theta - \mu + q(n\cos\theta - \nu) + s(k\cos\theta - \nu) = 0.$$

The first condition arises when, in the second condition, we substitute $\alpha - a$, $\beta - b$, $\gamma - c$, $\delta - d$, for λ, μ, ν, κ, respectively; hence, if

$$D = l(\alpha - a) + m(\beta - b) + n(\gamma - c) + k(\delta - d),$$

the second condition becomes

$$mD - (\beta - b) + q\{nD - (\gamma - c)\} + s\{kD - (\delta - d)\} = 0.$$

Consequently, we have

$$\frac{q}{\begin{vmatrix} D, & \delta - d, & \beta - b \\ 1, & k, & m \\ \cos\theta, & \kappa, & \mu \end{vmatrix}} = \frac{s}{\begin{vmatrix} D, & \beta - b, & \gamma - c \\ 1, & m, & n \\ \cos\theta, & \mu, & \nu \end{vmatrix}} = \frac{1}{\begin{vmatrix} D, & \gamma - c, & \delta - d \\ 1, & n, & k \\ \cos\theta, & \nu, & \kappa \end{vmatrix}}.$$

Similarly, by using the relations

$$q = \frac{1}{m}(n - pl), \quad s = \frac{1}{m}(k - rl),$$

so that

$$ps - qr = \frac{1}{m}(pk - rn),$$

the two conditions have the form

$$l\cos\theta - \lambda + p(n\cos\theta - \nu) + r(k\cos\theta - \nu) = 0,$$

$$lD - (\alpha - a) + p\{nD - (\gamma - c)\} + r\{kD - (\delta - d)\} = 0;$$

and therefore we have

$$\frac{p}{\begin{vmatrix} D, & \delta - d, & \alpha - a \\ 1, & k, & l \\ \cos\theta, & \kappa, & \lambda \end{vmatrix}} = \frac{r}{\begin{vmatrix} D, & \alpha - a, & \gamma - c \\ 1, & l, & n \\ \cos\theta, & \lambda, & \nu \end{vmatrix}} = \frac{1}{\begin{vmatrix} D, & \gamma - c, & \delta - d \\ 1, & n, & k \\ \cos\theta, & \nu, & \kappa \end{vmatrix}}.$$

It is easy to infer that each of these fractions is equal to

$$\frac{ps - qr}{\begin{vmatrix} D, & \alpha - a, & \beta - b \\ 1, & l, & m \\ \cos\theta, & \lambda, & \mu \end{vmatrix}}.$$

Thus the values of p, q, r, s, are determined uniquely; and therefore a plane can be drawn through the first line, so that the second line projects upon the plane into that first line.

Similarly, a single plane can be drawn through the second line, so that upon it the first line is projected into that second line. The parameters of this plane can be derived, by symmetry, from the parameters of the former plane.

Inclination of a plane to a flat.

75. Before discussing the inclination of one plane to another plane, we discuss the inclination of a plane to a flat, chiefly because of the analogy between this question and that of the inclination of a line to a plane. The orientation of a flat is usually characterised by its association with one particular and unique direction—that of the normal to the flat.

Two extreme instances can be treated summarily.

In one instance, the plane might lie in the flat. In that event, the normal to the flat, being perpendicular to every direction in the flat, is perpendicular to every direction in a contained plane. If therefore the flat be

$$Lx + My + Nz + Kv = P,$$

and the plane be

$$\left\|\begin{matrix} x - a, & y - b, & z - c, & v - d \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{matrix}\right\| = 0,$$

the necessary and sufficient conditions, that the plane should lie in the flat, are

$$Ll_1 + Mm_1 + Nn_1 + Kk_1 = 0,$$

$$Ll_2 + Mm_2 + Nn_2 + Kk_2 = 0,$$

with the relation $La + Mb + Nc + Kd = P$, which does not affect properties of inclination.

In the other extreme instance, the plane contains the direction of the normal to the flat. When the flat and the plane are given by the respective preceding equations, and when the plane contains the direction normal to the flat, then (§ 38) the relations

$$\left\| \begin{matrix} L, & M, & N, & K \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{matrix} \right\| = 0$$

must be satisfied.

In the former instance, we could say that the plane lies in the flat or is parallel to the flat. In the latter instance, we could say that the plane is perpendicular to the flat: it contains one direction which is perpendicular to every line in the flat.

Excluding these extreme cases, we still take the preceding equations to represent the flat and the plane respectively. Any direction $\lambda, \mu, \nu, \kappa$, in the plane is such that

$$\lambda = pl_1 + ql_2, \quad \mu = pm_1 + qm_2, \quad \nu = pn_1 + qn_2, \quad \kappa = pk_1 + qk_2,$$

where the parameters p and q must satisfy the relation

$$p^2 + q^2 + 2pq \cos \theta_{12} = 1,$$

θ_{12} being such that $\cos \theta_{12} = \Sigma l_1 l_2$. Let l, m, n, k, be any direction in the flat, so that

$$Ll + Mm + Nn + Kk = 0,$$

together with the permanent relation

$$l^2 + m^2 + n^2 + k^2 = 1.$$

Now, as the plane is supposed not to lie in the flat, the plane and the flat intersect in a straight line. If then the direction $\lambda, \mu, \nu, \kappa$, in the plane and the direction l, m, n, k, in the flat be chosen parallel to the straight line of intersection, the value of $\cos^2 \psi$, where

$$\cos \psi = l\lambda + m\mu + n\nu + k\kappa,$$

could be unity. On the other hand, a direction l, m, n, k, could always be chosen in the flat perpendicular to the plane: for the necessary conditions are

$$ll_1 + mm_1 + nn_1 + kk_1 = 0,$$

$$ll_2 + mm_2 + nn_2 + kk_2 = 0,$$

together with

$$lL + mM + nN + kK = 0:$$

and these equations suffice to determine l, m, n, k, when (as is now supposed) the plane does not contain the normal to the flat. For such a direction, the value of $\cos^2\psi$ is zero.

We therefore seek a stationary value for $\cos^2\psi$, rejecting a unit value and a zero value if and when they arise in the analysis. The corresponding value of ψ is the inclination of the plane to the normal to the flat; and we can then take $\frac{1}{2}\pi - \psi$ as the angle between the plane and the flat. The critical equations, assigning a stationary value to $\cos^2\psi$, where

$$\cos\psi = \Sigma l\lambda = l(pl_1 + ql_2) + m(pm_1 + qm_2) + n(pn_1 + qn_2) + k(pk_1 + qk_2),$$

subject to the limiting relations

$$p^2 + q^2 + 2pq\cos\theta_{12} = 1,$$
$$Ll + Mm + Nn + Kk = 0,$$
$$l^2 + m^2 + n^2 + k^2 = 1,$$

are

$$\Sigma ll_1 = C(p + q\cos\theta_{12}), \quad \Sigma ll_2 = C(p\cos\theta_{12} + q),$$
$$pl_1 + ql_2 = AL + Bl,$$
$$pm_1 + qm_2 = AM + Bm,$$
$$pn_1 + qn_2 = AN + Bn,$$
$$pk_1 + qk_2 = AK + Bk,$$

arising respectively from variations of p, q, l, m, n, k: and, initially, A, B, C, are indeterminate multipliers.

Multiplying the first and the second of these critical equations by p and by q respectively, and adding, we have

$$C = \cos\psi.$$

Multiplying the remaining four by l, m, n, k, respectively, and adding, we find

$$B = \cos\psi.$$

Squaring those four, and adding, we have

$$1 = A^2 + B^2;$$

and therefore, without loss of generality, we may take

$$A = \sin\psi.$$

Thus the six critical equations become

$$\Sigma ll_1 = (p + q\cos\theta_{12})\cos\psi, \quad \Sigma ll_2 = (p\cos\theta_{12} + q)\cos\psi,$$
$$pl_1 + ql_2 = L\sin\psi + l\cos\psi,$$
$$pm_1 + qm_2 = M\sin\psi + m\cos\psi,$$
$$pn_1 + qn_2 = N\sin\psi + n\cos\psi,$$
$$pk_1 + qk_2 = K\sin\psi + k\cos\psi;$$

and there still remain the permanent relations of condition, viz.

$$p^2 + q^2 + 2pq \cos \theta_{12} = 1,$$

$$Ll + Mm + Nn + Kk = 0, \quad l^2 + m^2 + n^2 + k^2 = 1.$$

For further results, we shall require the inclinations α' and β' of L, M, N, K, to l_1, m_1, n_1, k_1, and l_2, m_2, n_2, k_2, respectively, so that

$$\Sigma Ll_1 = \cos \alpha', \quad \Sigma Ll_2 = \cos \beta';$$

thus α' and β' are known quantities.

76. We first obtain the expression for the inclination ψ. Multiply the last four equations by l_1, m_1, n_1, k_1, respectively and add: then

$$\begin{aligned} p + q \cos \theta_{12} &= \Sigma Ll_1 \sin \psi + \Sigma ll_1 \cos \psi \\ &= \cos \alpha' \sin \psi + (p + q \cos \theta_{12}) \cos^2 \psi, \end{aligned}$$

and therefore

$$(p + q \cos \theta_{12}) \sin \psi = \cos \alpha'.$$

Multiply the same four equations by l_2, m_2, n_2, k_2, respectively and add: then

$$\begin{aligned} p \cos \theta_{12} + q &= \Sigma Ll_2 \sin \psi + \Sigma ll_2 \cos \psi \\ &= \cos \beta' \sin \psi + (p \cos \theta_{12} + q) \cos^2 \psi, \end{aligned}$$

and therefore

$$(p \cos \theta_{12} + q) \sin \psi = \cos \beta'.$$

Now

$$p^2 + q^2 + 2pq \cos \theta_{12} = 1,$$

consequently

$$\sin^2 \theta_{12} \sin^2 \psi = \cos^2 \alpha' + \cos^2 \beta' - 2 \cos \alpha' \cos \beta' \cos \theta_{12},$$

giving the angle of inclination between the flat and the plane.

Next, we have

$$\begin{aligned} \lambda &= pl_1 + ql_2 = L \sin \psi + l \cos \psi, \\ \mu &= pm_1 + qm_2 = M \sin \psi + m \cos \psi, \\ \nu &= pn_1 + qn_2 = N \sin \psi + n \cos \psi, \\ \kappa &= pk_1 + qk_2 = K \sin \psi + k \cos \psi; \end{aligned}$$

and therefore

$$\left\| \begin{matrix} \lambda, & \mu, & \nu, & \kappa \\ l, & m, & n, & k \\ L, & M, & N, & K \end{matrix} \right\| = 0:$$

that is, the selected line in the plane, the selected line in the flat, and the normal to the flat are complanar.

77. We next obtain the positions of these selected lines. From the relations

$$(p + q \cos \theta_{12}) \sin \psi = \cos \alpha', \quad (p \cos \theta_{12} + q) \sin \psi = \cos \beta',$$

we have

$$\begin{aligned} p \sin^2 \theta_{12} \sin \psi &= \cos \alpha' - \cos \beta' \cos \theta_{12}, \\ q \sin^2 \theta_{12} \sin \psi &= \cos \beta' - \cos \alpha' \cos \theta_{12}; \end{aligned}$$

and therefore

$$\begin{aligned}
\lambda \sin^2\theta_{12}\sin\psi &= (pl_1 + ql_2)\sin^2\theta_{12}\sin\psi \\
&= (l_1 - l_2\cos\theta_{12})\cos\alpha' + (l_2 - l_1\cos\theta_{12})\cos\beta', \\
\mu \sin^2\theta_{12}\sin\psi &= (m_1 - m_2\cos\theta_{12})\cos\alpha' + (m_2 - m_1\cos\theta_{12})\cos\beta', \\
\nu \sin^2\theta_{12}\sin\psi &= (n_1 - n_2\cos\theta_{12})\cos\alpha' + (n_2 - n_1\cos\theta_{12})\cos\beta', \\
\kappa \sin^2\theta_{12}\sin\psi &= (k_1 - k_2\cos\theta_{12})\cos\alpha' + (k_2 - k_1\cos\theta_{12})\cos\beta'.
\end{aligned}$$

Now (§ 72) the four quantities on the right-hand side are proportional to the direction-cosines of the projection of the direction L, M, N, K, upon the plane Hence the selected line $\lambda, \mu, \nu, \kappa$, in the plane is the projection, upon the plane, of a normal to the flat at any point along the line of intersection of the plane and the flat.

As this normal, the selected line in the plane, and the selected line in the flat (necessarily perpendicular to the normal to the flat), are complanar, the selected line in the flat is obtained by drawing, in the plane through L, M, N, K, which projects the normal upon the given plane, a line perpendicular to that normal. Algebraically, its direction-cosines l, m, n, k, are given by

$$\left.\begin{aligned}
l\cos\psi &= \lambda - L\sin\psi \\
m\cos\psi &= \mu - M\sin\psi \\
n\cos\psi &= \nu - N\sin\psi \\
k\cos\psi &= \kappa - K\sin\psi
\end{aligned}\right\},$$

with the foregoing values of $\lambda, \mu, \nu, \kappa$.

Finally, the plane through the normal and the two selected lines is perpendicular to the line which is the intersection of the plane and the flat. Let this common line have direction-cosines

$$l', m', n', k', = rl_1 + sl_2,\ rm_1 + sm_2,\ rn_1 + sn_2,\ rk_1 + sk_2,$$

where

$$r^2 + s^2 + 2rs\cos\theta_{12} = 1:$$

as it lies in the flat, we have

$$\Sigma L(rl_1 + sl_2) = 0,$$

so that

$$r\cos\alpha' + s\cos\beta' = 0,$$

and therefore

$$\frac{r}{\cos\beta'} = \frac{s}{-\cos\alpha'} = \frac{1}{(\cos^2\alpha' + \cos^2\beta' - 2\cos\alpha'\cos\beta'\cos\theta_{12})^{\frac{1}{2}}}$$

$$= \frac{1}{\sin\theta_{12}\sin\psi}.$$

Now the plane through L, M, N, K, and λ, μ, ν, κ, is

$$\left\|\begin{matrix} x-a', & y-b', & z-c', & v-d' \\ L\ , & M\ , & N\ , & K \\ \lambda\ , & \mu\ , & \nu\ , & \kappa \end{matrix}\right\| = 0.$$

We have just seen that

$$\Sigma L\,(rl_1+sl_2)=0.$$

Again,

$$\Sigma\lambda\,(rl_1+sl_2)=r\Sigma\lambda l_1+s\Sigma\lambda l_2$$

$$=\frac{1}{\sin^2\theta_{12}\sin\psi}\left[r\sin^2\theta_{12}\cos\alpha'+s\sin^2\theta_{12}\cos\beta'\right]$$

$$=0;$$

and therefore the plane in question is perpendicular to the line rl_1+sl_2, rm_1+sm_2, rn_1+sn_2, rk_1+sk_2, that is, it is perpendicular to the intersection of the plane and the flat.

Ex. 1. Shew that, if the line of intersection of the plane and the flat be taken as one of the guiding lines of the plane, so that the equations of the plane are

$$\left\|\begin{matrix} x-a', & y-b', & z-c', & v-d' \\ l'\ , & m'\ , & n'\ , & k' \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{matrix}\right\| = 0,$$

while $\Sigma Ll'=0$, the inclination is given by

$$\sin\psi\sin\gamma=\cos\beta'\,;$$

and the selected line in the plane has direction-cosines

$$\frac{l_2-l'\cos\gamma}{\sin\gamma},\quad \frac{m_2-m'\cos\gamma}{\sin\gamma},\quad \frac{n_2-n'\cos\gamma}{\sin\gamma},\quad \frac{k_2-k'\cos\gamma}{\sin\gamma},$$

where

$$\cos\beta'=\Sigma Ll_2,\quad \cos\gamma=\Sigma l'l_2.$$

Ex. 2. Shew that, if the plane is given by the equations

$$L_1x+M_1y+N_1z+K_1v=P_1,\quad L_2x+M_2y+N_2z+K_2v=P_2,$$

its inclination ψ to the flat $Lx+My+Nz+Kv=P$ is given by the equation

$$\cos^2\psi\sin^2\eta=\cos^2\alpha+\cos^2\beta-2\cos\alpha\cos\beta\cos\eta,$$

where

$$\cos\eta=\Sigma L_1L_2,\quad \cos\alpha=\Sigma LL_1,\quad \cos\beta=\Sigma LL_2.$$

Projection of a plane upon a flat.

78. It is natural to consider the projection of the plane into the flat, just as we consider the projection of a line upon a plane or into the flat. We have already seen that the projection of a line into the flat is another line: the projection of the plane can be obtained by regarding the plane as made up of an aggregate of lines.

When the plane is

$$\left\|\begin{matrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{matrix}\right\| = 0,$$

any point in it is given by

$$\xi = a + pl_1 + ql_2, \quad \eta = b + pm_1 + qm_2, \quad \zeta = c + pn_1 + qn_2, \quad v = d + pk_1 + qk_2.$$

Let the flat be

$$Lx + My + Nz + Kv = P;$$

and let X, Y, Z, V, be the foot of the perpendicular on the flat from ξ, η, ζ, v. If D be the length of this perpendicular, we have

$$X - \xi = LD, \quad Y - \eta = MD, \quad Z - \zeta = ND, \quad V - v = KD,$$

and therefore

$$\Sigma LX - \Sigma L\xi = D.$$

Now $\Sigma LX = P$; and

$$\Sigma L\xi = \Sigma La + p\Sigma Ll_1 + q\Sigma Ll_2 = \Sigma La + p\cos\alpha' + q\cos\beta',$$

with the former significance (§ 72) for α' and β', hence

$$D = P - \Sigma La - p\cos\alpha' - q\cos\beta'.$$

Consequently X, Y, Z, V, are given by

$$\begin{aligned} X &= \xi + LD = a + L\ (P - \Sigma La) + p\,(l_1\ - L\ \cos\alpha') + q\,(l_2\ - L\ \cos\beta'), \\ Y &= \eta + MD = b + M\,(P - \Sigma La) + p\,(m_1 - M\cos\alpha') + q\ (m_2 - M\cos\beta'), \\ Z &= \zeta + ND = c + N\ (P - \Sigma La) + p\,(n_1\ - N\ \cos\alpha') + q\,(n_2\ - N\cos\beta'), \\ V &= v + KD = d + K\,(P - \Sigma La) + p\,(k_1\ - K\ \cos\alpha') + q\ (k_2\ - K\cos\beta'); \end{aligned}$$

and therefore the locus of X, Y, Z, V, is the plane

$$\left\|\begin{matrix} x-\alpha\quad , & y-\beta\quad , & z-\gamma\quad , & v-\delta \\ l_1 - L\cos\alpha', & m_1 - M\cos\alpha', & n_1 - N\cos\alpha', & k_1 - K\cos\alpha' \\ l_2 - L\cos\beta', & m_2 - M\cos\beta', & n_2 - N\cos\beta', & k_2 - K\cos\beta' \end{matrix}\right\| = 0,$$

which accordingly is the projection of the given plane, the values of $\alpha, \beta, \gamma, \delta$, being given by

$$\frac{\alpha - a}{L} = \frac{\beta - b}{M} = \frac{\gamma - c}{N} = \frac{\delta - d}{K} = P - \Sigma La.$$

The coordinates of any point in the projection satisfy the equation

$$Lx + My + Nz + Kv = P$$

of the flat, as is to be expected from the fact of projection into the flat.

Again, having regard to the guiding lines of the original plane and to the guiding lines in the equations of the projection of the plane, we note that the condition

$$\begin{vmatrix} l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_1 - L\cos\alpha', & m_1 - M\cos\alpha', & n_1 - N\cos\alpha', & k_1 - K\cos\alpha' \\ l_2 - L\cos\beta', & m_2 - M\cos\beta', & n_2 - N\cos\beta', & k_2 - K\cos\beta' \end{vmatrix} = 0$$

is satisfied. Consequently (§ 43) the two planes lie in one and the same flat; and it is easy to verify that this flat, containing the original plane and its projection, is

$$\begin{vmatrix} x-a, & y-b, & z-c, & v-d \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ L, & M, & N, & K \end{vmatrix} = 0.$$

Moreover, as the two planes lie in one flat, they intersect in a line and not in a point only (§ 43): this line is their common line, and its equations are

$$\frac{x - a - \frac{l_1}{\Sigma L l_1}(P - \Sigma La)}{l_2 \Sigma L l_1 - l_1 \Sigma L l_2} = \frac{y - b - \frac{m_1}{\Sigma L l_1}(P - \Sigma La)}{m_2 \Sigma L l_1 - m_1 \Sigma L l_2}$$

$$= \frac{z - c - \frac{n_1}{\Sigma L l_1}(P - \Sigma La)}{n_2 \Sigma L l_1 - n_1 \Sigma L l_2} = \frac{v - d - \frac{k_1}{\Sigma L l_1}(P - \Sigma La)}{k_2 \Sigma L l_1 - k_1 \Sigma L l_2}.$$

Angle between a plane and its projection upon a flat.

79. We could regard the angle between the plane and its projection as giving the angle between the plane and the flat. To obtain the inclination of the plane to its projection, we can use the forthcoming analysis of §§ 83, 84. As the direction-cosines of the guiding lines in one plane are

$$l_1, m_1, n_1, k_1, \text{ and } l_2, m_2, n_2, k_2,$$

with ω as the angle between them; and as the direction-cosines of the guiding lines in the other plane are

$$\frac{l_1 - L\cos\alpha'}{\sin\alpha'}, \quad \frac{m_1 - M\cos\alpha'}{\sin\alpha'}, \quad \frac{n_1 - N\cos\alpha'}{\sin\alpha'}, \quad \frac{k_1 - K\cos\alpha'}{\sin\alpha'},$$

and

$$\frac{l_2 - L\cos\beta'}{\sin\beta'}, \quad \frac{m_2 - M\cos\beta'}{\sin\beta'}, \quad \frac{n_2 - N\cos\beta'}{\sin\beta'}, \quad \frac{k_2 - K\cos\beta'}{\sin\beta'},$$

with η as the angle between them, where

$$\cos\eta = \frac{\cos\alpha - \cos\alpha' \cos\beta'}{\sin\alpha' \sin\beta'},$$

the angle between the planes can be obtained by using the formula of § 83.

Again, in order to obtain this angle between the plane and its projection, we can use the fact that the plane and its projection intersect in a line. The equations of this line have been given. In the plane, take a direction perpendicular to this common line; in the projection, take a direction perpendicular to this common line; the angle between these directions is the required inclination (see also § 89).

Both processes, by their respective methods, are left as exercises to construct the value of ψ which has already (§ 76) been given.

Ex. 1 Prove that the projection of the plane

$$\left.\begin{array}{l} L_1x+M_1y+N_1z+K_1v=P_1 \\ L_2x+M_2y+N_2z+K_2v=P_2 \end{array}\right\}$$

on the flat

$$Lx+My+Nz+Kv=P$$

is the intersection of the last flat by

$$\frac{\Sigma L_1x-P_1}{\Sigma LL_1}=\frac{\Sigma L_2x-P_2}{\Sigma LL_2};$$

and verify that the two planes meet in the line, common to the original plane and the original flat

Ex 2. A plane is given by the equations

$$x=a+pz+qv, \quad y=b+rz+sv;$$

prove that its projection on the flat

$$Lx+My+Nz+Kv=P$$

is the plane

$$\begin{vmatrix} x-a+Lc, & y-b+Mc, & z+Nc, & v+Kc \\ p-Lu\ , & r-Mu\ , & 1-Nu, & -Ku \\ q-Lv\ , & s-Mv\ , & -Nv, & 1-Kv \end{vmatrix}=0,$$

where

$$c=La+Mb-P, \quad u=Lp+Mr+N, \quad v=Lq+Ms+K.$$

Verify that the two planes intersect in a line and not in a point merely; and obtain the equations of the line.

Inclination of two flats.

80. The orientation of a flat is considered most simply by reference to the direction of its normal; and consequently the inclination of two flats is estimated by—and is taken to be—the angle between the normals to the flats. When the equations of the flats are

$$Lx\ +My\ +Nz\ +Kv\ =P\,,$$

$$L'x+M'y+N'z+K'v=P',$$

respectively, and when the inclination of the flats is denoted by χ, we have

$$\cos\chi=LL'+MM'+NN'+KK',$$

on the assumption that L, M, N, K, and L', M', N', K', are the actual direction-cosines of the normals.

The intersection of the two flats is a plane—their plane of cleavage. The properties of the plane of cleavage, in relation to the flats, have already (§ 55) been discussed.

An estimate of the inclination of two flats, by choosing a line in one flat and a line in the other flat in such a way as to secure a maximum or a minimum inclination θ of the two lines, or a maximum or minimum value of $\cos^2\theta$, has already been made in § 65. From the analysis there given, it appears that the angle between lines, selected so as to satisfy the critical conditions for a maximum or a minimum inclination, is given by the equation

$$\cos\theta = LL' + MM' + NN' + KK'.$$

Accordingly the estimate, thus obtained, agrees with the estimate provided by the angle between the directions of normals to the two flats.

CHAPTER VI.

INCLINATIONS OF PLANES: ORTHOGONALITY.

Summary of conditions, for parallel planes, for orthogonal planes.

81. Before entering on the discussion of the inclination of two planes to one another, it is convenient to summarise the analytical results affecting respectively the two extreme instances: (i), when the two planes are parallel; (ii), when the two planes are orthogonal.

I. *Parallel planes.*

(i) The conditions for parallelism, when the planes are given by the equations

$$\begin{Vmatrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{Vmatrix} = 0, \quad \begin{Vmatrix} x-a', & y-b', & z-c', & v-d' \\ l_3\ , & m_3\ , & n_3\ , & k_3 \\ l_4\ , & m_4\ , & n_4\ , & k_4 \end{Vmatrix} = 0,$$

can be taken either in the forms

$$l_3 = \alpha l_1 + \beta l_2, \quad m_3 = \alpha m_1 + \beta m_2, \quad n_3 = \alpha n_1 + \beta n_2, \quad k_3 = \alpha k_1 + \beta k_2,$$

$$l_4 = \gamma l_1 + \delta l_2\ , \quad m_4 = \gamma m_1 + \delta m_2\ , \quad n_4 = \gamma n_1 + \delta n_2\ , \quad k_4 = \gamma k_1 + \delta k_2\ ,$$

where $\alpha\delta - \beta\gamma$ does not vanish, while α, β, γ, δ, are otherwise arbitrary; or in the equivalent forms

$$\begin{Vmatrix} l_3, & m_3, & n_3, & k_3 \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{Vmatrix} = 0, \quad \begin{Vmatrix} l_4, & m_4, & n_4, & k_4 \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{Vmatrix} = 0$$

(ii) When the planes are given by the equations

$$\begin{Vmatrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{Vmatrix} = 0, \quad \left.\begin{aligned} L_1x + M_1y + N_1z + K_1v = P_1 \\ L_2x + M_2y + N_2z + K_2v = P_2 \end{aligned}\right\},$$

the conditions for parallelism are

$$L_1l_1 + M_1m_1 + N_1n_1 + K_1k_1 = 0, \quad L_1l_2 + M_1m_2 + N_1n_2 + K_1k_2 = 0,$$

$$L_2l_1 + M_2m_1 + N_2n_1 + K_2k_1 = 0, \quad L_2l_2 + M_2m_2 + N_2n_2 + K_2k_2 = 0.$$

(iii) When the planes are given by the equations

$$\begin{Vmatrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{Vmatrix} = 0, \quad \left.\begin{aligned} z &= f + px + qy \\ v &= h + rx + sy \end{aligned}\right\},$$

the conditions for parallelism are

$$n_1 = pl_1 + qm_1, \quad k_1 = rl_1 + sm_1,$$
$$n_2 = pl_2 + qm_2, \quad k_2 = rl_2 + sm_2.$$

(iv) When the planes are given by the equations

$$\left.\begin{array}{l} L_1x + M_1y + N_1z + K_1v = P_1 \\ L_2x + M_2y + N_2z + K_2v = P_2 \end{array}\right\}, \quad \left.\begin{array}{l} L_3x + M_3y + N_3z + K_3v = P_3 \\ L_4x + M_4y + N_4z + K_4v = P_4 \end{array}\right\},$$

the conditions for parallelism are

$$\begin{Vmatrix} L_3, & M_3, & N_3, & K_3 \\ L_1, & M_1, & N_1, & K_1 \\ L_2, & M_2, & N_2, & K_2 \end{Vmatrix} = 0, \quad \begin{Vmatrix} L_4, & M_4, & N_4, & K_4 \\ L_1, & M_1, & N_1, & K_1 \\ L_2, & M_2, & N_2, & K_2 \end{Vmatrix} = 0.$$

(v) When the planes are given by the equations

$$\left.\begin{array}{l} L_1x + M_1y + N_1z + K_1v = P_1 \\ L_2x + M_2y + N_2z + K_2v = P_2 \end{array}\right\}, \quad \left.\begin{array}{l} z = f + px + qy \\ v = h + rx + sy \end{array}\right\},$$

the conditions for parallelism are

$$L_1 + N_1p + K_1r = 0, \quad L_2 + N_2p + K_2r = 0,$$
$$M_1 + N_1q + K_1s = 0, \quad M_2 + N_2q + K_2s = 0$$

(vi) When the planes are given by the equations

$$\left.\begin{array}{l} z = f + px + qy \\ v = h + rx + sy \end{array}\right\}, \quad \left.\begin{array}{l} z = f' + p'x + q'y \\ v = h' + r'x + s'y \end{array}\right\},$$

the conditions for parallelism are

$$p' = p, \quad q' = q, \quad r' = r, \quad s' = s.$$

II. *Orthogonal planes.*

The six possible combinations, arising out of the different forms of equations, are numbered as for the preceding combinations.

(i) The conditions, that the planes be orthogonal, are

$$l_3l_1 + m_3m_1 + n_3n_1 + k_3k_1 = 0, \quad l_4l_1 + m_4m_1 + n_4n_1 + k_4k_1 = 0,$$
$$l_3l_2 + m_3m_2 + n_3n_2 + k_3k_2 = 0, \quad l_4l_2 + m_4m_2 + n_4n_2 + k_4k_2 = 0.$$

(ii) The conditions, that the planes be orthogonal, are

$$\begin{Vmatrix} L_1, & M_1, & N_1, & K_1 \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{Vmatrix} = 0, \quad \begin{Vmatrix} L_2, & M_2, & N_2, & K_2 \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{Vmatrix} = 0.$$

(iii) The conditions, that the planes be orthogonal, are

$$l_1 + n_1p + k_1r = 0, \quad l_2 + n_2p + k_2r = 0,$$
$$m_1 + n_1q + k_1s = 0, \quad m_2 + n_2q + k_2s = 0.$$

(iv) The conditions, that the planes be orthogonal, are

$$L_3L_1 + M_3M_1 + N_3N_1 + K_3K_1 = 0, \qquad L_4L_1 + M_4M_1 + N_4N_1 + K_4K_1 = 0,$$
$$L_3L_2 + M_3M_2 + N_3N_2 + K_3K_2 = 0, \qquad L_4L_2 + M_4M_2 + N_4N_2 + K_4K_2 = 0.$$

(v) The conditions, that the planes be orthogonal, are

$$N_1 = L_1p + M_1q, \qquad N_2 = L_2p + M_2q,$$
$$K_1 = L_1r + M_1s, \qquad K_2 = L_2r + M_2s.$$

(vi) The conditions, that the planes be orthogonal, are

$$pq' + rs' = 0, \qquad 1 + pp' + rr' = 0,$$
$$qp' + sr' = 0, \qquad 1 + qq' + ss' = 0.$$

Ex. 1. Shew that, if two planes intersect in a line, they cannot be orthogonal.

Ex. 2. Given four flats F_1, F_2, F_3, F_4, prove that, if the cleavage planes of F_1 and F_2, and of F_3 and F_4, are parallel, the cleavage planes of F_1 and F_3, and of F_2 and F_4, also are parallel, and the cleavage planes of F_2 and F_3, and of F_1 and F_4, are parallel.

Ex. 3. Given four flats F_1, F_2, F_3, F_4, such that the cleavage plane of F_1 and F_2 is orthogonal to the cleavage plane of F_3 and F_4, in what circumstances is the cleavage plane of F_1 and F_4 orthogonal to the cleavage plane of F_2 and F_3?

Relation between four directions in one flat.

82. We have seen (§ 43) that, when the two planes

$$\left\| \begin{matrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{matrix} \right\| = 0, \qquad \left\| \begin{matrix} x-f, & y-g, & z-h, & v-i \\ l_3\ , & m_3\ , & n_3\ , & k_3 \\ l_4\ , & m_4\ , & n_4\ , & k_4 \end{matrix} \right\| = 0,$$

meet in a line and not in a point alone, the condition

$$R = \begin{vmatrix} l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \\ l_4, & m_4, & n_4, & k_4 \end{vmatrix} = 0$$

is satisfied. Further, we have seen that the two planes then lie in one flat. It therefore follows that, when four directions are taken in one and the same flat, there exists a relation among their direction-cosines: it is a well-known property of homaloidal triple space.

This relation can be exhibited as an identical equation among the six inclinations of the four directions when combined in pairs. Let the angle between the positive senses of the directions l_r, m_r, n_r, k_r, and l_s, m_s, n_s, k_s, be denoted by θ_{rs}, so that

$$\cos\theta_{rs} = l_rl_s + m_rm_s + n_rn_s + k_rk_s,$$

for $r, s, = 1, 2, 3, 4$. Then, as

$$\begin{vmatrix} l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \\ l_4, & m_4, & n_4, & k_4 \end{vmatrix}^2 = \begin{vmatrix} \Sigma l_1^2\ , & \Sigma l_1 l_2, & \Sigma l_1 l_3, & \Sigma l_1 l_4 \\ \Sigma l_1 l_2, & \Sigma l_2^2\ , & \Sigma l_2 l_3, & \Sigma l_2 l_4 \\ \Sigma l_1 l_3, & \Sigma l_2 l_3, & \Sigma l_3^2\ , & \Sigma l_3 l_4 \\ \Sigma l_1 l_4, & \Sigma l_2 l_4, & \Sigma l_3 l_4, & \Sigma l_4^2 \end{vmatrix},$$

the foregoing relation implies the equation

$$\begin{vmatrix} 1\ \ , & \cos\theta_{12}, & \cos\theta_{13}, & \cos\theta_{14} \\ \cos\theta_{12}, & 1\ \ , & \cos\theta_{23}, & \cos\theta_{24} \\ \cos\theta_{13}, & \cos\theta_{23}, & 1\ \ , & \cos\theta_{34} \\ \cos\theta_{14}, & \cos\theta_{24}, & \cos\theta_{34}, & 1 \end{vmatrix} = 0.$$

Let this determinant be denoted by Θ; when it is expanded, we have

$$\begin{aligned} \Theta = 1 &- \cos^2\theta_{23} - \cos^2\theta_{31} - \cos^2\theta_{12} - \cos^2\theta_{14} - \cos^2\theta_{24} - \cos^2\theta_{34} \\ &+ \cos^2\theta_{23}\cos^2\theta_{14} + \cos^2\theta_{31}\cos^2\theta_{24} + \cos^2\theta_{12}\cos^2\theta_{34} \\ &+ 2\,(\cos\theta_{23}\cos\theta_{34}\cos\theta_{42} + \cos\theta_{34}\cos\theta_{41}\cos\theta_{13} + \cos\theta_{41}\cos\theta_{12}\cos\theta_{24} \\ &\qquad\qquad + \cos\theta_{12}\cos\theta_{23}\cos\theta_{31}) \\ &- 2\,(\cos\theta_{12}\cos\theta_{13}\cos\theta_{24}\cos\theta_{34} + \cos\theta_{21}\cos\theta_{23}\cos\theta_{14}\cos\theta_{34} \\ &\qquad\qquad + \cos\theta_{13}\cos\theta_{23}\cos\theta_{14}\cos\theta_{24}). \end{aligned}$$

Then the condition, satisfied by four directions when they lie in one and the same flat, is

$$\Theta = 0.$$

The four directions can be taken arbitrarily, subject to the sole restriction that they shall lie in one flat, if this condition is to be satisfied. But if, chosen arbitrarily in homaloidal quadruple space, they do not observe the restriction of being contained in some single flat, the condition $\Theta = 0$ is not satisfied.

Ex. If the fourth line makes an angle $\frac{1}{2}\pi - \phi$ with the normal to the flat through the first three, shew that

$$\Theta = (1 - \cos^2\theta_{21} - \cos^2\theta_{31} - \cos^2\theta_{12} + 2\cos\theta_{23}\cos\theta_{31}\cos\theta_{12})\sin^2\phi.$$

Inclination of two planes by reference to projection of areas.

83. Instead of estimating the inclination of two planes by proceeding from the inclination of lines lying in the respective planes, we can frame an estimate by a comparison of areas in the planes, one of which will be deduced from the other—and it will appear that, as obviously should be the fact, the same result is obtained whichever plane is first selected—by projecting its boundary, through perpendiculars drawn from the points of that boundary, upon that other plane.

Accordingly, let it be required to find the inclination of the plane

$$\begin{Vmatrix} x-a, & y-b, & z-c, & v-d \\ l_3\ , & m_3\ , & n_3\ , & k_3 \\ l_4\ , & m_4\ , & n_4\ , & k_4 \end{Vmatrix} = 0$$

to the plane

$$\begin{Vmatrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{Vmatrix} = 0.$$

The planes will be supposed not to be parallel to one another. The conditions for complete orthogonality have been stated; and the possible event of orthogonality will therefore be omitted as the full conditions are known. Two planes certainly have a point of intersection: this common point has been taken to be a, b, c, d. The two planes possibly intersect in a line and not merely in a point: the condition for linear intersection is

$$\begin{vmatrix} l_3, & m_3, & n_3, & k_3 \\ l_4, & m_4, & n_4, & k_4 \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{vmatrix} = 0,$$

but such eventuality will be left open at this stage, so that its influence may be discussed in the course of the investigation.

As already indicated, we shall proceed by projections. Let the direction l_3, m_3, n_3, k_3, through the common point a, b, c, d, be projected on the latter plane: its projection, also passing through that common point and making an angle ϑ_3 with that direction, has direction-cosines $L_3/\Omega_3, M_3/\Omega_3, N_3/\Omega_3, K_3/\Omega_3$, where

$$\left.\begin{aligned} L_3 &= e_{ll}\, l_3 + e_{lm}\, m_3 + e_{ln}\, n_3 + e_{lk}\, k_3 \\ M_3 &= e_{lm}\, l_3 + e_{mm}\, m_3 + e_{mn}\, n_3 + e_{mk}\, k_3 \\ N_3 &= e_{ln}\, l_3 + e_{mn}\, m_3 + e_{nn}\, n_3 + e_{nk}\, k_3 \\ K_3 &= e_{lk}\, l_3 + e_{mk}\, m_3 + e_{nk}\, n_3 + e_{kk}\, k_3 \end{aligned}\right\},$$

$$\Omega_3 = \cos\vartheta_3 \sin^2\omega = (\cos^2\theta_{13} + \cos^2\theta_{23} - 2\cos\theta_{13}\cos\theta_{23}\cos\omega)^{\frac{1}{2}} \sin\omega,$$

with the same notation as before (§ 69) for ω; for θ_{13} and θ_{23}; and for the symbols e_{pq}, where $p, q = l, m, n, k$. Similarly when the direction l_4, m_4, n_4, k_4, is projected also on the second plane, its projection through a, b, c, d, and making an angle ϑ_4 with that direction, has direction-cosines L_4/Ω_4, M_4/Ω_4, N_4/Ω_4, K_4/Ω_4, where

$$\left.\begin{aligned} L_4 &= e_{ll}\, l_4 + e_{lm}\, m_4 + e_{ln}\, n_4 + e_{lk}\, k_4 \\ M_4 &= e_{lm}\, l_4 + e_{mm}\, m_4 + e_{mn}\, n_4 + e_{mk}\, k_4 \\ N_4 &= e_{ln}\, l_4 + e_{mn}\, m_4 + e_{nn}\, n_4 + e_{nk}\, k_4 \\ K_4 &= e_{lk}\, l_4 + e_{mk}\, m_4 + e_{nk}\, n_4 + e_{kk}\, k_4 \end{aligned}\right\},$$

$$\Omega_4 = \cos\vartheta_4 \sin^2\omega = (\cos^2\theta_{14} + \cos^2\theta_{24} - 2\cos\theta_{14}\cos\theta_{24}\cos\omega)^{\frac{1}{2}} \sin\omega.$$

Measure a length r along l_3, m_3, n_3, k_3, and a length s along l_4, m_4, n_4, k_4; the lengths along L_3, M_3, N_3, K_3, and L_4, M_4, N_4, K_4, of the projected segments are

$$r\cos\vartheta_3, \quad s\cos\vartheta_4.$$

Let Ω denote the angle between these projected segments, so that

$$\Omega_3\Omega_4\cos\Omega = L_3L_4 + M_3M_4 + N_3N_4 + K_3K_4.$$

The area of the original triangle, two of the sides of which are r and s, is

$$\tfrac{1}{2}rs\sin\eta,$$

where

$$\cos\eta = l_3l_4 + m_3m_4 + n_3n_4 + k_3k_4.$$

The area of the projected triangle is

$$\tfrac{1}{2}r\cos\vartheta_3 \,.\, s\cos\vartheta_4 \,.\, \sin\Omega.$$

If therefore we denote by ϕ the inclination of the two planes to one another, and we take the inclination of the projected area to the original area as the inclination of the planes, we have

$$\cos\phi = \frac{\tfrac{1}{2}r\cos\vartheta_3 \,.\, s\cos\vartheta_4 \,.\, \sin\Omega}{\tfrac{1}{2}rs\sin\eta}$$

$$= \frac{\cos\vartheta_3\cos\vartheta_4\sin\Omega}{\sin\eta}$$

Now

$$\Omega_3^2 = L_3^2 + M_3^2 + N_3^2 + K_3^2, \quad \Omega_4^2 = L_4^2 + M_4^2 + N_4^2 + K_4^2;$$

and therefore

$$\Omega_3\Omega_4\sin\Omega = \square^{\frac{1}{2}},$$

where

$$\square = (L_3M_4 - M_3L_4)^2 + (L_3N_4 + N_3L_4)^2 + (L_3K_4 - K_3L_4)^2$$
$$+ (M_3N_4 - N_3M_4)^2 + (M_3K_4 - K_3M_4)^2 + (N_3K_4 - K_3N_4)^2,$$

consequently

$$\cos\phi = \frac{\square^{\frac{1}{2}}}{\sin\eta\sin^4\omega}$$

It is necessary to evaluate $\square^{\frac{1}{2}}$.

In the various terms in $\square$, we substitute for L_3, M_3, N_3, K_3, and L_4, M_4, N_4, K_4. We have

$$L_3M_4 - M_3L_4 = \begin{vmatrix} e_{ll}\,l_3 + e_{lm}\,m_3 + e_{ln}\,n_3 + e_{lk}\,k_3, & e_{ll}\,l_4 + e_{lm}\,m_4 + e_{ln}\,n_4 + e_{lk}\,k_4 \\ e_{lm}l_3 + e_{mm}m_3 + e_{mn}n_3 + e_{mk}k_3, & e_{lm}l_4 + e_{mm}m_4 + e_{mn}n_4 + e_{mk}k_4 \end{vmatrix}$$

By direct substitution for the quantities $e_{ll}, e_{lm}, \ldots$, we find $e_{l\theta}e_{m\phi} - e_{l\phi}e_{m\theta}$

$$= \begin{vmatrix} l_1\theta_1 + l_2\theta_2 - (l_1\theta_2 + l_2\theta_1)\cos\omega, & l_1\phi_1 + l_2\phi_2 - (l_1\phi_2 + l_2\phi_1)\cos\omega \\ m_1\theta_1 + m_2\theta_2 - (m_1\theta_2 + m_2\theta_1)\cos\omega, & m_1\phi_1 + m_2\phi_2 - (m_1\phi_2 + m_2\phi_1)\cos\omega \end{vmatrix}$$

$$= (l_1m_2 - l_2m_1)(\theta_1\phi_2 - \theta_2\phi_1)\sin^2\omega,$$

for all the combinations θ, $\phi = l, m, n, k$. But

$$L_3M_4 - M_3L_4 = \Sigma \begin{vmatrix} \theta_3, & \phi_3 \\ \theta_4, & \phi_4 \end{vmatrix} \begin{vmatrix} e_{l\theta}, & e_{l\phi} \\ e_{m\theta}, & e_{m\phi} \end{vmatrix},$$

the summation being taken over the six combinations of θ and ϕ; hence

$$L_3M_4 - M_3L_4 = (l_1m_2 - l_2m_1)\sin^2\omega\, \Sigma \begin{vmatrix} \theta_1, & \phi_1 \\ \theta_2, & \phi_2 \end{vmatrix} \begin{vmatrix} \theta_3, & \phi_3 \\ \theta_4, & \phi_4 \end{vmatrix}.$$

With our former notation (§ 47),

$$\begin{vmatrix} m_1, & n_1 \\ m_2, & n_2 \end{vmatrix} = a_{12}, \quad \begin{vmatrix} l_1, & k_1 \\ l_2, & k_2 \end{vmatrix} = f_{12},$$

and so for the other combinations: also

$$\Sigma a_{12}^2 = \Sigma (l_1m_2 - l_2m_1)^2 = \sin^2\omega, \quad \Sigma a_{34}^2 = \Sigma (l_3m_4 - l_4m_3)^2 = \sin^2\eta$$

We write

$$a_{12}a_{34} + b_{12}b_{34} + c_{12}c_{34} + f_{12}f_{34} + g_{12}g_{34} + h_{12}h_{34} = \Delta;$$

and now

$$L_3M_4 - M_3L_4 = \Delta(l_1m_2 - l_2m_1)\sin^2\omega.$$

Similarly for the other combinations $L_3N_4 - L_4N_3, \ldots, N_3K_4 - N_4K_3$. Hence

$$\Box = \Sigma (L_3M_4 - M_3L_4)^2$$
$$= \Delta^2 \sin^4\omega\, \Sigma (l_1m_2 - l_2m_1)^2 = \Delta^2\sin^6\omega.$$

Therefore

$$\cos\phi = \frac{\Box^{\frac{1}{2}}}{\sin\eta\sin^4\omega}$$
$$= \frac{\Delta}{\sin\eta\sin\omega}$$
$$= \frac{\Sigma a_{12}a_{34}}{(\Sigma a_{12}^2)^{\frac{1}{2}}(\Sigma a_{34}^2)^{\frac{1}{2}}}.$$

Expressions for the inclination of two planes.

84 We thus have the inclination of the two planes.

Various corollaries can be derived.

(i) Any two non-coincident lines in the plane can be substituted for the two guiding lines l_1, m_1, n_1, k_1, and l_2, m_2, n_2, k_2, of the plane

$$\begin{Vmatrix} x-a, & y-b, & z-c, & v-d \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{Vmatrix} = 0,$$

without affecting the value of ϕ. For any two such lines are given by

$$l_1' = \alpha l_1 + \beta l_2, \quad m_1' = \alpha m_1 + \beta m_2, \quad n_1' = \alpha n_1 + \beta n_2, \quad k_1' = \alpha k_1 + \beta k_2,$$

$$l_2' = \gamma l_1 + \delta l_2, \quad m_2' = \gamma m_1 + \delta m_2, \quad n_2' = \gamma n_1 + \delta n_2, \quad k_2' = \gamma k_1 + \delta k_2,$$

where $\alpha\delta - \beta\gamma$ is not zero: then

$$l_1' m_2' - l_2' m_1' = (\alpha\delta - \beta\gamma)(l_1 m_2 - l_2 m_1),$$

and so for the other quantities, so that every magnitude

$$\frac{\iota_{12}}{(\Sigma a_{12}^2)^{\frac{1}{2}}},$$

for $\iota = a, b, c, f, g, h$, is unaltered. The value of $\cos\phi$ is unchanged.

Similarly any two non-coincident lines in the plane

$$\begin{Vmatrix} x-a, & y-b, & z-c, & v-d \\ l_3\ , & m_3\ , & n_3\ , & k_3 \\ l_4\ , & m_4\ , & n_4\ , & k_4 \end{Vmatrix} = 0$$

can be substituted for the guiding lines l_3, m_3, n_3, k_3, and l_4, m_4, n_4, k_4, without affecting the value of ϕ.

(ii) The expression for $\cos\phi$ is symmetrical, as between the parameters defining the two planes: thus verifying the expectation that the same result would have been attained, had an area in the second plane been projected upon the first plane.

(iii) The quantities $a_{34}, b_{34}, c_{34}, f_{34}, g_{34}, h_{34}$, are (§ 30) proportional to the quantities $-p, -q, 1, s, -r, ps-qr$, arising when the equations of the plane are taken in the canonical form

$$\left.\begin{aligned} z &= f + px + qy \\ v &= h + rx + sy \end{aligned}\right\};$$

and similarly for the quantities $a_{34}, b_{34}, c_{34}, f_{34}, g_{34}, h_{34}$, when the equations of the other plane occur in the canonical form

$$\left.\begin{aligned} z &= f' + p'x + q'y \\ v &= h' + r'x + s'y \end{aligned}\right\}$$

If then

$$S^2 = 1 + p^2 + q^2 + r^2 + s^2 + (ps - qr)^2,$$

$$S'^2 = 1 + p'^2 + q'^2 + r'^2 + s'^2 + (p's' - q'r')^2,$$

$$T = 1 + pp' + qq' + rr' + ss' + (ps - qr)(p's' - q'r'),$$

the inclination of the two planes, when their equations occur in the foregoing canonical forms, is given by

$$SS' \cos\phi = T.$$

(iv) When the equations of the two planes are given in the form

$$\left.\begin{array}{l} A_1x+B_1y+C_1z+D_1v=E_1 \\ A_2x+B_2y+C_2z+D_2v=E_2 \end{array}\right\},\quad \left.\begin{array}{l} A_3x+B_3y+C_3z+D_3v=E_3 \\ A_4x+B_4y+C_4z+D_4v=E_4 \end{array}\right\},$$

their inclination is given by

$$\Delta_{12}\Delta_{34}\cos\phi=U,$$

where

$$\begin{aligned}\Delta_{12}{}^2=(A_1B_2-A_2B_1)^2&+(A_1C_2-A_2C_1)^2+(A_1D_2-A_2D_1)^2\\&+(B_1C_2-B_2C_1)^2+(B_1D_2-B_2D_1)^2+(C_1D_2-C_2D_1)^2,\\ \Delta_{34}{}^2=(A_3B_4-A_4B_3)^2&+(A_3C_4-A_4C_3)^2+(A_3D_4-A_4D_3)^2\\&+(B_3C_4-B_4C_3)^2+(B_3D_4-B_4D_3)^2+(C_3D_4-C_4D_3)^2,\\ U=(A_1B_2-A_2B_1)&(A_3B_4-A_4B_3)+(A_1C_2-A_2C_1)(A_3C_4-A_4C_3)\\&+(A_1D_2-A_2D_1)(A_3D_4-A_4D_3)+(B_1C_2-B_2C_1)(B_3C_4-B_4C_3)\\&+(B_1D_2-B_2D_1)(B_3D_4-B_4D_3)+(C_1D_2-C_2D_1)(C_3D_4-C_4D_3).\end{aligned}$$

Ex. Obtain an expression for the inclination ϕ of the planes

$$\left\|\begin{array}{cccc} x, & y, & z, & v \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{array}\right\|=0,\quad \left.\begin{array}{l} z=px+qy \\ v=rx+sy \end{array}\right\};$$

likewise for the inclination ψ of the planes

$$\left\|\begin{array}{cccc} x, & y, & z, & v \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{array}\right\|=0,\quad \left.\begin{array}{l} A_1x+B_1y+C_1z+D_1v=0 \\ A_2x+B_2y+C_2z+D_2v=0 \end{array}\right\},$$

in the respective forms

$$\begin{aligned}S\sin\omega\cos\phi=(l_1m_2-m_1l_2)&-p(m_1n_2-n_1m_2)-q(n_1l_2-l_1n_2)\\&+s(l_1k_2-k_1l_2)-r(m_1k_2-k_1m_2)+(ps-qr)(n_1k_2-k_1n_2),\end{aligned}$$

where

$$\left.\begin{aligned}S^2&=1+p^2+q^2+r^2+s^2+(pr-qs)^2\\ \cos\omega&=l_1l_2+m_1m_2+n_1n_2+k_1k_2\end{aligned}\right\};$$

and

$$\begin{aligned}\sin\omega\sin\eta\cos\psi=\ &(l_1m_2-m_1l_2)(C_1D_2-D_1C_2)+(n_1k_2-k_1n_2)(A_1B_2-B_1A_2)\\&+(m_1n_2-n_1m_2)(A_1D_2-D_1A_2)+(l_1k_2-k_1l_2)(B_1C_2-C_1B_2)\\&+(n_1l_2-l_1n_2)(B_1D_2-D_1B_2)+(m_1k_2-k_1m_2)(C_1A_2-A_1C_2),\end{aligned}$$

where ω has the foregoing value, and

$$\cos\eta=L_1L_2+M_1M_2+N_1N_2+K_1K_2.$$

Orientation coordinates of a plane.

85. Now consider the inclinations of the plane to the planes of reference in the system of coordinate axes. As the direction-cosines of the axes are

$$\begin{array}{lllll} 1, & 0, & 0, & 0, & \text{for } OX, \\ 0, & 1, & 0, & 0, & \ldots\ OY, \\ 0, & 0, & 1, & 0, & \ldots\ OZ, \\ 0, & 0, & 0, & 1, & \ldots\ OV, \end{array}$$

the equations of the various planes, through two of these axes, in all possible pairs, are

$$\left\| \begin{matrix} x, & y, & z, & v \\ 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \end{matrix} \right\| = 0, \text{ for } XOY; \qquad \left\| \begin{matrix} x, & y, & z, & v \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{matrix} \right\| = 0, \text{ for } ZOV,$$

$$\left\| \begin{matrix} x, & y, & z, & v \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \end{matrix} \right\| = 0, \text{ for } YOZ; \qquad \left\| \begin{matrix} x, & y, & z, & v \\ 1, & 0, & 0, & 0 \\ 0, & 0, & 0, & 1 \end{matrix} \right\| = 0, \text{ for } XOV;$$

$$\left\| \begin{matrix} x, & y, & z, & v \\ 1, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0 \end{matrix} \right\| = 0, \text{ for } ZOX, \qquad \left\| \begin{matrix} x, & y, & z, & v \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 1 \end{matrix} \right\| = 0, \text{ for } YOV.$$

Let ϕ_{XY} denote the inclination of the plane

$$\left\| \begin{matrix} x, & y, & z, & v \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{matrix} \right\| = 0$$

to the plane XOY; and let ϕ_{ZX}, ϕ_{YZ}, ϕ_{XV}, ϕ_{YV}, ϕ_{ZV}, bear the similar significance for its inclinations to the other planes respectively. From the preceding result, we have

$$\mathrm{a} = \cos\phi_{YZ} = \frac{m_1n_2 - n_1m_2}{\sin\omega} = \frac{a_{12}}{\sin\omega},$$

$$\mathrm{b} = \cos\phi_{ZX} = \frac{n_1l_2 - l_1n_2}{\sin\omega} = \frac{b_{12}}{\sin\omega},$$

$$\mathrm{c} = \cos\phi_{XY} = \frac{l_1m_2 - m_1l_2}{\sin\omega} = \frac{c_{12}}{\sin\omega},$$

$$\mathrm{f} = \cos\phi_{XV} = \frac{l_1k_2 - k_1l_2}{\sin\omega} = \frac{f_{12}}{\sin\omega},$$

$$\mathrm{g} = \cos\phi_{YV} = \frac{m_1k_2 - k_1m_2}{\sin\omega} = \frac{g_{12}}{\sin\omega},$$

$$\mathrm{h} = \cos\phi_{ZV} = \frac{n_1k_2 - k_1n_2}{\sin\omega} = \frac{h_{12}}{\sin\omega}.$$

Thus the quantities a, b, c, f, g, h, may be regarded as *orientation-cosines* or *orientation coordinates* of the plane. Now we have

$$a_{12}f_{12} + b_{12}g_{12} + c_{12}h_{12} = 0,$$

$$a_{12}^2 + b_{12}^2 + c_{12}^2 + f_{12}^2 + g_{12}^2 + h_{12}^2 = \sin^2\omega;$$

and therefore the orientation-cosines of a plane satisfy the two universal relations

$$\mathrm{af} + \mathrm{bg} + \mathrm{ch} = 0,$$

$$\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2 + \mathrm{f}^2 + \mathrm{g}^2 + \mathrm{h}^2 = 1.$$

86. If it proves desirable to indicate, specially, guiding directions of lines the construction of the plane, such as the directions l_1, m_1, n_1, k_1, and n_2, n_2, k_2, the need is met by using the symbols a_{12}, b_{12}, c_{12}, f_{12}, g_{12}, h_{12}. , with this notation, it must be remembered that any two non-coincident ctions in the plane

$$l_1' = \alpha l_1 + \beta l_2,\quad m_1' = \alpha m_1 + \beta m_2,\quad n_1' = \alpha n_1 + \beta n_2,\quad k_1' = \alpha k_1 + \beta k_2,$$
$$l_2' = \gamma l_1 + \delta l_2,\quad m_2' = \gamma m_1 + \delta m_2,\quad n_2' = \gamma n_1 + \delta n_2,\quad k_2' = \gamma k_1 + \delta k_2,$$

re $\alpha\delta - \beta\gamma$ is not zero, can be selected as guiding lines: the orientation-nes, by this change, are invariantive. For, as the new quantities are ction-cosines, we have

$$\Sigma l_1'^2 = 1,\quad \Sigma l_2'^2 = 1,$$

hat

$$\alpha^2 + \beta^2 + 2\alpha\beta\cos\omega = 1,\quad \gamma^2 + \delta^2 + 2\gamma\delta\cos\omega = 1.$$

en ω' denotes the angle between the two new lines,

$$\text{in}^2\,\omega' = \Sigma\,(l_1'm_2' - l_2'm_1')^2 = (\alpha\delta - \beta\gamma)^2\,\Sigma\,(l_1 m_2 - l_2 m_1)^2 = (\alpha\delta - \beta\gamma)^2 \sin^2\omega.$$

is

$$\mathrm{a}_{12}' = \frac{1}{\sin\omega'}(m_1'n_2' - n_1'm_2') = \frac{(\alpha\delta - \beta\gamma)(m_1 n_2 - n_1 m_2)}{(\alpha\delta - \beta\gamma)\sin\omega} = \mathrm{a}_{12},$$

l so for the rest: that is, the orientation-cosines of the plane are invariantive all variations in the choice of guiding lines.

When the equations of the plane are given in the form

$$\left.\begin{aligned} z &= f + px + qy \\ v &= h + rx + sy \end{aligned}\right\},$$

orientation-cosines are

$$\mathrm{a} = -\frac{p}{\Delta},\quad \mathrm{b} = -\frac{q}{\Delta},\quad \mathrm{c} = \frac{1}{\Delta},\quad \mathrm{f} = \frac{s}{\Delta},\quad \mathrm{g} = -\frac{r}{\Delta},\quad \mathrm{h} = \frac{ps - qr}{\Delta}$$

ere

$$\Delta^2 = 1 + p^2 + q^2 + r^2 + s^2 + (ps - qr)^2.$$

e two universal relations satisfied by orientation-cosines are satisfied ntically by these values of a, b, c, f, g, h.

When the equations of the plane are given in the form

$$\left.\begin{aligned} L_1 x + M_1 y + N_1 z + K_1 v &= P_1 \\ L_2 x + M_2 y + N_2 z + K_2 v &= P_2 \end{aligned}\right\},$$

orientation-cosines are

$$\mathrm{a} = \frac{L_1 K_2 - K_1 L_2}{\sin\Omega},\quad \mathrm{b} = \frac{M_1 K_2 - K_1 M_2}{\sin\Omega},\quad \mathrm{c} = \frac{N_1 K_2 - K_1 N_2}{\sin\Omega},$$
$$\mathrm{f} = \frac{M_1 N_2 - N_1 M_2}{\sin\Omega},\quad \mathrm{g} = \frac{N_1 L_2 - L_1 N_2}{\sin\Omega},\quad \mathrm{h} = \frac{L_1 M_2 - M_1 L_2}{\sin\Omega},$$

where

$$\cos\Omega = L_1L_2 + M_1M_2 + N_1N_2 + K_1K_2 .$$

The two universal relations remain as before.

Further, if there be two planes, with orientation-cosines a, b, c, f, g, h, and a′, b′, c′, f′, g′, h′, respectively, their inclination ϕ to one another is given by

$$\cos\phi = \mathrm{aa'} + \mathrm{bb'} + \mathrm{cc'} + \mathrm{ff'} + \mathrm{gg'} + \mathrm{hh'}.$$

Again, the equation of the flat through the plane

$$z = f + px + qy, \quad v = h + rx + sy,$$

and a direction λ, μ, ν, κ, not lying in the plane, is

$$\frac{px + qy - (z - f)}{p\lambda + q\mu - \nu} = \frac{rx + sy - (v - h)}{r\lambda + s\mu - \kappa},$$

that is,

$$\begin{aligned} &x\{(ps - qr)\mu + r\nu - p\kappa\} + y\{(qr - ps)\lambda + s\nu - q\kappa\} \\ &\quad + (z - f)(-r\lambda - s\mu + \kappa) + (v - h)(p\lambda + q\mu - \nu) = 0, \end{aligned}$$

and therefore the direction-cosines L, M, N, K, of its normal are

$$\left.\begin{aligned} L\Theta^{\frac{1}{2}} &= \phantom{-\lambda\mathrm{h}\ } \mu\mathrm{h} - \nu\mathrm{g} + \kappa\mathrm{a} \\ M\Theta^{\frac{1}{2}} &= -\lambda\mathrm{h} \phantom{- \mu\mathrm{f}\ } + \nu\mathrm{f} + \kappa\mathrm{b} \\ N\Theta^{\frac{1}{2}} &= \lambda\mathrm{g} - \mu\mathrm{f} \phantom{+ \nu\mathrm{f}\ } + \kappa\mathrm{c} \\ K\Theta^{\frac{1}{2}} &= -\lambda\mathrm{a} - \mu\mathrm{b} - \nu\mathrm{c} \end{aligned}\right\},$$

where

$$\Theta = \Sigma\,(\mu\mathrm{h} - \nu\mathrm{g} + \kappa\mathrm{a})^2.$$

Ex. Identify this value of Θ with the value given in § 47

Orientation-cosines.

87. The inclination of two planes can be obtained likewise from the projections of an area—say a triangle—upon the coordinate planes of reference. Take any point O in the plane as origin; a line OP_1, of length r_1 and direction-cosines l_1, m_1, n_1, k_1, and a second line OP_2, of length r_2 and direction-cosines l_2, m_2, n_2, k_2; and consider the projection of the triangle OP_1P_2, which is of area

$$\tfrac{1}{2} r_1 r_2 \sin P_2OP_1 = \tfrac{1}{2} r_1 r_2 \sin\omega,$$

upon the planes of reference in turn.

In the plane XOV, the coordinates of the projection of P_1 are $\bar{x}_1 = r_1l_1$, $\bar{y}_1 = 0$, $\bar{z}_1 = 0$, $\bar{v}_1 = r_1k_1$: those of the projection of P_2 are $\bar{x}_2 = r_2l_2$, $\bar{y}_2 = 0$, $\bar{z}_2 = 0$, $\bar{v}_2 = r_2k_2$. The area, in the plane XOV, whose angular points, referred to the axes in that plane, have coordinates 0, 0; $\bar{x}_1$, $\bar{v}_1$; $\bar{x}_2$, $\bar{v}_2$; is

$$\tfrac{1}{2}(\bar{x}_1\bar{v}_2 - \bar{v}_1\bar{x}_2), = \tfrac{1}{2} r_1 r_2 (l_1k_2 - k_1l_2).$$

Hence the cosine of the angle between this projected area and the original area is

$$\frac{l_1k_2-k_1l_2}{\sin\omega};$$

that is, it is the magnitude denoted by f.

Similarly, for the planes of reference XOY, YOZ, ZOX, YOV, ZOV, the cosines of the angles, between the respective projections on these planes and the original triangle, are c, a, b, g, h: that is, the orientation-cosines of the plane are

$$\text{a, b, c, f, g, h.}$$

88. Now let these various projections on the six planes of reference be projected back, in successive addition, on the original plane P_1OP_2. When Δ denotes the area of the original triangle, the various projections are

$$\text{a}\Delta,\ \text{b}\Delta,\ \text{c}\Delta,\ \text{f}\Delta,\ \text{g}\Delta,\ \text{h}\Delta.$$

When these magnitudes are projected back into the plane P_1OP_2, the respective magnitudes of the re-projections are

$$\text{a}\,.\,\text{a}\Delta,\ \text{b}\,.\,\text{b}\Delta,\ \text{c}\,.\,\text{c}\Delta,\ \text{f}\,.\,\text{f}\Delta,\ \text{g}\,.\,\text{g}\Delta,\ \text{h}\,.\,\text{h}\Delta,$$

that is,

$$\text{a}^2\Delta,\ \text{b}^2\Delta,\ \text{c}^2\Delta,\ \text{f}^2\Delta,\ \text{g}^2\Delta,\ \text{h}^2\Delta.$$

But $\text{a}^2+\text{b}^2+\text{c}^2+\text{f}^2+\text{g}^2+\text{h}^2=1$; and therefore the sum of the re-projections is equal to the area of the original triangle.

Again, project the triangle P_1OP_2 in the plane with orientation-cosines a, b, c, f, g, h, into a triangle Q_1OQ_2 in another plane, passing through the point O and having orientation-cosines a′, b′, c′, f′, g′, h′. When we take the projections of P_1OP_2 upon the planes of reference, viz. $\text{a}\Delta$, $\text{b}\Delta$, $\text{c}\Delta$, $\text{f}\Delta$, $\text{g}\Delta$, $\text{h}\Delta$, and then project these constituent elements upon the plane Q_1OQ_2, obtaining respectively

$$\text{a}'.\text{a}\Delta,\ \text{b}'.\text{b}\Delta,\ \text{c}'.\text{c}\Delta,\ \text{f}'.\text{f}\Delta,\ \text{g}'.\text{g}\Delta,\ \text{h}'.\text{h}\Delta,$$

the new aggregate is

$$(\text{aa}'+\text{bb}'+\text{cc}'+\text{ff}'+\text{gg}'+\text{hh}')\,\Delta.$$

But $\text{aa}'+\text{bb}'+\text{cc}'+\text{ff}'+\text{gg}'+\text{hh}'=\cos\phi$, where ϕ is the inclination of the two planes; and therefore this aggregate is $\Delta\cos\phi$, that is, the area of the projected triangle.

Hence plane areas in quadruple space can be projected upon the coordinate planes, and can have these projections combined by further projection upon any plane, in a manner analogous to the manner in which lines are projected upon the coordinate axes and have their projections combined by further projection upon any line.

Inclination of two planes in the same flat.

89. A review of the preceding analysis shews that no account is taken of the range of intersection of the two planes; that is, the range may be only the point a, b, c, d, or it may be a line through that point. With the equations as given, the value obtained for $\cos\phi$ is valid, for either type of range.

But when the intersection is a line, and when this line is in evidence in the equations of the planes, a substantial change can be effected in the expression for $\cos\phi$, which then subsides into the customary expression for the angle of a triangle in three-dimensional spherical trigonometry. To establish this, let the planes be

$$\left\|\begin{matrix} x-a', & y-b', & z-c', & v-d' \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{matrix}\right\|=0, \quad \left\|\begin{matrix} x-a', & y-b', & z-c', & v-d' \\ l_1, & m_1, & n_1, & k_1 \\ l_3, & m_3, & n_3, & k_3 \end{matrix}\right\|=0,$$

so that the line, common to the planes, is

$$\frac{x-a'}{l_1}=\frac{y-b'}{m_1}=\frac{z-c'}{n_1}=\frac{v-d'}{k_1}.$$

Also, let $\theta_{12}=c$, $\theta_{31}=b$, $\theta_{23}=a$; so that

$$\cos a=\Sigma l_2 l_3, \quad \cos b=\Sigma l_3 l_1, \quad \cos c=\Sigma l_1 l_2,$$

and the angles ω and η of the preceding investigation become $\omega=c$ and $\eta=b$. Then, in these circumstances, we have

$$\begin{aligned} & a_{12}a_{13}+b_{12}b_{13}+c_{12}c_{13}+f_{12}f_{13}+g_{12}g_{13}+h_{12}h_{13} \\ &= \Sigma(l_1m_2-m_1l_2)(l_1m_3-m_1l_3) \\ &= \Sigma l_1^2 . \Sigma l_2 l_3-\Sigma l_1 l_2 . \Sigma l_1 l_3 \\ &= \cos a-\cos c\cos b; \end{aligned}$$

and our formula becomes

$$\sin b\sin c\cos\phi=\cos a-\cos b\cos c:$$

that is, the formula gives the angle ϕ, between the planes containing the sides b and c, as equal to the angle A, of a spherical triangle with sides a, b, c, in a three-dimensional space

And this result is to be expected. The two planes, now having a line in common, lie in one flat (§ 43). Accordingly, in this flat, take a sphere centre the origin: draw three radii l_1, m_1, n_1, k_1; l_2, m_2, n_2, k_2; l_3, m_3, n_3, k_3; cutting the sphere in the points A, B, C, respectively. The section of the sphere by the first plane is the arc AB of a great circle, and its section by the second plane is the arc AC of another great circle, while the angle between the planes is the angle A at which these two arcs cut. The customary formula of spherical trigonometry at once leads to the particularised result—a mode of derivation of the inclination which manifestly depends on the circumstance that the two planes happen to intersect in a line and not merely in a point.

Moreover, we can associate the inclination of a line to a plane with this construction. Let the direction of the line through the centre of the sphere meet the sphere in A: with the former notation,

$$AB = \alpha', \quad AC = \beta', \quad BC = \omega, \quad AN = \theta,$$

while AN is the perpendicular from A on BC. In this spherical triangle, a customary formula for AN is

$$\cos^2 AN \sin^2 BC$$
$$= \cos^2 AB + \cos^2 AC - 2 \cos AB \cos AC \cos BC,$$

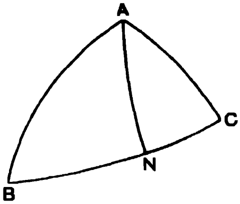

Fig. 4

that is,

$$\cos^2 \theta \sin^2 \omega = \cos^2 \alpha' + \cos^2 \beta' - 2 \cos \alpha' \cos \beta' \cos \omega,$$

the former result. Also

$$\sin^2 \theta \sin^2 \omega = 1 - \cos^2 \alpha' - \cos^2 \beta' - \cos^2 \omega + 2 \cos \alpha' \cos \beta' \cos \omega,$$

which is only another form of the usual relation in a spherical triangle

$$\sin^2 p \sin^2 a = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c,$$

where p is the (angular) perpendicular from A on BC.

Application of the method of § 68, *with a restriction.*

90. It is to be noticed that this result, as regards the inclination of two planes intersecting in a line, can be brought into relation with the earlier method (§ 68) of finding the inclination between a line and a plane.

Take any point on the line as an origin of reference: let the line have direction-cosines l, m, n, k; and let the planes be

$$\left\| \begin{matrix} x, & y, & z, & v \\ l, & m, & n, & k \\ l_1, & m_1, & n_1, & k_1 \end{matrix} \right\| = 0, \qquad \left\| \begin{matrix} x, & y, & z, & v \\ l, & m, & n, & k \\ l_2, & m_2, & n_2, & k_2 \end{matrix} \right\| = 0.$$

The direction-cosines of any line in the first plane are

$$\lambda = pl + ql_1, \quad \mu = pm + qm_1, \quad \nu = pn + qn_1, \quad \kappa = pk + qk_1,$$

with the condition

$$p^2 + q^2 + 2pq \cos c = 1,$$

where $\Sigma ll_1 = \cos c = \cos \alpha'$; and those of any line in the second plane are

$$\lambda' = rl + sl_2, \quad \mu' = rm + sm_2, \quad \nu' = rn + sn_2, \quad \kappa' = rk + sk_2,$$

with the condition

$$r^2 + s^2 + 2rs \cos b = 1,$$

where $\Sigma ll_2 = \cos b = \cos \beta'$. The inclination of these two new lines is

$$\cos \theta = \lambda\lambda' + \mu\mu' + \nu\nu' + \kappa\kappa'$$
$$= pr + qr \cos c + ps \cos b + qs \cos a,$$

where $\Sigma l_1 l_2 = \cos\omega = \cos a$: and the estimate of the inclination of the planes is obtained by making $\cos^2\theta$ a maximum, that is, for values of p, q, r, s, subject to the two conditions

$$p^2 + q^2 + 2pq\cos c = 1, \qquad r^2 + s^2 + 2rs\cos b = 1.$$

The critical equations are

$$\begin{aligned} r + s\cos b &= P\,(p + q\cos c), \\ r\cos c + s\cos a &= P\,(p\cos c + q), \\ p + q\cos c &= Q\,(r + s\cos b), \\ p\cos b + q\cos a &= Q\,(r\cos b + s). \end{aligned}$$

By the customary analysis, we find

$$P = \cos\theta, \qquad Q = \cos\theta.$$

Then

$$r + s\cos b = P\,(p + q\cos c) = PQ\,(r + s\cos b) = (r + s\cos b)\cos^2\theta.$$

Either

$$\cos^2\theta = 1:$$

the planes are parallel; and the line l, m, n, k, lies in the same plane as the lines l_1, m_1, n_1, k_1, and l_2, m_2, n_2, k_2, under the relation

$$1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\cos b\cos c = 0,$$

which requires $a \pm b \pm c = n\pi$, where n is an integer. Or

$$r + s\cos b = 0, \qquad p + q\cos c = 0;$$

and the equations then give

$$\begin{aligned} \sin b\sin c\cos\theta &= \cos a - \cos b\cos c \\ &= \sin b\sin c\cos A, \end{aligned}$$

leading to the former result

$$\theta = A,$$

as the angle between the planes.

Further, we notice that the plane, through the two directions λ, μ, ν, κ, and λ', μ', ν', κ', in the respective planes, is

$$\left\|\begin{matrix} x, & y, & z, & v \\ \lambda, & \mu, & \nu, & \kappa \\ \lambda', & \mu', & \nu', & \kappa' \end{matrix}\right\| = 0.$$

Now

$$\begin{aligned} l\lambda + m\mu + n\nu + k\kappa &= p\Sigma l^2 + q\Sigma ll_1 = p + q\cos c = 0, \\ l\lambda' + m\mu' + n\nu' + k\kappa' &= r\Sigma l^2 + s\Sigma ll_2 = r + s\cos b = 0: \end{aligned}$$

that is, the plane through these two directions is perpendicular to the supposed line of intersection of the two planes—there being, of course, only one plane, at once perpendicular to this line and lying in the flat containing our two original planes.

91. After the last result, it might almost be expected that the inclination of two planes, when they intersect in a point only and not in a line, could be obtained by reference to selected lines: but the expectation is not justified.

We take the one point, common to the planes, as the origin, and we assume that their equations are now

$$\left\| \begin{matrix} x, & y, & z, & v \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{matrix} \right\| = 0, \quad \left\| \begin{matrix} x, & y, & z, & v \\ l_3, & m_3, & n_3, & k_3 \\ l_4, & m_4, & n_4, & k_4 \end{matrix} \right\| = 0.$$

Because the two planes do not meet in a line and so do not coexist in any the same flat, the determinant

$$\Theta^{\frac{1}{2}} = \left| \begin{matrix} l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \\ l_4, & m_4 & n_4, & k_4 \end{matrix} \right|$$

does not vanish.

Two simple propositions must first be established.

(i) it is impossible to draw a line in one plane and a line in the other plane which shall be parallel to one another: and

(ii) it is possible, in an unlimited number of ways, to draw a line in one plane and a line in the other plane which shall be perpendicular to one another.

To establish the prior proposition, we take some line

$$pl_1 + ql_2, \quad pm_1 + qm_2, \quad pn_1 + qn_2, \quad pk_1 + qk_2,$$

in the first plane, and some line

$$rl_3 + sl_4, \quad rm_3 + sm_4, \quad rn_3 + sn_4, \quad rk_3 + sk_4,$$

in the second plane. If for any values of p and q, and for any values of r and s, these lines can be parallel, we should have

$$\begin{aligned} pl_1 + ql_2 &= \mu (rl_3 + sl_4), \\ pm_1 + qm_2 &= \mu (rm_3 + sm_4), \\ pn_1 + qn_2 &= \mu (rn_3 + sn_4), \\ pk_1 + qk_2 &= \mu (rk_3 + sk_4), \end{aligned}$$

where the value of μ is

$$(p^2 + q^2 + 2pq \cos \theta_{12})^{\frac{1}{2}} (r^2 + s^2 + 2rs \cos \theta_{34})^{-\frac{1}{2}}.$$

In order that the four relations may hold, we must have

$$\left| \begin{matrix} l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \\ l_4, & m_4, & n_4, & k_4 \end{matrix} \right| = 0.$$

that is, $\Theta = 0$, a condition which is excluded. There are no non-zero values of p, q, r, s, satisfying the four relations. It is therefore not possible to draw two parallel lines lying in the respective planes. (Should the planes have a common line, directions in the respective planes parallel to that line are parallel to one another.)

To establish the later proposition, we postulate the same two lines in the respective planes. In order that they may be perpendicular, we must have

$$\Sigma (pl_1 + ql_2)(rl_3 + sl_4) = 0 ,$$

and therefore

$$r(p \cos \theta_{13} + q \cos \theta_{23}) + s(p \cos \theta_{14} + q \cos \theta_{24}) = 0.$$

Also

$$r^2 + s^2 + 2rs \cos \theta_{34} = 1 ;$$

it is therefore possible to determine r and s, whenever p and q are assigned, or we can always draw a line in the second plane perpendicular to an arbitrarily assumed line in the first plane. Similarly, if a line be arbitrarily assumed in the second plane, it is always possible to draw a perpendicular line in the first plane.

Least angular distance between lines in two planes.

92. We now take a line

$$l = pl_1 + ql_2, \quad m = pm_1 + qm_2, \quad n = pn_1 + qn_2, \quad k = pk_1 + qk_2,$$

in the first plane, and a line

$$\lambda = rl_3 + sl_4, \quad \mu = rm_3 + sm_4, \quad \nu = rn_3 + sn_4, \quad \kappa = rk_3 + sk_4,$$

in the second plane, where the disposable parameters p, q, r, s, are subject to the two relations

$$p^2 + q^2 + 2pq \cos \theta_{12} = 1, \quad r^2 + s^2 + 2rs \cos \theta_{34} = 1.$$

The inclination of the two lines is given by

$$\cos \psi = l\lambda + m\mu + n\nu + k\kappa.$$

After the preceding proposition, ψ cannot be 0 or π, so that $\cos^2 \psi$ cannot attain the value unity; but, by appropriate choice, ψ may be made $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$, so that $\cos^2 \psi$ can attain the value zero. We therefore seek a maximum value for $\cos^2 \psi$; and we denote, by θ, the associated value of ψ.

This maximum value is attained for the possible range of values of p, q, r, s, which are subject to the two specified relations. The value of $\cos \psi$ is

$$\cos \theta = pr \cos \theta_{13} + ps \cos \theta_{14} + qr \cos \theta_{23} + qs \cos \theta_{24}.$$

Thus the critical equations are

$$\begin{aligned} r\cos\theta_{13} + s\cos\theta_{14} &= A\,(p + q\cos\theta_{12}),\\ r\cos\theta_{23} + s\cos\theta_{24} &= A\,(p\cos\theta_{12} + q),\\ p\cos\theta_{13} + q\cos\theta_{23} &= B\,(r + s\cos\theta_{34}),\\ p\cos\theta_{14} + q\cos\theta_{24} &= B\,(r\cos\theta_{34} + s),\end{aligned}$$

where initially A and B are two indeterminate multipliers, the values of which can however be determined at once.

We multiply the first and second of the critical equations by p and q, and add: we multiply the third and fourth of the critical equations by r and s, and add: the results are

$$\cos\theta = A, \quad \cos\theta = B.$$

When the values of A and B are inserted, the critical equations become

$$\left.\begin{aligned} (p + q\cos\theta_{12})\cos\theta &= r\cos\theta_{13} + s\cos\theta_{14}\\ (p\cos\theta_{12} + q)\cos\theta &= r\cos\theta_{23} + s\cos\theta_{24}\\ p\cos\theta_{13} + q\cos\theta_{23} &= (r + s\cos\theta_{34})\cos\theta\\ p\cos\theta_{14} + q\cos\theta_{24} &= (r\cos\theta_{34} + s)\cos\theta \end{aligned}\right\}.$$

Accordingly, the value of $\cos\theta$ is given by the equation

$$\begin{vmatrix} \cos\theta, & \cos\theta\cos\theta_{12}, & \cos\theta_{13}, & \cos\theta_{14} \\ \cos\theta\cos\theta_{12}, & \cos\theta, & \cos\theta_{23}, & \cos\theta_{24} \\ \cos\theta_{13}, & \cos\theta_{23}, & \cos\theta, & \cos\theta\cos\theta_{34} \\ \cos\theta_{14}, & \cos\theta_{24}, & \cos\theta\cos\theta_{34}, & \cos\theta \end{vmatrix} = 0.$$

When this determinant is expanded, we have

$$(1-\cos^2\theta_{12})(1-\cos^2\theta_{34})\cos^4\theta - \Omega\cos^2\theta + (\cos\theta_{13}\cos\theta_{24} - \cos\theta_{14}\cos\theta_{23})^2 = 0,$$

where

$$\begin{aligned}\Omega = \cos^2\theta_{13} &+ \cos^2\theta_{14} + \cos^2\theta_{23} + \cos^2\theta_{24}\\ &+ 2\cos\theta_{12}\cos\theta_{34}(\cos\theta_{13}\cos\theta_{24} + \cos\theta_{14}\cos\theta_{23})\\ &- 2\cos\theta_{12}\cos\theta_{23}\cos\theta_{31} - 2\cos\theta_{23}\cos\theta_{34}\cos\theta_{42}\\ &- 2\cos\theta_{31}\cos\theta_{14}\cos\theta_{43} - 2\cos\theta_{14}\cos\theta_{42}\cos\theta_{21}.\end{aligned}$$

Accordingly, this is the equation determining $\cos^2\theta$. It is a quadratic in $\cos^2\theta$, not linear, as might have been expected from the earlier analysis.

We proceed to some inferences from the equation.

93. When we recur to the expression (§ 81) for the magnitude Θ, which vanishes when the four directions l_p, m_p, n_p, k_p (for $p = 1, 2, 3, 4$) lie in one flat and which (being equal to R^2) otherwise is positive, we have

$$\begin{aligned}\Omega + \Theta = 1 &- \cos^2\theta_{12} - \cos^2\theta_{34} \\ &+ \cos^2\theta_{12}\cos^2\theta_{34} + \cos^2\theta_{13}\cos^2\theta_{24} + \cos^2\theta_{14}\cos^2\theta_{23} \\ &- 2\cos\theta_{13}\cos\theta_{14}\cos\theta_{23}\cos\theta_{24} \\ = (1 &- \cos^2\theta_{12})(1-\cos^2\theta_{34}) + (\cos\theta_{13}\cos\theta_{24} - \cos\theta_{14}\cos\theta_{23})^2.\end{aligned}$$

Also

$$\begin{aligned}\cos\theta_{13}\cos\theta_{24} &- \cos\theta_{14}\cos\theta_{23} \\ &= \Sigma l_1 l_3 . \Sigma l_2 l_4 - \Sigma l_1 l_4 . \Sigma l_2 l_3 \\ &= a_{12}a_{34} + b_{12}b_{34} + c_{12}c_{34} + f_{12}f_{34} + g_{12}g_{34} + h_{12}h_{34} \\ &= \sin\theta_{12}\sin\theta_{34}\cos\phi,\end{aligned}$$

where ϕ is the inclination of the planes. Hence

$$\Omega + \Theta = \sin^2\theta_{12}\sin^2\theta_{34}(1+\cos^2\phi).$$

The result, obtained in § 92, was

$$\sin^2\theta_{12}\sin^2\theta_{34}\cos^4\theta - \Omega\cos^2\theta + \sin^2\theta_{12}\sin^2\theta_{34}\cos^2\phi = 0.$$

We therefore have

$$\frac{\Theta\cos^2\theta}{\sin^2\theta_{12}\sin^2\theta_{34}} = (1-\cos^2\theta)(\cos^2\theta - \cos^2\phi).$$

Now when Θ is not zero, so that the two planes do not lie in one and the same flat and no line in one plane can be drawn parallel to a line in the other, whatever lines be chosen, θ cannot be equal to zero, thus $1-\cos^2\theta$ cannot be zero. Hence

$$\cos^2\theta > \cos^2\phi;$$

and therefore, taking for ϕ an inclination less than $\frac{1}{2}\pi$,

$$0 < \theta < \phi < \tfrac{1}{2}\pi < \pi - \phi < \pi - \theta < \pi.$$

But whatever value be chosen for ϕ, we cannot have θ equal to ϕ, that is, *the minimum angle between two lines appropriately chosen in the respective planes is not equal to the inclination of the planes when they meet in a point and not in a line.*

Next, suppose that the planes do meet in a line and not merely in a point. Then

$$\Theta = 0:$$

consequently either $\cos^2\theta = 1$, or $\cos^2\theta = \cos^2\phi$. Now, on the present hypothesis that the planes meet in a line, $\cos^2\theta = 1$ provides a maximum value for $\cos^2\psi$, where ψ is the inclination of two lines drawn in the respective planes: it arises when two lines are drawn parallel to the direction of intersection. Manifestly it is a value irrelevant to the inclination of the planes. The alternative is

$$\cos^2\phi = \cos^2\theta,$$

or $\phi = \theta$: that is, the inclination of the planes is measured by the inclination of the two directions providing a minimum value of $\cos^2\psi$. And it has already (p. 148) been pointed out that, in such circumstances, the plane through the two selected directions is perpendicular to the direction of intersection.

We shall return later (§ 109) to the consideration of the critical equations which serve to determine p, q, r, s.

Ex. 1. Two planes, which meet in the origin only and not in a line through the origin, are given by the equations

$$\left.\begin{array}{l} z=px+qy \\ v=rx+sy \end{array}\right\}, \quad \left.\begin{array}{l} z=p'x+q'y \\ v=r'x+s'y \end{array}\right\}$$

A line is drawn in the first plane, and another in the second plane, and their inclination is θ; prove that the minimum value of $\cos^2\theta$ is given by

$$(AB-C^2)(A'B'-C'^2)\cos^4\theta-\Omega\cos^2\theta+(ad-bc)^2=0,$$

where

$$A=1+p^2+r^2, \quad C=pq+rs, \quad B=1+q^2+s^2,$$
$$A'=1+p'^2+r'^2, \quad C'=p'q'+r's', \quad B'=1+q'^2+s'^2,$$
$$a=1+pp'+rr', \quad b=pq'+rs', \quad c=qp'+sr', \quad d=1+qq'+ss',$$

and Ω denotes

$$A(A'd^2-2C'cd+B'c^2)-2C\{A'bd-C'(ad+bc)+B'ac\}+B(A'b^2-2C'ab+B'a^2).$$

Obtain the relations

$$ad-bc=T, \quad AB-C^2=S, \quad A'B'-C'^2=S',$$

where S, S', T are the quantities of § 84, and bring the foregoing equation into the form

$$\frac{\begin{vmatrix} p'-p, & q'-q \\ r'-r, & s'-s \end{vmatrix}^2}{SS'}=(1-\cos^2\theta)(\cos^2\theta-\cos^2\phi),$$

where ϕ is the inclination of the two planes to one another.

Ex. 2. Find the angle between the planes XOY and OCa in the figure on p. 7

Another expression for the inclination of two planes.

94. The expression for the inclination of two planes, of which the respective orientation-cosines are a, b, c, f, g, h, and a′, b′, c′, f′, g′, h′, has been obtained in the form

$$\cos\phi=\mathrm{aa}'+\mathrm{bb}'+\mathrm{cc}'+\mathrm{ff}'+\mathrm{gg}'+\mathrm{hh}'.$$

When the planes are given in their canonical forms

$$\left.\begin{array}{l} z=f+px+qy \\ v=h+rx+sy \end{array}\right\}, \quad \left.\begin{array}{l} z=f'+p'x+q'y \\ v=h'+r'x+s'y \end{array}\right\},$$

the expression is

$$SS'\cos\phi=T,$$

where

$$S^2=1+p^2+q^2+r^2+s^2+(ps-qr)^2,$$
$$T=1+pp'+qq'+rr'+ss'+(ps-qr)(p's'-q'r'),$$
$$S'^2=1+p'^2+q'^2+r'^2+s'^2+(p's'-q'r')^2.$$

But an expression, having a somewhat more obviously geometrical form, can be given for the former value of $\cos\phi$, when the planes have their equations in either of the customary non-canonical forms. When their equations are

$$\left\|\begin{matrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{matrix}\right\| = 0, \qquad \left\|\begin{matrix} x-a', & y-b', & z-c', & v-d' \\ l_3\ , & m_3\ , & n_3\ , & k_3 \\ l_4\ , & m_4\ , & n_4\ , & k_4 \end{matrix}\right\| = 0,$$

we denote by θ_{ij} the inclination of the directions l_i, m_i, n_i, k_i, and l_j, m_j, n_j, k_j, so that

$$\cos\theta_{ij} = l_i l_j + m_i m_j + n_i n_j + k_i k_j,$$

for all the values of i and j. Then, as

$$a = \frac{m_1 n_2 - n_1 m_2}{\sin\theta_{12}}, \quad a' = \frac{m_3 n_4 - n_3 m_4}{\sin\theta_{34}},$$

and so for the other orientation-cosines, we have

$$\sin\theta_{12}\sin\theta_{34}\cos\phi = U,$$

where

$$U = \Sigma\,(l_1 m_2 - m_1 l_2)(l_3 m_4 - m_3 l_4),$$

the summation being over all the pairs of corresponding symbols from the two planes. Now

$$\begin{aligned}&(l_1 m_2 - m_1 l_2)(l_3 m_4 - m_3 l_4)\\ &\quad = (l_1 l_3 + m_1 m_3)(l_2 l_4 + m_2 m_4) - (l_2 l_3 + m_2 m_3)(l_1 l_4 + m_1 m_4),\end{aligned}$$

and similarly for each of the other products; hence

$$\begin{aligned}U &= \Sigma l_1 l_3 \,.\, \Sigma l_2 l_4 - \Sigma l_2 l_3 \,.\, \Sigma l_1 l_4\\ &= \cos\theta_{13}\cos\theta_{24} - \cos\theta_{23}\cos\theta_{14}.\end{aligned}$$

Consequently the inclination ϕ of two planes, one through the directions l_1, m_1, n_1, k_1, and l_2, m_2, n_2, k_2, the other through the directions l_3, m_3, n_3, k_3, and l_4, m_4, n_4, k_4, is given by

$$\sin\theta_{12}\sin\theta_{34}\cos\phi = \cos\theta_{13}\cos\theta_{24} - \cos\theta_{23}\cos\theta_{14}.$$

Similarly, when the two planes are given by

$$\left.\begin{aligned}L_1x + M_1y + N_1z + K_1v = P_1\\ L_2x + M_2y + N_2z + K_2v = P_2\end{aligned}\right\}, \quad \left.\begin{aligned}L_3x + M_3y + N_3z + K_3v = P_3\\ L_4x + M_4y + N_4z + K_4v = P_4\end{aligned}\right\},$$

and when ϑ_{ij} is the inclination between the normal to the flat $\Sigma L_i x = P_i$ and the normal to the flat $\Sigma L_j x = P_j$, so that

$$\cos\vartheta_{ij} = L_i L_j + M_i M_j + N_i N_j + K_i K_j,$$

the inclination ϕ of the planes is given by

$$\sin\vartheta_{12}\sin\vartheta_{34}\cos\phi = \cos\vartheta_{13}\cos\vartheta_{24} - \cos\vartheta_{23}\cos\vartheta_{14}.$$

Ex. Shew that if the planes 12 and 34 meet in a point only, the planes 23 and 14 also meet in a point only, and likewise the planes 31 and 24.

Prove that, if the respective inclinations of these pairs of planes are ϕ, χ, ψ, they satisfy the relation

$$\sin 12 \sin 34 \cos\phi + \sin 23 \sin 14 \cos\chi + \sin 31 \sin 24 \cos\psi = 0.$$

The relation of parallelism between two planes.

95. The conditions for the parallelism of two planes have already (§§ 43, 81) been set out; consequently, their discussion in connection with the expression for the inclination of the two planes can be brief. The condition for parallelism, in this connection, is

$$\phi = 0,$$

and therefore

$$\cos \phi = 1.$$

When the equations are given in their canonical forms, this condition requires

$$SS' = T,$$

that is,

$$\{1 + p^2 + q^2 + r^2 + s^2 + (ps - qr)^2\}\{1 + p'^2 + q'^2 + r'^2 + s'^2 + (p's' - q'r')^2\}$$
$$= \{1 + pp' + qq' + rr' + ss' + (ps - qr)(p's' - q'r')\}^2,$$

a relation which can only be satisfied if

$$p' = p, \quad q' = q, \quad r' = r, \quad s' = s.$$

These are the conditions already (§ 81) stated for this form.

For any non-canonical form, the condition of parallelism is

$$\mathrm{aa}' + \mathrm{bb}' + \mathrm{cc}' + \mathrm{ff}' + \mathrm{gg}' + \mathrm{hh}' = 1,$$

with the conditions

$$\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2 + \mathrm{f}^2 + \mathrm{g}^2 + \mathrm{h}^2 = 1 = \mathrm{a}'^2 + \mathrm{b}'^2 + \mathrm{c}'^2 + \mathrm{f}'^2 + \mathrm{g}'^2 + \mathrm{h}'^2.$$

These equations require the relations

$$\mathrm{a}' = \mathrm{a}, \quad \mathrm{b}' = \mathrm{b}, \quad \mathrm{c}' = \mathrm{c}, \quad \mathrm{f}' = \mathrm{f}, \quad \mathrm{g}' = \mathrm{g}, \quad \mathrm{h}' = \mathrm{h};$$

that is, for one of the usual non-canonical forms,

$$\frac{m_3n_4 - n_3m_4}{\sin\theta_{34}} = \frac{m_1n_2 - n_1m_2}{\sin\theta_{12}}, \quad \frac{l_3k_4 - k_3l_4}{\sin\theta_{34}} = \frac{l_1k_2 - k_1l_2}{\sin\theta_{12}}$$
$$\frac{n_3l_4 - l_3n_4}{\sin\theta_{34}} = \frac{n_1l_2 - l_1n_2}{\sin\theta_{12}}, \quad \frac{m_3k_4 - k_3m_4}{\sin\theta_{34}} = \frac{m_1k_2 - k_1m_2}{\sin\theta_{12}},$$
$$\frac{l_3m_4 - m_3l_4}{\sin\theta_{34}} = \frac{l_1m_2 - m_1l_2}{\sin\theta_{12}} \quad \frac{n_3k_4 - k_3n_4}{\sin\theta_{34}} = \frac{n_1k_2 - k_1n_2}{\sin\theta_{12}}.$$

Hence

$$\left\|\begin{matrix} \alpha l_3 + \beta l_4, & \alpha m_3 + \beta m_4, & \alpha n_3 + \beta n_4, & \alpha k_3 + \beta k_4 \\ l_1 \quad , & m_1 \quad , & n_1 \quad , & k_1 \\ l_2 \quad , & m_2 \quad , & n_2 \quad , & k_2 \end{matrix}\right\| = 0:$$

and therefore the parallel to any direction in the second plane, which has l_3, m_3, n_3, k_3, and l_4, m_4, n_4, k_4, for guiding lines, is contained also in the first plane: a property characteristic, unique and complete, of the parallelism.

If the two planes meet in a point only, there is no direction in either plane which has a parallel in the other. When such directions are possible we have

$$\begin{aligned}\gamma l_1 + \delta l_2 &= \sigma(\alpha l_3 + \beta l_4),\\ \gamma m_1 + \delta m_2 &= \sigma(\alpha m_3 + \beta m_4),\\ \gamma n_1 + \delta n_2 &= \sigma(\alpha n_3 + \beta n_4),\\ \gamma k_1 + \delta k_2 &= \sigma(\alpha k_3 + \beta k_4),\end{aligned}$$

where

$$\sigma = \left(\frac{\gamma^2 + \delta^2 + 2\gamma\delta\cos\theta_{12}}{\alpha^2 + \beta^2 + 2\alpha\beta\cos\theta_{34}}\right)^{\frac{1}{2}}.$$

In order that these equations may coexist, it is necessary that the relation

$$\begin{vmatrix} l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \\ l_4, & m_4, & n_4, & k_4 \end{vmatrix} = 0$$

shall be satisfied. and the relation is not satisfied when the two planes meet in a point only. In that circumstance, neither plane contains a direction parallel to a direction in the other plane.

If the two planes do not meet in a point only, but meet in a line (which may lie at infinity), a guiding direction common to the two planes can be selected, being the direction of this line; let it be l, m, n, k. When another guiding direction for the first plane is l_2, m_2, n_2, k_2, and another guiding direction for the second plane is l_4, m_4, n_4, k_4, a direction in the first plane can be parallel to the second plane, if

$$\begin{aligned}\lambda l + \mu l_2 &= \sigma(\alpha l + \beta l_4),\\ \lambda m + \mu m_2 &= \sigma(\alpha m + \beta m_4),\\ \lambda n + \mu n_2 &= \sigma(\alpha n + \beta n_4),\\ \lambda k + \mu k_2 &= \sigma(\alpha k + \beta k_4).\end{aligned}$$

These equations can be satisfied, if

$$\lambda = \sigma\alpha, \quad \mu = 0, \quad \beta = 0.$$

that is, if all parallel directions in the two planes are parallel to l, m, n, k; that is effectively, if there is no such direction which is not parallel to the common line. Or they can be satisfied, if

$$\begin{Vmatrix} l_4, & m_4, & n_4, & k_4 \\ l\ , & m\ , & n\ , & k \\ l_2, & m_2, & n_2, & k_2 \end{Vmatrix} = 0$$

so that, now, every line in the second plane finds a parallel direction in the first plane: the two planes are coincident, if their common line is in the finite part of space: they are parallel, if their common line lies at infinity.

These results may therefore be summarised as follows:

(i) if two planes meet in a point only, there exists no direction in either parallel to a direction in the other.

(ii) if two planes meet in a line, every direction in one parallel to that line is parallel to every direction in the other parallel to the line, and such directions may be the only parallel directions:

(iii) if two planes meet in a line not at infinity, they may be coincident·

(iv) if two planes meet in a line at infinity, they may be parallel.

The relations between two planes at right angles.

96. The condition, that two planes may be regarded as being at right angles, is that the quantity ϕ shall be $\frac{1}{2}\pi$; and therefore we have

$$\cos\theta_{13}\cos\theta_{24} - \cos\theta_{23}\cos\theta_{14} = 0$$

as a necessary condition; with, of course, the equivalent condition $T = 0$, when the equations of the plane are in canonical form.

We have already seen (§ 67) that there are two kinds of relation in which planes can be regarded as at right angles. In one type of relation, every line in either plane is perpendicular to every line in the other plane, in the other type of relation, one direction certainly (and consequently every parallel direction) is perpendicular to every direction in the other plane. We may note a third type of relation: because, given any arbitrary direction in one plane, there will be some direction perpendicular to it in the other plane. It therefore is desirable to investigate the types of relation in which two planes must stand so that they can be regarded as being at right angles

We shall take a common point of the planes as origin; and we represent the two planes by the two sets of equations

$$\left\| \begin{matrix} x, & y, & z, & v \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{matrix} \right\| = 0, \qquad \left\| \begin{matrix} x, & y, & z, & v \\ l_3, & m_3, & n_3, & k_3 \\ l_4, & m_4, & n_4, & k_4 \end{matrix} \right\| = 0.$$

Typical directions in the first plane and in the second plane are given by

$$\lambda = \alpha l_1 + \beta l_2, \quad \mu = \alpha m_1 + \beta m_2, \quad \nu = \alpha n_1 + \beta n_2, \quad \kappa = \alpha k_1 + \beta k_2,$$
$$\lambda' = \gamma l_3 + \delta l_4, \quad \mu' = \gamma m_3 + \delta m_4, \quad \nu' = \gamma n_3 + \delta n_4, \quad \kappa' = \gamma k_3 + \delta k_4,$$

respectively.

The third type of relation requires only brief consideration. When a direction λ, μ, ν, κ, is arbitrarily assumed in the first plane, a perpendicular direction λ', μ', ν', κ', can be obtained in the second plane. The necessary condition is

$$\gamma(\alpha\cos\theta_{13} + \beta\cos\theta_{23}) + \delta(\alpha\cos\theta_{14} + \beta\cos\theta_{24}) = 0;$$

and therefore, as

$$\gamma^2 + \delta^2 + 2\gamma\delta\cos\theta_{34} = 1,$$

we have values of γ and δ, that are unique (save as to sign): that is, we have a direction λ', μ', ν', κ', in the second plane, unique (save as to sense), and perpendicular to λ, μ, ν, κ.

Similarly, if a direction λ', μ', ν', κ', is arbitrarily assumed in the second plane, we can find a direction in the first plane, unique (save as to sense), and perpendicular to λ', μ', ν', κ'.

Thus there is nothing distinctive, arising out of this kind of relation: there is no limitation of any nature upon the planes. This property is merely an incident common to any pair of planes: it is ignorable as regards any relation of perpendicularity for the whole of the first plane, or the whole of the second plane, or both, because it is concerned only with single lines.

There remains the consideration of the other two types of relation.

Perpendicular planes.

97. We have seen (§§ 33, 67) that a line can be perpendicular to every direction in a plane, and yet be not unique in the possession of that property. We have also seen (§ 22) that the locus of lines through a point perpendicular to a given line is a flat, so that every plane in the flat is perpendicular to the line: yet no plane in the flat is unique in the possession of that property. We therefore proceed to enquire in what circumstances a line or lines in one plane can be perpendicular to every direction in the other plane.

Consider the direction λ', μ', ν', κ', in the second plane. If it is perpendicular to every direction in the first plane, the relation

$$\lambda' (\alpha l_1 + \beta l_2) + \mu' (\alpha m_1 + \beta m_2) + \nu' (\alpha n_1 + \beta n_2) + \kappa' (\alpha k_1 + \beta k_2) = 0$$

must be satisfied for all values of α and β, subject to

$$\alpha^2 + \beta^2 + 2\alpha\beta \cos \theta_{12} = 1.$$

Among these possible values are $\alpha = 1$, $\beta = 0$; and $\alpha = 0$, $\beta = 1$, which lead to the respective relations

$$\Sigma\lambda' l_1 = 0, \quad \Sigma\lambda' l_2 = 0.$$

Conversely, when these are satisfied, we have

$$\Sigma\lambda' (\alpha l_1 + \beta l_2) = 0,$$

for all values of α and β: that is, the two relations secure the property that the direction λ', μ', ν', κ', in the second plane is perpendicular to every direction in the first plane.

When we substitute the values of λ', μ', ν', κ', the relations become

$$\left.\begin{array}{l} \gamma \cos \theta_{13} + \delta \cos \theta_{14} = 0 \\ \gamma \cos \theta_{23} + \delta \cos \theta_{24} = 0 \end{array}\right\},$$

while

$$\gamma^2 + \delta^2 + 2\gamma\delta \cos \theta_{34} = 1,$$

an equation which precludes the possibility

$$\gamma = 0, \quad \delta = 0.$$

It follows that, if both relations exist, we must have

$$\cos \theta_{13} \cos \theta_{24} - \cos \theta_{23} \cos \theta_{14} = 0,$$

and when this relation is satisfied, the two relations are equivalent to one only. That single surviving equation, together with the relation

$$\gamma^2 + \delta^2 + 2\gamma\delta \cos \theta_{34} = 1,$$

determines values of γ and δ that are unique save as to sign; and consequently, under the assumptions made, there is a single direction in the second plane perpendicular to every direction in the first plane. And the assumptions, which have been made, are, firstly,

$$\cos \theta_{13} \cos \theta_{24} - \cos \theta_{23} \cos \theta_{14} = 0,$$

and secondly, that both the relations

$$\gamma \cos \theta_{13} + \delta \cos \theta_{14} = 0, \quad \gamma \cos \theta_{23} + \delta \cos \theta_{24} = 0,$$

exist, that is, are not evanescent.

Now it might happen that one of these two relations is evanescent. If it were the first, we should have

$$\cos \theta_{13} = 0, \quad \cos \theta_{14} = 0,$$

and if it were the second, we should have

$$\cos \theta_{23} = 0, \quad \cos \theta_{24} = 0,$$

all four conditions being necessary to reduce both relations to evanescence. We might have the first relation evanescent, and then the direction l_1, m_1, n_1, k_1, is perpendicular to the second plane: that is, every direction in the second plane is perpendicular to the particular direction l_1, m_1, n_1, k_1, in the first plane. In order that the particular direction $\lambda', \mu', \nu', \kappa'$, may be perpendicular to the direction l_2, m_2, n_2, k_2, (and therefore, now, perpendicular to the first plane), we must have

$$\gamma \cos \theta_{23} + \delta \cos \theta_{24} = 0 \cdot$$

a non-evanescent relation which serves for a determination of γ and δ, unique save as to sign: that is, there is a single line $\lambda', \mu', \nu', \kappa'$, perpendicular to the first plane.

We could even have either $\cos \theta_{23} = 0$, and then $\delta = 0$, that is, the direction l_3, m_3, n_3, k_3, is perpendicular to the first plane: or $\cos \theta_{24} = 0$, and then $\gamma = 0$, that is, the direction l_4, m_4, n_4, k_4, is perpendicular to the first plane. In each set of conditions, the relation

$$\cos \theta_{13} \cos \theta_{24} - \cos \theta_{23} \cos \theta_{14} = 0$$

is satisfied, and the two relations

$$\gamma \cos\theta_{13} + \delta\cos\theta_{14} = 0, \quad \gamma\cos\theta_{23} + \delta\cos\theta_{24} = 0,$$

do not simultaneously become evanescent: and, for each set of conditions, there is one (and there is only one) direction in the second plane perpendicular to the first plane.

98. But now, when these conditions are satisfied, there is a reciprocal inference: there is one (and there is only one) direction in the first plane perpendicular to the second plane. The direction λ, μ, ν, κ, in the first plane is perpendicular to the direction l_3, m_3, n_3, k_3, in the second plane if

$$\Sigma\lambda l_3 = \alpha\cos\theta_{13} + \beta\cos\theta_{23} = 0.$$

But owing to the relation

$$\cos\theta_{13}\cos\theta_{24} - \cos\theta_{23}\cos\theta_{14} = 0,$$

we now have

$$\alpha\cos\theta_{14} + \beta\cos\theta_{24} = 0.$$

Hence the direction λ, μ, ν, κ, chosen perpendicular to the direction l_3, m_3, n_3, k_3, is perpendicular also to the direction l_4, m_4, n_4, k_4: that is, it is perpendicular to the second plane. Moreover, the relations

$$\alpha\cos\theta_{13} + \beta\cos\theta_{23} = 0, \quad \alpha^2 + \beta^2 + 2\alpha\beta\cos\theta_{12} = 1,$$

determine values of α and β, which are unique save as to sign; that is, under the set of conditions, there is one (and there is only one) direction in the first plane perpendicular to the second plane.

But, in order that these results may follow, it is necessary that not both the equations

$$\gamma\cos\theta_{13} + \delta\cos\theta_{14} = 0, \quad \gamma\cos\theta_{23} + \delta\cos\theta_{24} = 0,$$

shall be evanescent—one may be evanescent, but not both. There is also a necessity that not both the equations

$$\alpha\cos\theta_{13} + \beta\cos\theta_{23} = 0, \quad \alpha\cos\theta_{14} + \beta\cos\theta_{24} = 0,$$

shall be evanescent—one may be evanescent, but not both.

Orthogonal planes.

99. If both equations in the first pair are evanescent, then

$$\cos\theta_{13} = 0, \quad \cos\theta_{14} = 0, \quad \cos\theta_{23} = 0, \quad \cos\theta_{24} = 0.$$

Should these conditions be satisfied, then both the equations in the second pair are evanescent. The conditions are necessary and sufficient to secure that both pairs of equations are evanescent; and, of course, the condition

$$\cos\theta_{13}\cos\theta_{24} - \cos\theta_{23}\cos\theta_{14} = 0$$

is satisfied. We now have

$$\begin{aligned}\Sigma\lambda\lambda' &= \Sigma\,(\alpha l_1 + \beta l_2)(\gamma l_3 + \delta l_4)\\ &= \alpha\gamma\cos\theta_{13} + \beta\gamma\cos\theta_{23} + \alpha\delta\cos\theta_{14} + \beta\delta\cos\theta_{24}\\ &= 0:\end{aligned}$$

or every direction in the first plane is perpendicular to every direction in the second plane.

Thus there are two kinds of relation of perpendicularity possible between two planes. In one kind of relation, each plane contains one (and only one) direction perpendicular to every direction in the other plane. In the alternative kind of relation, every direction in either plane is perpendicular to every direction in the other plane.

Accordingly, we define two planes as *orthogonal* to one another, when every direction in either plane is at right angles to every direction in the other plane; and we define two planes as *perpendicular* to one another when each plane contains one (but only one) direction at right angles to every direction in the other plane. Finally, when two planes are such that each contains only one direction at right angles to any arbitrarily assumed direction in the other—a selection that always is possible—we do not regard this incident of isolated perpendicularity, which is common to all planes in pairs, as constituting a relation of perpendicularity.

The *conditions for orthogonality* are

$$\cos\theta_{13} = 0, \quad \cos\theta_{14} = 0, \quad \cos\theta_{23} = 0, \quad \cos\theta_{24} = 0.$$

If the four quantities $\cos\theta_{13}$, $\cos\theta_{14}$, $\cos\theta_{23}$, $\cos\theta_{24}$, do not necessarily vanish individually, yet satisfy the equation

$$\cos\theta_{13}\cos\theta_{24} - \cos\theta_{23}\cos\theta_{14} = 0,$$

the planes are perpendicular. This single relation is the *condition for perpendicularity*.

NOTE. We have seen that it is possible to draw a line in one plane parallel to a line in another plane, only if they lie in one flat: an extreme case, when the directions are not unique, arises when the two planes are parallel.

On the other hand, it is always possible to find one line in a plane and one line in another plane such that the two lines are at right angles. When the planes are perpendicular, any line can be taken in one plane, and there is one line in the other at right angles to every such line. When the planes are orthogonal, every line in either is at right angles to every line in the other.

If, then, χ denote the inclination of a line in one plane to a line in another plane, usually $\cos^2\chi$ cannot be equal to unity: a condition is required if $\cos^2\chi$ can be equal to unity. But lines can always be chosen so that $\cos^2\chi$ can be equal to zero. It is therefore to be expected that, when two planes are arbitrarily chosen, there will be a minimum value of $\cos^2\chi$ different from zero; the investigation of this value has already (§ 93) been effected, and it shews that the angle χ is not equal to the inclination of the two planes when they do not lie in one and the same flat.

Ex 1 Shew that the planes

$$\left.\begin{aligned} z&=px+qy\\ v&=rx+sy\end{aligned}\right\},\quad \left.\begin{aligned} z&=p'x+q'y\\ v&=r'x+s'y\end{aligned}\right\},$$

are orthogonal, provided

$$\begin{aligned} 1+pp'+qq'&=0, & pq'+rs'&=0,\\ 1+rr'+ss'&=0, & qp'+sr'&=0.\end{aligned}$$

Ex 2. A plane is drawn through $\alpha, \beta, \gamma, \delta$, orthogonal to the plane $x=pz+qv, y=rz+sv$; obtain its equations in the form

$$\left.\begin{aligned} (ps-qr)(x-\alpha)+s(z-\gamma)-r(v-\delta)&=0\\ (ps-qr)(y-\beta)-q(z-\gamma)+p(v-\delta)&=0\end{aligned}\right\}.$$

Ex. 3. If two planes meet in a line, the planes through any point respectively orthogonal to them also meet in a line.

Ex 4. If two planes are orthogonal, a plane meeting each of them in a line is perpendicular to both

Ex. 5. Prove that all planes, through a direction l_3, m_3, n_3, k_3, and perpendicular at the origin to the plane

$$\left\|\begin{matrix} x, & y, & z, & v\\ l_1, & m_1, & n_1, & k_1\\ l_2, & m_2, & n_2, & k_2\end{matrix}\right\|=0,$$

lie in the flat

$$(l_2x+m_2y+n_2z+k_2v)\cos\theta_{13}=(l_1x+m_1y+n_1z+k_1v)\cos\theta_{23}.$$

Ex. 6. The planes, having orientation-cosines a, b, c, f, g, h, and a′, b′, c′, f′, g′, h′, respectively, are orthogonal to one another; prove that

$$\frac{\mathrm{a}'}{\mathrm{f}}=\frac{\mathrm{b}'}{\mathrm{g}}=\frac{\mathrm{c}'}{\mathrm{h}}=\frac{\mathrm{f}'}{\mathrm{a}}=\frac{\mathrm{g}'}{\mathrm{b}}=\frac{\mathrm{h}'}{\mathrm{c}}.$$

Hence (or otherwise) shew that, if two planes are perpendicular, the planes respectively orthogonal to them also are perpendicular to one another.

Ex. 7. Shew that the two planes

$$\left.\begin{aligned} \frac{x}{a}-\frac{z}{c}\sin\alpha-\frac{v}{d}\cos\alpha&=0\\ \frac{y}{b}+\frac{z}{c}\cos\alpha-\frac{v}{d}\sin\alpha&=0\end{aligned}\right\},\quad \left.\begin{aligned} \frac{x}{a}+\frac{z}{c}\sin\beta-\frac{v}{d}\cos\beta&=0\\ \frac{y}{b}-\frac{z}{c}\cos\beta-\frac{v}{d}\sin\beta&=0\end{aligned}\right\},$$

constitute the intersection of the region

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-\frac{v^2}{d^2}=0$$

by the flat

$$\frac{x}{a}\cos\tfrac{1}{2}(\alpha+\beta)+\frac{y}{b}\sin\tfrac{1}{2}(\alpha+\beta)-\frac{z}{c}\sin\tfrac{1}{2}(\alpha-\beta)-\frac{v}{d}\cos\tfrac{1}{2}(\alpha-\beta)=0,$$

and that they themselves meet in the line

$$\frac{x}{a}\sec\tfrac{1}{2}(\alpha+\beta)=\frac{y}{b}\operatorname{cosec}\tfrac{1}{2}(\alpha+\beta)=\frac{z}{c}\operatorname{cosec}\tfrac{1}{2}(\alpha-\beta)=\frac{v}{d}\sec\tfrac{1}{2}(\alpha-\beta).$$

Prove that two planes of the first family, for different values of α, meet in the origin only; and likewise two planes of the second family, for different values of β.

Prove also that the two given planes are perpendicular to one another, for parametric values of α and β connected by the relation

$$c^2d^2 - a^2b^2 = (a^2d^2 - b^2c^2) \sin\alpha \sin\beta + (b^2d^2 - a^2c^2) \cos\alpha \cos\beta.$$

Ex. 8. Prove that there are no real values of the quantities $\alpha, \beta, \gamma, \delta$, for which the region

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} + \frac{z^2}{\gamma} + \frac{v^2}{\delta} = \epsilon$$

can contain planes, unless $\epsilon = 0$: and that, if $\epsilon = 0$, and if the region contains straight lines, two of the four quantities $\alpha, \beta, \gamma, \delta$, are positive and two are negative.

Prove also that, when the last requirements are satisfied, two (and only two) contained lines pass through every point of the region.

Ex. 9. Given five flats

$$L_r x + M_r y + N_r z + K_r v - P_r = 0,$$

($r = 1, 2, 3, 4, 5$), no four of which intersect in a line: prove that, if p_r be the perpendicular upon the flat $\Sigma L_r x = P_r$ from the intersection of the remaining four,

$$p_r \begin{vmatrix} L_s, & L_t, & L_u, & L_v \\ M_s, & M_t, & M_u, & M_v \\ N_s, & N_t, & N_u, & N_v \\ K_s, & K_t, & K_u, & K_v \end{vmatrix} = \begin{vmatrix} P_1, & L_1, & M_1, & N_1, & K_1 \\ P_2, & L_2, & M_2, & N_2, & K_2 \\ P_3, & L_3, & M_3, & N_3, & K_3 \\ P_4, & L_4, & M_4, & N_4, & K_4 \\ P_5, & L_5, & M_5, & N_5, & K_5 \end{vmatrix},$$

where $L_r^2 + M_r^2 + N_r^2 + K_r^2 = 1$, and r, s, t, u, v, are 1, 2, 3, 4, 5, taken cyclically.

Shew that the inclination of the edge

$$\Sigma L_3 x - P_3 = 0, \quad \Sigma L_4 x - P_4 = 0, \quad \Sigma L_5 x - P_5 = 0$$

of the pentahedroid to the opposite face

$$\Sigma L_1 x - P_1 = 0, \quad \Sigma L_2 x - P_2 = 0,$$

is given by

$$\sin^2\theta \sin^2\omega = \cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos\omega,$$

where

$$\cos\omega = L_1 L_2 + M_1 M_2 + N_1 N_2 + K_1 K_2,$$

$$\Delta \cos\alpha = \begin{vmatrix} L_1, & L_3, & L_4, & L_5 \\ M_1, & M_3, & M_4, & M_5 \\ N_1, & N_3, & N_4, & N_5 \\ K_1, & K_3, & K_4, & K_5 \end{vmatrix}, \quad \Delta \cos\beta = \begin{vmatrix} L_2, & L_3, & L_4, & L_5 \\ M_2, & M_3, & M_4, & M_5 \\ N_2, & N_3, & N_4, & N_5 \\ K_2, & K_3, & K_4, & K_5 \end{vmatrix},$$

and

$$\Delta^2 = \begin{vmatrix} 1, & \Sigma L_3 L_4, & \Sigma L_3 L_5 \\ \Sigma L_3 L_4, & 1, & \Sigma L_4 L_5 \\ \Sigma L_3 L_5, & \Sigma L_4 L_5, & 1 \end{vmatrix}.$$

Orthogonal planes: orthogonal frames of reference.

100. One important property of any couple of orthogonal planes is an inference from the conditions, first as regards orthogonality, and next as regards the modification of guiding lines in a plane without affecting the orientation of the plane.

Let the two planes, as before, be

$$\left\| \begin{matrix} x & y, & z, & v \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{matrix} \right\| = 0, \quad \left\| \begin{matrix} x, & y, & z, & v \\ l_3, & m_3, & n_3, & k_3 \\ l_4, & m_4, & n_4, & k_4 \end{matrix} \right\| = 0.$$

The conditions of orthogonality are

$$\cos\theta_{13} = \Sigma l_1 l_3 = 0, \quad \cos\theta_{14} = \Sigma l_1 l_4 = 0, \quad \cos\theta_{23} = \Sigma l_2 l_3 = 0, \quad \cos\theta_{24} = \Sigma l_2 l_4 = 0;$$

and whatever guiding lines be substituted for those specified in the first pair of equations, with equally free substitution of guiding lines for those specified in the second pair of equations, the condition of orthogonality

$$\Sigma(\alpha l_1 + \beta l_2)(\gamma l_3 + \delta l_4) = 0$$

is satisfied.

We can substitute for the two directions l_1, m_1, n_1, k_1, and l_2, m_2, n_2, k_2, any pair of perpendicular lines lying in the first plane. The choice of perpendicular lines can be made in an unlimited number of ways: thus we could keep the direction l_1, m_1, n_1, k_1, unaltered, and could substitute, for l_2, m_2, n_2, k_2,

$$\frac{l_2 - l_1\cos\theta_{12}}{\sin\theta_{12}}, \quad \frac{m_2 - m_1\cos\theta_{12}}{\sin\theta_{12}}, \quad \frac{n_2 - n_1\cos\theta_{12}}{\sin\theta_{12}}, \quad \frac{k_2 - k_1\cos\theta_{12}}{\sin\theta_{12}},$$

a direction which is perpendicular to the retained direction: or we could retain the direction l_2, m_2, n_2, k_2, and similarly, for l_1, m_1, n_1, k_1, substitute the perpendicular direction

$$\frac{l_1 - l_2\cos\theta_{12}}{\sin\theta_{12}}, \quad \frac{m_1 - m_2\cos\theta_{12}}{\sin\theta_{12}}, \quad \frac{n_1 - n_2\cos\theta_{12}}{\sin\theta_{12}}, \quad \frac{k_1 - k_2\cos\theta_{12}}{\sin\theta_{12}}.$$

Similar changes can be made in the assignment of guiding lines for the second plane without affecting the orientation.

Consequently it may be assumed, without any loss of generality or effect upon the respective orientations of the two planes, that the two guiding lines for each plane are perpendicular to one another. Therefore, assuming that such guiding lines have been selected, we have

$$\cos\theta_{12} = 0, \quad \cos\theta_{34} = 0.$$

The orthogonality is not affected, so that the relations

$$\cos\theta_{13} = 0, \quad \cos\theta_{14} = 0, \quad \cos\theta_{23} = 0, \quad \cos\theta_{24} = 0,$$

still hold. Thus there are four directions through the origin, which are perpendicular to one another in all the six pairs.

We therefore infer that any two orthogonal planes belong to an orthogonal frame of reference.

Further, as l_4, m_4, n_4, k_4, is a direction perpendicular to the other three directions, it is normal to the flat through the origin determined by those three directions. Similarly for each of the four directions in turn. Hence the flats of reference, in any orthogonal frame to which two orthogonal planes belong, are the flats perpendicular to the four determining directions; their four equations are

$$l_r x + m_r y + n_r z + k_r v = 0,$$

for $r = 1, 2, 3, 4$, the origin being at the intersection of the planes.

Amplitudes generated by the projecting lines in the process of projection.

101. Brief notice may be taken of the amplitudes which can be constituted from the array of lines projecting a given line, point by point, upon another amplitude.

(A) When the line

$$\frac{x-\alpha}{\lambda} = \frac{y-\beta}{\mu} = \frac{z-\gamma}{\nu} = \frac{v-\delta}{\kappa}$$

is projected upon the *flat*

$$Lx + My + Nz + Kv = P,$$

the direction-cosines of any projecting line are L, M, N, K. When a point $\xi, \eta, \zeta, \upsilon$, on the given line is taken, so that

$$\xi = \alpha + \lambda\rho, \quad \eta = \beta + \mu\rho, \quad \zeta = \gamma + \nu\rho, \quad \upsilon = \delta + \kappa\rho,$$

the equations of the projecting line from $\xi, \eta, \zeta, \upsilon$, are

$$x - \xi = LR, \quad y - \eta = MR, \quad z - \zeta = NR, \quad v - \upsilon = KR,$$

and therefore x, y, z, v, lies upon the plane

$$\left\| \begin{matrix} x-\alpha, & y-\beta, & z-\gamma, & v-\delta \\ \lambda\ , & \mu\ , & \nu\ , & \kappa \\ L\ , & M\ , & N\ , & K \end{matrix} \right\| = 0,$$

which accordingly is the amplitude generated by the projecting lines. Manifestly it is the plane through the given line and a direction through $\alpha, \beta, \gamma, \delta$, normal to the flat.

(B) When the line

$$\frac{x-\alpha}{\lambda} = \frac{y-\beta}{\mu} = \frac{z-\gamma}{\nu} = \frac{v-\delta}{\kappa}$$

is projected upon the *plane*

$$\left\| \begin{matrix} x-a, & y-b, & z-c, & v-d \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{matrix} \right\| = 0$$

the foot of the perpendicular upon the plane from any point

$$\xi, \eta, \zeta, v, = \alpha + \lambda\rho,\ \ \beta + \mu\rho,\ \ \gamma + \nu\rho,\ \ \delta + \kappa\rho,$$

upon the line has X, Y, Z, V, for coordinates, where (§ 69)

$$X = a + pl_1 + ql_2, \quad Y = b + pm_1 + qm_2, \quad Z = c + pn_1 + qn_2, \quad V = d + pk_1 + qk$$

and

$$p = \frac{1}{\sin^2\omega}\left[\Sigma l_1(\alpha - a) - \{\Sigma l_2(\alpha - a)\}\cos\omega + \rho(\cos\gamma - \cos\delta\cos\omega)\right] = A + \rho C,$$

$$q = \frac{1}{\sin^2\omega}\left[\Sigma l_2(\alpha - a) - \{\Sigma l_1(\alpha - a)\}\cos\omega + \rho(\cos\delta - \cos\gamma\cos\omega)\right] = B + \rho D,$$

$$\cos\gamma = \Sigma l_1\lambda, \quad \cos\delta = \Sigma l_2\lambda, \quad \cos\omega = \Sigma l_1 l_2.$$

Now along the line of projection, any point is given by

$$\frac{x - \xi}{X - \xi} = \frac{y - \eta}{Y - \eta} = \frac{z - \zeta}{Z - \zeta} = \frac{v - v}{V - v} = Q,$$

so that

$$\begin{aligned} x &= \xi(1 - Q) + QX \\ &= \alpha(1 - Q) + Q\{a + (l_1 A + l_2 B)\} + Q\rho(l_1 C + l_2 D) + \lambda\rho(1 - Q), \end{aligned}$$

that is,

$$x - \alpha = Q\{(l_1 A + l_2 B) - (\alpha - a)\} + Q\rho(l_1 C + l_2 D) + \rho(1 - Q)\lambda,$$

with three similar equations Hence

$$\begin{aligned} \Sigma l_1(x - \alpha) = Q\{(A + B\cos\omega) &- \Sigma l_1(\alpha - a)\} \\ &+ Q\rho(C + D\cos\omega) + \rho(1 - Q)\cos\gamma \\ = \rho\cos\gamma; \end{aligned}$$

and similarly

$$\Sigma l_2(x - \alpha) = \rho\cos\delta.$$

Hence the array of projecting lines lies in the flat

$$(\Sigma l_2\lambda)\,\Sigma l_1(x - \alpha) = (\Sigma l_1\lambda)\,\Sigma l_2(x - \alpha).$$

Again, writing the four equations in the form

$$\begin{aligned} x &= QL + Q\rho L' + \rho(1 - Q)\lambda + \alpha(1 - Q), \\ y &= QM + Q\rho M' + \rho(1 - Q)\mu + \beta(1 - Q), \\ z &= QN + Q\rho N' + \rho(1 - Q)\nu + \gamma(1 - Q), \\ v &= QK + Q\rho K' + \rho(1 - Q)\kappa + \delta(1 - Q), \end{aligned}$$

we have

$$\Theta Q = \Delta_1, \quad \Theta\rho Q = \Delta_2, \quad \Theta\rho(1 - Q) = \Delta_3, \quad \Theta(1 - Q) = \Delta_4,$$

where

$$\Theta = \begin{vmatrix} L, & L', & \lambda, & \alpha \\ M, & M', & \mu, & \beta \\ N, & N', & \nu, & \gamma \\ K, & K', & \kappa, & \delta \end{vmatrix},$$

$$\Delta_1 = \begin{vmatrix} x, & L', & \lambda, & \alpha \\ y, & M', & \mu, & \beta \\ z, & N', & \nu, & \gamma \\ v, & K', & \kappa, & \delta \end{vmatrix},$$

with similar expressions for Δ_2, Δ_3, Δ_4. Consequently

$$\Delta_1 + \Delta_4 = \Theta,$$

$$\Delta_2 \Delta_3 - \Delta_1 \Delta_4 = 0.$$

The former can be changed so as to become the foregoing equation of the containing flat. As each of the expressions Δ_1, Δ_2, Δ_3, Δ_4, is linear in x, y, z, v, the latter is a quadratic region.

Hence *the locus of the projecting lines is* the section of this quadratic region by the flat · that is, it is a quadric surface, which manifestly must be *a ruled quadric surface in a space of three dimensions.*

(C) When the line

$$\frac{x-\alpha}{\lambda} = \frac{y-\beta}{\mu} = \frac{z-\gamma}{\nu} = \frac{v-\delta}{\kappa}$$

is projected upon the *line*

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = \frac{v-d}{k},$$

the foot of the perpendicular, upon the second line from any point $\xi, \eta, \zeta, \upsilon, = \alpha + \lambda\rho,\ \beta + \mu\rho,\ \gamma + \nu\rho,\ \delta + \kappa\rho$, on the first line, has for its coordinates X, Y, Z, V, where (§ 24)

$$\frac{X-a}{l} = \frac{Y-b}{m} = \frac{Z-c}{n} = \frac{V-d}{k} = \Sigma l(\xi - a) = \Sigma l(\alpha - a) + \rho \cos\theta,$$

and $\cos\theta = l\lambda + m\mu + n\nu + k\kappa$. Along the projecting line, we have

$$\frac{x-\xi}{X-\xi} = \frac{y-\eta}{Y-\eta} = \frac{z-\zeta}{Z-\zeta} = \frac{v-\upsilon}{V-\upsilon} = R;$$

that is,

$$x = \xi(1-R) + RX,$$

or

$$\begin{aligned} x - \alpha &= \lambda\rho(1-R) + R(X-\alpha) \\ &= \lambda\rho(1-R) + \{a - \alpha + l\Sigma l(\alpha - a)\}R + R\rho l\cos\theta, \end{aligned}$$

with three similar expressions.

Eliminating $\rho(1-R)$, R, $R\rho$, we have

$$\begin{vmatrix} x-\alpha, & \lambda, & a-\alpha+l\,\Sigma l(\alpha-a), & l \\ y-\beta, & \mu, & b-\beta+m\Sigma l(\alpha-a), & m \\ z-\gamma, & \nu, & c-\gamma+n\,\Sigma l(\alpha-a), & n \\ v-\delta, & \kappa, & d-\delta+k\,\Sigma l(\alpha-a), & k \end{vmatrix}=0.$$

Thus the generating lines lie in a flat, which manifestly contains also both the given lines.

Again, writing the four equations in the form

$$x=\alpha(1-P)+P\{a+l\Sigma l(\alpha-a)\}+\lambda\rho(1-P)+P\rho l\cos\theta,$$

with three others, and resolving for $1-P$, P, $\rho(1-P)$, ρP, we have relations of the form

$$\Theta(1-P)=\Delta_1,\quad \Theta P=\Delta_2,\quad \Theta\rho(1-P)=\Delta_3,\quad \Theta\rho P=\Delta_4,$$

where Δ_1, Δ_2, Δ_3, Δ_4, are linear in x, y, z, v, and Θ is independent of x, y, z, v. Consequently

$$\Delta_1+\Delta_2=\Theta,$$

$$\Delta_2\Delta_3-\Delta_1\Delta_4=0.$$

The former can be changed into the equation of the containing flat. The latter is a quadratic region.

Hence *the locus of the projecting lines is* the section of the quadratic region by the flat: that is, it is *a ruled quadric surface in a space of three dimensions.*

NOTE. The identity of the results in (B) and (C) can be explained by the following consideration.

The projection of a line upon a plane, as in (B), is a line in that plane, and if a point P on the line is projected into a point Q in the plane, PQ is perpendicular to the projected line. This characteristic is exactly the same as for the projection in (C) Hence the configuration, constituted by the original line, the projected line, and the array of projecting lines, is the same for the two problems.

Moreover, a flat can be drawn through any two lines in quadruple space. Hence the whole configuration lies in a flat: and thus both problems, in essence, become problems in three dimensions.

CHAPTER VII.

GLOBULAR REPRESENTATION: PROJECTIONS: ROTATIONS.

Globular representation.

102. It is convenient to introduce the trigonometry of a globular configuration, in connection with the inclinations of various amplitudes to one another, whether they be of the same dimensions or of different dimensions.

A line, by itself, provides one direction. A flat, by itself, provides three independent directions, which initially are not unique: but no two of them may coincide, and the three may not be complanar: and the flat contains a double infinitude of directions, linearly congruent with the three guiding directions A plane, by itself, provides two independent directions, which also initially are not unique, but they may not coincide; and the plane contains a single infinitude of directions, linearly congruent with the two guiding directions. When two planes are given, they either intersect in a line; and then the four guiding directions can be reduced to three, by taking the common direction as one of the two guiding lines for each plane or, if this arrangement is not adopted, the four guiding lines lie in one flat: or the two planes intersect in a point only, and then they provide four guiding lines which do not simultaneously lie in one flat. Further, external directions connected with the trilinear flat are most conveniently estimated by reference to a normal to the flat, that is, by reference to a line which is perpendicular to every direction in the flat.

It therefore appears that four directions, chosen so that they do not simultaneously lie in one flat, will provide a configuration adequate for the consideration of the inclination between a line, and a plane, and a flat (through its normal), and of each such homaloid with another of its own type. Thus, in all, there are six possible cases: but, effectively, they reduce in number, because of the relation between a flat and its normal line.

Let the four directions be denoted, temporarily, by 1, 2, 3, 4. Then there are six planes, determined by pairs of these directions taken to be represented by four lines concurrent in the origin; and there are four flats, determined by triads of these directions. We can denote the flats by 234, 341, 412, 123.

The six planes are 12, 13, 14, 23, 24, 34. Of these, 12 and 13 intersect in the line 1, and lie in the flat 123, which also contains 23 intersecting 12 in the line 2 and 13 in the line 3; similarly 23, 34, 42, lie in the flat 234; 34, 41, 13, in the flat 341, and 41, 12, 24, lie in the flat 412. But the planes 12, 34, intersect in the origin only and do not lie in one flat: likewise the planes 23, 14; and likewise the planes 31, 24.

As the cosines determining any direction satisfy an equation

$$l^2 + m^2 + n^2 + k^2 = 1,$$

we take a triple region in the quadruple space: this region, called a unit *globe*, is defined by the equation

$$x^2 + y^2 + z^2 + v^2 = 1,$$

the centre of the globe being the concurrent point of the four directions.

Sections of a globe.

103. (i) The section of the globe by a central flat is a surface, which is easily seen to be a sphere (that is, the surface of a sphere in the three-dimensional space of the flat). For any central flat is represented by an equation

$$Lx + My + Nz + Kv = 0,$$

where L, M, N, K, are the direction-cosines of its normal Take an orthogonal frame of four lines, perpendicular to one another in pairs, the normal to the flat being one of these lines, and let the lines of this orthogonal frame be taken as axes of X, Y, Z, V, respectively. Then one of the new coordinates—let it be V—is

$$V = Lx + My + Nz + Kv.$$

In the new frame, the globe is

$$X^2 + Y^2 + Z^2 + V^2 = 1.$$

Consequently, the section of the globe by the central flat $V = 0$ is the sphere

$$X^2 + Y^2 + Z^2 = 1,$$

lying in the flat region $V = 0$, that is, in an ordinary three-dimensional Euclidean space; and the radius of the sphere is unity, being the radius of the globe.

For our immediate purpose, the section of the globe by a non-central flat

$$Lx + My + Nz + Kv = \cos\alpha$$

is not required: with the same change in the frame of reference it would be a sphere

$$X^2 + Y^2 + Z^2 = \sin^2\alpha,$$

existing in the flat $V = \cos\alpha$: but we are not concerned with non-central sections of the globe at this stage.

(ii) The section of the globe by a central plane is a curve, which is easily seen to be a circle (that is, the circumference of a circle lying in the plane). For any central plane is represented by the equations

$$Lx + My + Nz + Kv = 0, \quad L'x + M'y + N'z + K'v = 0;$$

and therefore, with the previous change of frame of reference, it comes to be represented by the equations

$$V=0, \quad \lambda X+\mu Y+\nu Z+\kappa V=0,$$

(the two flats in the equation of the plane are not supposed to be necessarily perpendicular): that is, by the equations

$$V=0, \quad \lambda X+\mu Y+\nu Z=0.$$

As before, the globe is

$$X^2+Y^2+Z^2+V^2=1:$$

and therefore its section by the specified central plane is the section of the sphere

$$X^2+Y^2+Z^2=1$$

by the amplitude $\lambda X+\mu Y+\nu Z=0$, both lying in the flat $V=0$: that is, the section is a great circle of this sphere. We may regard it as a great circle of the globe.

Again, the section of the globe by a non-central plane is not wanted for our immediate purpose: it would be a plane section of a 'small sphere'

$$X^2+Y^2+Z^2=\sin^2\alpha,$$

and it might be either a great circle or a small circle on that sphere, according to the position of the plane: that is, we could regard it as a small circle on the globe.

(iii) The section of the globe by any central line, regarded as a direction, is a point. (Strictly, it would be two points, if opposite senses were allowed to the line. But we regard the line as a single direction in one sense, emanating from the origin; the portion of the complete line, in the other sense, would be another single direction in the same diameter. With this convention, a direction meets the globe in only one point.)

The globular quadrilateral.

104. Accordingly, we have a configuration. The points 1, 2, 3, 4, being the ends of radii of the globular region, represent the four directions l_i, m_i, n_i, k_i, for $i=1, 2, 3, 4$. The arcs 12, 13, 14, 23, 24, 34, being arcs of circles with their common centre at the centre of the region, represent the six planes, determined by the directions, taken in pairs. Any three non-complanar directions determine a flat; and thus the surfaces of the four curvilinear triangles 234, 341, 412, 123, represent the four flats, determined by the four directions, taken in triads.

Moreover, each triangle is a part of the spherical surface which is the section of the globe by its flat: thus there are four such spherical sections. As we shall be dealing with the inclination of the planes 12 and 34 to one another, we drop arc-perpendiculars on the representative arcs: on 12, the perpendicular 37 in the spherical surface 123 and the perpendicular 48 in the spherical surface 124; on 34, the perpendicular 15 in the spherical surface 134 and the perpendicular 26 in the spherical surface 234. Thus the arc 56 represents the projection of the plane 12 on the plane 34, and the arc 78 represents the projection of the plane 34 on the plane 12; each of these intercepted arcs will be connected with the measure of the inclination of the planes 12 and 34 to one another.

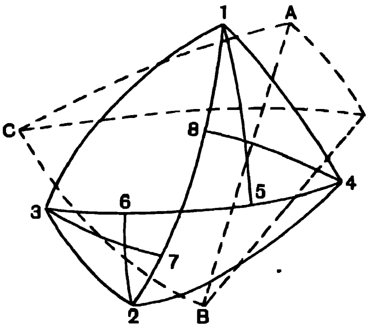

Fig. 5.

The polar quadrilateral.

105. Further, each of the four flats has a unique direction normal to itself: the radius parallel to this direction will be taken to represent the normal. Thus we shall have two points in the globular region corresponding to this normal. That one of the two points, which lies on the same side of a flat through three directions as the remaining point-direction lies, will be called the *pole* of the flat. Let A, B, C, D, be the respective poles of the flats 234, 341, 412, 123; then A and 1 lie in the same vicinity, likewise B and 2, likewise C and 3, and likewise D and 4. Also, the arc $1B$ is a quadrant of a circle because B is the pole of 341, the arc $1C$ is a quadrant because C is the pole of 412, and the arc $1D$ is a quadrant because D is the pole of 123; consequently, as each of the arcs $1B$, $1C$, $1D$ is a quadrant, the point 1 is the pole of the flat BCD. Similarly, the point 2 is the pole of the flat CDA, the point 3 is the pole of the flat DAB, and the point 4 is the pole of the flat ABC. Thus the polar relation of the two quadrilaterals 1234 and $ABCD$ is reciprocal.

Normals to the flats.

106. The direction-cosines of the normals to the four flats—being the coordinates of the four poles in the globular region—have been given already by implication. Let

$$\left.\begin{aligned}
\Delta_1^2 &= 1 - \cos^2\theta_{23} - \cos^2\theta_{34} - \cos^2\theta_{42} + 2\cos\theta_{23}\cos\theta_{34}\cos\theta_{42}\\
\Delta_2^2 &= 1 - \cos^2\theta_{34} - \cos^2\theta_{41} - \cos^2\theta_{13} + 2\cos\theta_{34}\cos\theta_{41}\cos\theta_{13}\\
\Delta_3^2 &= 1 - \cos^2\theta_{41} - \cos^2\theta_{12} - \cos^2\theta_{24} + 2\cos\theta_{41}\cos\theta_{12}\cos\theta_{24}\\
\Delta_4^2 &= 1 - \cos^2\theta_{12} - \cos^2\theta_{23} - \cos^2\theta_{31} + 2\cos\theta_{12}\cos\theta_{23}\cos\theta_{31}
\end{aligned}\right\};$$

and let the coordinates of the poles be L_1, M_1, N_1, K_1, for A; L_2, M_2, N_2, K_2, for B; L_3, M_3, N_3, K_3, for C; and L_4, M_4, N_4, K_4, for D. Then

$$\left.\begin{array}{l}
\dfrac{L_1}{|\,m_2,\ n_3,\ k_4\,|} = \dfrac{M_1}{-|\,n_2,\ k_3,\ l_4\,|} = \dfrac{N_1}{|\,k_2,\ l_3,\ m_4\,|} = \dfrac{K_1}{-|\,l_2,\ m_3,\ n_4\,|} = \dfrac{1}{\Delta_1} \\
\dfrac{L_2}{-|\,m_3,\ n_4,\ k_1\,|} = \dfrac{M_2}{|\,n_3,\ k_4,\ l_1\,|} = \dfrac{N_2}{-|\,k_3,\ l_4,\ m_1\,|} = \dfrac{K_2}{|\,l_3,\ m_4,\ n_1\,|} = \dfrac{1}{\Delta_2} \\
\dfrac{L_3}{|\,m_4,\ n_1,\ k_2\,|} = \dfrac{M_3}{-|\,n_4,\ k_1,\ l_2\,|} = \dfrac{N_3}{|\,k_4,\ l_1,\ m_2\,|} = \dfrac{K_3}{-|\,l_4,\ m_1,\ n_2\,|} = \dfrac{1}{\Delta_3} \\
\dfrac{L_4}{-|\,m_1,\ n_2,\ k_3\,|} = \dfrac{M_4}{|\,n_1,\ k_2,\ l_3\,|} = \dfrac{N_4}{-|\,k_1,\ l_2,\ m_3\,|} = \dfrac{K_4}{|\,l_1,\ m_2,\ n_3\,|} = \dfrac{1}{\Delta_4}
\end{array}\right\}.$$

Also, the inclination of the normal of the flat 234 to the direction 1 is represented by the arc $A1$: therefore

$$\cos A1 = L_1 l_1 + M_1 m_1 + N_1 n_1 + K_1 k_1$$

$$= \frac{1}{\Delta_1}\begin{vmatrix} l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \\ l_4, & m_4, & n_4, & k_4 \end{vmatrix} = \frac{\Theta^{\frac{1}{2}}}{\Delta_1},$$

where, as in § 82,

$$\begin{aligned}
\Theta = 1 &- \cos^2\theta_{12} - \cos^2\theta_{23} - \cos^2\theta_{31} - \cos^2\theta_{14} - \cos^2\theta_{24} - \cos^2\theta_{34} \\
&+ \cos^2\theta_{12}\cos^2\theta_{34} + \cos^2\theta_{23}\cos^2\theta_{14} + \cos^2\theta_{31}\cos^2\theta_{24} \\
&+ 2\cos\theta_{12}\cos\theta_{23}\cos\theta_{31} + 2\cos\theta_{23}\cos\theta_{34}\cos\theta_{42} \\
&+ 2\cos\theta_{34}\cos\theta_{41}\cos\theta_{13} + 2\cos\theta_{41}\cos\theta_{12}\cos\theta_{24} \\
&- 2\cos\theta_{12}\cos\theta_{13}\cos\theta_{24}\cos\theta_{34} - 2\cos\theta_{21}\cos\theta_{23}\cos\theta_{14}\cos\theta_{34} \\
&- 2\cos\theta_{13}\cos\theta_{23}\cos\theta_{14}\cos\theta_{24};
\end{aligned}$$

and Θ is not zero, because the four directions 1, 2, 3, 4, do not lie in one flat Similarly, for the arcs $B2$, $C3$, $D4$. We thus have

$$\Delta_1 \cos A1 = \Delta_2 \cos B2 = \Delta_3 \cos C3 = \Delta_4 \cos D4 = \Theta^{\frac{1}{2}}.$$

It is easy to verify that

$$\left.\begin{aligned}
L_1 &= \frac{\Delta_1}{\Theta^{\frac{1}{2}}} l_1 + Q_1 l_2 + R_1 l_3 + S_1 l_4 \\
M_1 &= \frac{\Delta_1}{\Theta^{\frac{1}{2}}} m_1 + Q_1 m_2 + R_1 m_3 + S_1 m_4 \\
N_1 &= \frac{\Delta_1}{\Theta^{\frac{1}{2}}} n_1 + Q_1 n_2 + R_1 n_3 + S_1 n_4 \\
K_1 &= \frac{\Delta_1}{\Theta^{\frac{1}{2}}} k_1 + Q_1 k_2 + R_1 k_3 + S_1 k_4
\end{aligned}\right\},$$

where

$$-\Theta^{\frac{1}{2}}\Delta_1 Q_1 = \begin{vmatrix} \cos\theta_{12}, & \cos\theta_{23}, & \cos\theta_{24} \\ \cos\theta_{13}, & 1\ , & \cos\theta_{34} \\ \cos\theta_{14}, & \cos\theta_{34}, & 1 \end{vmatrix},$$

$$-\Theta^{\frac{1}{2}}\Delta_1 R_1 = \begin{vmatrix} 1, & \cos\theta_{12}, & \cos\theta_{24} \\ \cos\theta_{23}, & \cos\theta_{13}, & \cos\theta_{34} \\ \cos\theta_{24}, & \cos\theta_{14}, & 1 \end{vmatrix},$$

$$-\Theta^{\frac{1}{2}}\Delta_1 S_1 = \begin{vmatrix} 1 & \cos\theta_{23}, & \cos\theta_{12} \\ \cos\theta_{23}, & 1\ , & \cos\theta_{13} \\ \cos\theta_{24}, & \cos\theta_{34}, & \cos\theta_{14} \end{vmatrix}.$$

Similarly for the direction-coordinates of the other poles.

Again, L_1, M_1, N_1, K_1, are the minors of l_1, m_1, n_1, k_1, in $\Theta^{\frac{1}{2}}$; and likewise for the direction-coordinates of the other poles. Hence

$$\Delta_1\Delta_2\,(L_1M_2 - M_1L_2) = \Theta^{\frac{1}{2}}\,(n_3k_4 - n_4k_3),$$

and so for other second minors of the determinant

$$\begin{vmatrix} L_1, & M_1, & N_1, & K_1 \\ L_2, & M_2, & N_2, & K_2 \\ L_3, & M_3, & N_3, & K_3 \\ L_4, & M_4, & N_4, & K_4 \end{vmatrix},$$

and therefore

$$\Delta_1^2\Delta_2^2\,\Sigma\,(L_1M_2 - M_1L_2)^2 = \Theta\,\Sigma\,(n_3k_4 - n_4k_3)^2,$$

that is,

$$\Delta_1^2\Delta_2^2\sin^2 AB = \Theta\sin^2\theta_{34},$$

or

$$\Delta_1\Delta_2\sin AB = \Theta^{\frac{1}{2}}\sin\theta_{34}.$$

Further, we have

$$\begin{aligned}\Delta_1\Delta_2\cos AB &= (L_1L_2 + M_1M_2 + N_1N_2 + K_1K_2)\,\Delta_1\Delta_2 \\ &= -\Sigma \begin{vmatrix} m_2, & n_2, & k_2 \\ m_3, & n_3, & k_3 \\ m_4, & n_4, & k_4 \end{vmatrix} \begin{vmatrix} m_3, & n_3, & k_3 \\ m_4, & n_4, & k_4 \\ m_1, & n_1, & k_1 \end{vmatrix} \\ &= -\begin{vmatrix} \cos\theta_{12}, & \cos\theta_{13}, & \cos\theta_{14} \\ \cos\theta_{23}, & 1\ , & \cos\theta_{34} \\ \cos\theta_{24}, & \cos\theta_{34}, & 1 \end{vmatrix}.\end{aligned}$$

Similarly for other quantities connected with the arcs of the polar quadrilateral *ABCD*. These polar arcs measure the inclinations of the flats: thus *AB* is the supplement of the inclination of the two flats which intersect in the plane 34.

Inclination of two planes meeting in a line.

107. The inclinations of the planes which in pairs intersect in a line, viz. the planes 12, 13, 14, all of which pass through the direction 1; the planes 23, 21, 24, all of which pass through the direction 2; the planes 34, 31, 32, all of which pass through the direction 3; and the planes 41, 42, 43, all of which pass through the direction 4: are given by the customary relations of the four triangles 234, 341, 412, 123, on the spherical surfaces in the respective flats. Thus if p, q, r, be any three of the four numbers 1, 2, 3, 4, denoting directions, the inclination qpr of the planes pq and pr is given by

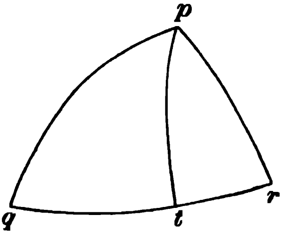

Fig. 6.

$$\cos qpr = \frac{\cos qr - \cos pq \cos pr}{\sin pq \sin pr},$$

$$\sin qpr = \frac{1}{\sin pq \sin pr}(1 - \cos^2 pq - \cos^2 qr - \cos^2 rp + 2 \cos pq \cos qr \cos rp)^{\frac{1}{2}}.$$

Further, if O be the centre of the globe, and if pt be the arc-perpendicular in the spherical surface from p upon the opposite arc qr, so that Ot is the projection of the direction Op upon the plane qr, we have, as usual,

$$\cos pt \sin qr = (\cos^2 pq + \cos^2 pr - 2 \cos pq \cos pr \cos qr)^{\frac{1}{2}},$$

$$\sin pt \sin qr = (1 - \cos^2 pq - \cos^2 pr - \cos^2 qr + 2 \cos pq \cos pr \cos qr)^{\frac{1}{2}},$$

$$\frac{\cos rt}{\cos pr \sin qr} = \frac{\sin rt}{\cos pq - \cos pr \cos qr} = \frac{\cos qt}{\cos pq \sin qr} = \frac{\sin qt}{\cos pr - \cos pq \cos qr}$$

$$= \frac{1}{(\cos^2 pq + \cos^2 pr - 2 \cos pq \cos pr \cos qr)^{\frac{1}{2}}},$$

results to be used immediately, in connection with the inclination of two planes which meet in a point only (the centre of the globe) and not in a line, that is, which have no point of intersection to be represented on the globular quadrilateral.

Inclination of two planes meeting in a point only.

108. The planes 12 and 34 do not meet except at the centre of the globe: they have no line common. The same holds of the planes 13 and 24, and of the planes 23 and 14. Thus the arcs 12 and 34 do not meet; nor the arcs 13 and 24; nor the arcs 23 and 14.

It has been pointed out that the arc 56, intercepted on the arc 34, represents a projection of the plane 12 on the plane 34; and likewise the arc 78, intercepted on the arc 12, represents a projection of the plane 34 on the plane 12. Thus each intercepted arc is a measure, in some form, of the inclination of the two planes: the actual relations of these arcs to the angle ϕ prove simple.

Referring to the figure in § 104, we have

$$\frac{\cos 45}{\cos 14 \sin 34} = \frac{\sin 45}{\cos 13 - \cos 14 \cos 34} = \frac{1}{\Gamma_2^{\frac{1}{2}}}, \quad \cos 15 \sin 34 = \Gamma_2^{\frac{1}{2}},$$

$$\frac{\cos 46}{\cos 24 \sin 34} = \frac{\sin 46}{\cos 23 - \cos 24 \cos 34} = \frac{1}{\Gamma_1^{\frac{1}{2}}}, \quad \cos 26 \sin 34 = \Gamma_1^{\frac{1}{2}},$$

where

$$\Gamma_1 = \cos^2 23 + \cos^2 24 - 2 \cos 23 \cos 24 \cos 34,$$

$$\Gamma_2 = \cos^2 13 + \cos^2 14 - 2 \cos 13 \cos 14 \cos 34.$$

Hence

$$\begin{aligned} \sin 65 &= \sin(46 - 45) \\ &= \frac{\sin 34}{\Gamma_1^{\frac{1}{2}} \Gamma_2^{\frac{1}{2}}} (\cos 14 \cos 23 - \cos 13 \cos 24). \end{aligned}$$

But (§ 93)

$$\sin 12 \sin 34 \cos \phi = \cos 13 \cos 24 - \cos 14 \cos 23,$$

hence, substituting for $\Gamma_1^{\frac{1}{2}}$ and $\Gamma_2^{\frac{1}{2}}$ their respective values cos 26 sin 34 and cos 15 sin 34, we have

$$\cos \phi = \frac{\cos 15 \cos 26}{\sin 12} \sin 56,$$

the relation in question between the intercepted arc 56 and the inclination of the planes.

In the same way, we have

$$\frac{\cos 18}{\cos 14 \sin 12} = \frac{\sin 18}{\cos 24 - \cos 14 \cos 12} = \frac{1}{\Gamma_3^{\frac{1}{2}}}, \quad \cos 48 \sin 12 = \Gamma_3^{\frac{1}{2}},$$

$$\frac{\cos 17}{\cos 13 \sin 12} = \frac{\sin 17}{\cos 23 - \cos 13 \cos 12} = \frac{1}{\Gamma_4^{\frac{1}{2}}}, \quad \cos 37 \sin 12 = \Gamma_4^{\frac{1}{2}},$$

where

$$\Gamma_3 = \cos^2 14 + \cos^2 24 - 2 \cos 14 \cos 24 \cos 12,$$

$$\Gamma_4 = \cos^2 13 + \cos^2 23 - 2 \cos 13 \cos 23 \cos 12;$$

and we find

$$\cos \phi = \frac{\cos 37 \cos 48}{\sin 34} \sin 87,$$

the relation in question between the intercepted arc 78 and the inclination of the planes.

There is a simple verification in the fact that, if the planes are perpendicular so that $\phi = \frac{1}{2}\pi$, the points 5 and 6 coincide, so that the perpendicular from 1 upon 34 drawn in the triangle 134 meets 34 in the same point as the perpendicular from 2 upon 34 in the triangle 234; and similarly, in the same event of the planes being perpendicular, the points 7 and 8 coincide, being the feet of the perpendiculars from 3 and from 4 in their respective spherical surfaces that intersect in the arc 12.

Ex 1. Obtain the following results, in addition to those stated for the arcs 45, 46, 17, 18, viz.

$$\frac{\cos 35}{\cos 13 \sin 34} = \frac{\sin 35}{\cos 14 - \cos 13 \cos 34} = \frac{1}{\Gamma_2^{\frac{1}{2}}},$$

$$\frac{\cos 36}{\cos 23 \sin 34} = \frac{\sin 36}{\cos 24 - \cos 23 \cos 34} = \frac{1}{\Gamma_1^{\frac{1}{2}}},$$

$$\frac{\cos 27}{\cos 23 \sin 12} = \frac{\sin 27}{\cos 13 - \cos 23 \cos 12} = \frac{1}{\Gamma_4^{\frac{1}{2}}},$$

$$\frac{\cos 28}{\cos 24 \sin 12} = \frac{\sin 28}{\cos 14 - \cos 24 \cos 12} = \frac{1}{\Gamma_3^{\frac{1}{2}}}.$$

Ex. 2 Prove that the direction-cosines of $O5$, or the coordinates of 5, are

$$l_5 = \alpha l_3 + \beta l_4, \quad m_5 = \alpha m_3 + \beta m_4, \quad n_5 = \alpha n_3 + \beta n_4, \quad k_5 = \alpha k_3 + \beta k_4,$$

where

$$\alpha \Gamma_2^{\frac{1}{2}} = \sin 14 \cos 143, \quad \beta \Gamma_2^{\frac{1}{2}} = \sin 13 \cos 134,$$

that the coordinates of 6 are

$$l_6 = \alpha' l_3 + \beta' l_4, \quad m_6 = \alpha' m_3 + \beta' m_4, \quad n_6 = \alpha' n_3 + \beta' n_4, \quad k_6 = \alpha' k_3 + \beta' k_4,$$

where

$$\alpha' \Gamma_1^{\frac{1}{2}} = \sin 24 \cos 243, \quad \beta' \Gamma_1^{\frac{1}{2}} = \sin 23 \cos 234,$$

that the coordinates of 7 are

$$l_7 = \gamma l_1 + \delta l_2, \quad m_7 = \gamma m_1 + \delta m_2, \quad n_7 = \gamma n_1 + \delta n_2, \quad k_7 = \gamma k_1 + \delta k_2,$$

where

$$\gamma \Gamma_4^{\frac{1}{2}} = \sin 23 \cos 123, \quad \delta \Gamma_4^{\frac{1}{2}} = \sin 13 \cos 213;$$

and that the coordinates of 8 are

$$l_8 = \gamma' l_1 + \delta' l_2, \quad m_8 = \gamma' m_1 + \delta' m_2, \quad n_8 = \gamma' n_1 + \delta' n_2, \quad k_8 = \gamma' k_1 + \delta' k_2,$$

where

$$\gamma' \Gamma_3^{\frac{1}{2}} = \sin 24 \cos 124, \quad \delta' \Gamma_3^{\frac{1}{2}} = \sin 14 \cos 412.$$

Ex 3 Prove that

$$\frac{\sin 78}{\Gamma_1^{\frac{1}{2}} \Gamma_2^{\frac{1}{2}} \sin 12} = \frac{\sin 65}{\Gamma_3^{\frac{1}{2}} \Gamma_4^{\frac{1}{2}} \sin 34};$$

and shew that the quantities α, β, α', β', γ, δ, γ', δ', in the preceding example are such that

$$(\alpha\beta' - \alpha'\beta) \Gamma_1^{\frac{1}{2}} \Gamma_2^{\frac{1}{2}} = (\gamma\delta' - \gamma'\delta) \Gamma_3^{\frac{1}{2}} \Gamma_4^{\frac{1}{2}} = \cos 13 \cos 24 - \cos 14 \cos 23.$$

Least arc-distance between two planes meeting only in a point.

109. In § 92 an estimate was obtained for the least angular distance (and therefore, in the diagram the least arc-distance) between two planes which do not meet in a line. When these planes are, once more, the planes 12 and 34, and this least arc-distance passes from a direction l, m, n, k, (say N) in 12, to a direction $\lambda, \mu, \nu, \kappa$, (say N') in 34, where

$$l, m, n, k = (p, q)(l_1, m_1, n_1, k_1;\ l_2, m_2, n_2, k_2),$$

$$\lambda, \mu, \nu, \kappa = (r, s)(l_3, m_3, n_3, k_3;\ l_4, m_4, n_4, k_4),$$

the critical equations are

$$\begin{aligned}
r \cos 13 + s \cos 14 &= (p + q \cos 12) \cos \theta,\\
r \cos 23 + s \cos 24 &= (p \cos 12 + q) \cos \theta,\\
p \cos 13 + q \cos 23 &= (r + s \cos 34) \cos \theta,\\
p \cos 14 + q \cos 24 &= (r \cos 34 + s) \cos \theta,
\end{aligned}$$

where $\theta, = NN'$, is the arc-distance in question.

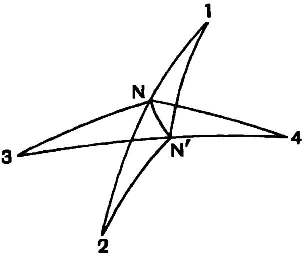

Fig. 7.

Now

$$\left.\begin{aligned}
\cos 1N &= \Sigma l_1 l = p + q \cos 12\\
\cos 2N &= \Sigma l_2 l = p \cos 12 + q\\
\cos 3N &= \Sigma l_3 l = p \cos 13 + q \cos 23\\
\cos 4N &= \Sigma l_4 l = p \cos 14 + q \cos 24
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\cos 1N' &= \Sigma l_1 \lambda = r \cos 13 + s \cos 14\\
\cos 2N' &= \Sigma l_2 \lambda = r \cos 23 + s \cos 24\\
\cos 3N' &= \Sigma l_3 \lambda = r + s \cos 34\\
\cos 4N' &= \Sigma l_4 \lambda = r \cos 34 + s
\end{aligned}\right\},$$

and therefore the critical equations are

$$\cos 1N' = \cos 1N \cos NN', \quad \cos 2N' = \cos 2N \cos NN',$$

$$\cos 3N = \cos 3N' \cos N'N, \quad \cos 4N = \cos 4N' \cos N'N.$$

Hence in the flat $1N'2N1$, the arc $N'N$ is at right angles to 12; and, in the flat $3N4N'3$, the arc NN' is at right angles to 34. Thus the arc NN', of least angular distance between the planes, is at right angles to the non-concurrent arcs 12 and 34, which represent the two planes 12 and 34 meeting in no point in the globular representation but only at the centre of the globe.

Projection of a line on a flat.

110. The projection of a line on a flat is determinate when we know the direction-cosines of the projection, the line of course not being contained in the flat. In the globular diagram, the flat is represented by a spherical surface; the given line is represented by a point which does not lie in that surface. The projection of the line, being a line in space, will be represented by a point; as the projection lies in the flat, it is represented by a point in the spherical surface.

The line and the flat are referred to their point of meeting, as the centre of the globe. The line is taken to be $O4$: the flat to have $O1$, $O2$, $O3$, for its guiding lines: that is, in the globular diagram, the flat is the spherical surface 123, and the line is the point 4. Let D be the pole of the flat 123. After the analysis of §§ 60–64, we obtain the projection of the line by drawing the plane through the normal OD to the flat and the line $O4$, and in this plane by then drawing the line $O4'$ perpendicular to OD. The line $O4'$ is the projection: the point $4'$ in the diagram represents the projection of the line $O4$, represented by the point 4, and $4'D$ is a right angle in arc, because D is the pole of the flat.

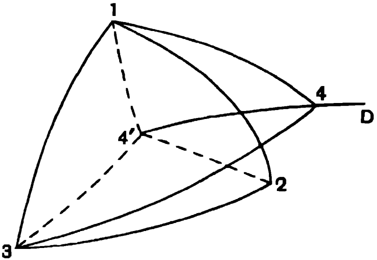

Fig. 8.

In the globular diagram $4'4$ is at right angles to $14'$, to $24'$, and to $34'$; and the arc $4'4$ is θ, the inclination of the line to the flat. But $4'4$ is the complement of $D4$; thus

$$\sin\theta = \cos D4$$

$$= \frac{\Theta^{\frac{1}{2}}}{\Delta_4},$$

with the preceding significance (§ 106) for Θ and Δ_4, and therefore

$$\cos\theta = \frac{(\Delta_4^2 - \Theta)^{\frac{1}{2}}}{\Delta_4}$$

Then we have

$$\cos 14' \cos 4'4 = \cos 14, \quad \cos 24' \cos 4'4 = \cos 24, \quad \cos 34' \cos 4'4 = \cos 34,$$

that is,

$$\frac{\cos 14'}{\cos 14} = \frac{\cos 24'}{\cos 24} = \frac{\cos 34'}{\cos 34} = \frac{1}{\cos\theta} = \frac{\Delta_4}{(\Delta_4^2 - \Theta)^{\frac{1}{2}}}.$$

Now if λ', μ', ν', κ', be the direction-cosines of $O4'$, that is, be the globular coordinates of $4'$, we have (§ 44)

$$\left.\begin{aligned} \lambda' &= \rho l_1 + \sigma l_2 + \tau l_3 \\ \mu' &= \rho m_1 + \sigma m_2 + \tau m_3 \\ \nu' &= \rho n_1 + \sigma n_2 + \tau n_3 \\ \kappa' &= \rho k_1 + \sigma k_2 + \tau k_3 \end{aligned}\right\},$$

so that

$$\cos 14' = \Sigma l_1 \lambda' = \rho + \sigma \cos 12 + \tau \cos 13,$$

$$\cos 24' = \Sigma l_2 \lambda' = \rho \cos 12 + \sigma + \tau \cos 23,$$

$$\cos 34' = \Sigma l_3 \lambda' = \rho \cos 13 + \sigma \cos 23 + \tau.$$

Thus the values of ρ, σ, τ, are known, being

$$\rho\Delta_4(\Delta_4^2 - \Theta)^{\frac{1}{2}} = \begin{vmatrix} \cos 14, & \cos 12, & \cos 13 \\ \cos 24, & 1\ , & \cos 23 \\ \cos 34, & \cos 23, & 1 \end{vmatrix},$$

$$\sigma\Delta_4(\Delta_4^2 - \Theta)^{\frac{1}{2}} = \begin{vmatrix} 1\ , & \cos 14, & \cos 13 \\ \cos 12, & \cos 24, & \cos 23 \\ \cos 13, & \cos 34, & 1 \end{vmatrix},$$

$$\tau\Delta_4(\Delta_4^2 - \Theta)^{\frac{1}{2}} = \begin{vmatrix} 1\ , & \cos 12, & \cos 14 \\ \cos 12, & 1\ , & \cos 24 \\ \cos 13, & \cos 23, & \cos 34 \end{vmatrix}.$$

Globular diagram.

111. The accompanying figure (p. 181) gives a globular representation connected with the parallelepiped in quadruple space shewn in the figure on p. 7.

The points O, A, B, C, D; α, β, γ, δ; and P; are the same in the two figures. To secure unit radius for the globe we take $a = l = OA$, $b = m = OB$, $c = n = OC$, $d = k = OD$. The points α, β, γ, δ, are the feet of the perpendiculars from P on the respective flats $x = 0$, $y = 0$, $z = 0$, $v = 0$.

There are four spherical surfaces in the diagram:

the surface $Y_\alpha Z_\alpha V_\alpha$, with equations $x = 0$, $y^2 + z^2 + v^2 = 1 - l^2$;

.... $Z_\beta V_\beta X_\beta$, ... $y = 0$, $x^2 + z^2 + v^2 = 1 - m^2$;

.... $V_\gamma X_\gamma Y_\gamma$, ... $z = 0$, $x^2 + y^2 + v^2 = 1 - n^2$;

.... $X_\delta Y_\delta Z_\delta$, ... $v = 0$, $x^2 + y^2 + z^2 = 1 - k^2$.

In the planes of reference, the sections of these spherical surfaces are quadrantal arcs, each joining points on adjacent axes.

In the right-angled spherical triangle $Y_\alpha Z_\alpha V_\alpha$, there are three arcs through the point α, being $Y_\alpha \alpha\alpha_2$, $Z_\alpha \alpha\alpha_3$, $V_\alpha \alpha\alpha_4$; and these arcs are the sections of the spherical surface by planes, through the point α and the respective axes OY_α, OZ_α, OV_α. Similarly for the arcs in the other three spherical triangles, concurrent in β, γ, δ, respectively.

The line $O\alpha_4\delta_1$ in the plane OYZ is the diagonal of the rectangle BOC in that plane: it meets the arc $Z_\alpha Y_\alpha$ at the point of intersection α_4 with the arc $V_\alpha \alpha$, and it meets the arc $Z_\delta Y_\delta$ at the point of intersection δ_1 with the arc $X_\delta \delta$. Similarly for the line $O\beta_4\delta_2$ in the plane OZX, for the line $O\gamma_4\delta_3$ in the plane OXY, for the line $O\beta_3\gamma_2$ in the plane OVX, for the line $O\gamma_1\alpha_3$ in the plane OVY, and for the line $O\alpha_2\beta_1$ in the plane OVZ.

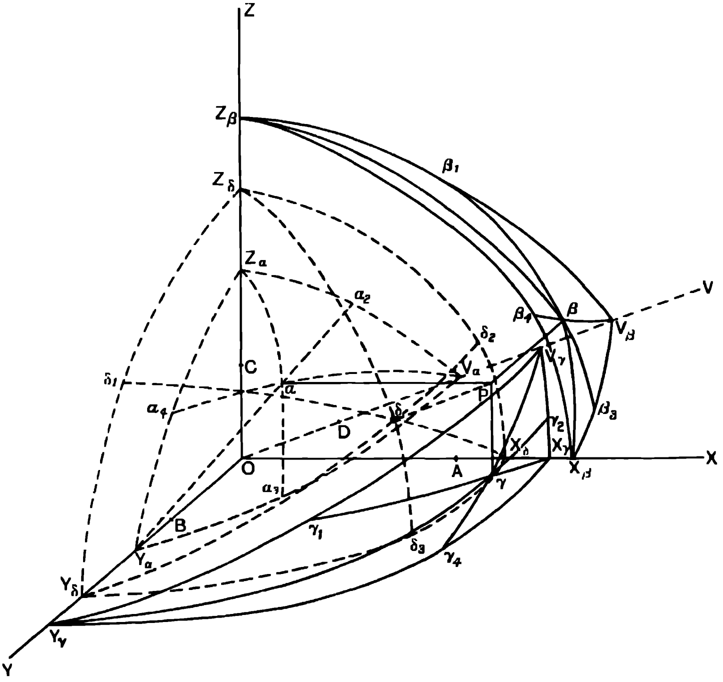

Fig. 9.

The sections of the globe by the planes of reference are not drawn in the diagram: when unit lengths OX, OY, OZ, OV, are taken, the quadrantal arcs are YZ, ZX, XY, VX, VY, VZ.

Conventions as to orientation coordinates of a plane.

112. Within the globular representation, we assign conventions as regards the senses of positive and negative directions, for rotations and for measurements of angles where mere magnitude is not the sole need.

Direction along a line is made precise, by assuming a centre of reference and defining direction along the segment of a line as the sense in which movement is made away from the centre of reference along the segment. Along one segment, direction is called positive: along the other, it is called negative. Thus for a line *ACB*, the direction *CB* from *C* towards *B* is taken to be positive (that is, with *C* as the centre of reference), and the direction from *C* towards *A* is taken to be negative (again with *C* as the centre of reference). The direction from *A* to *C* is positive (even with *A* as the centre of reference); and the direction from *B* to *C* is negative (even with *B* as the centre of reference).

The convention, as regards rotations, is less simple and less obvious: and its significance is of most importance in the application to the orientation-cosines of a plane. In the six planes of reference, the positive directions of rotation are selected by adopting, first of all, the convention that is customary in the geometry of three dimensions, and by superposing a convention for positive rotation from each of the three-dimensional axes to the remaining axis in four dimensions. Within this assumption, there is always a choice as to which axis shall be left as the fourth, we assume, as the fourth axis, the line which is associated with the dimension that, as the fourth, is imposed on the customary three dimensions. Accordingly, within the three planes of reference *YOZ*, *ZOX*, *XOY*, in the figure on p. 7, the positive direction of rotation in the plane *YOZ* is from *OY* to *OZ*, in the plane *ZOX* it is from *OZ* to *OX*, and in the plane *XOY* it is from *OX* to *OY*—the customary three-dimensional convention. In the plane *XOV*, the positive direction of rotation is taken to be from *OX* to *OV*; in the plane *YOV*, it is taken to be from *OY* to *OV*; and in the plane *ZOV*, it is taken to be from *OZ* to *OV*.

The orientation-cosines of the planes parallel to

$$\left\| \begin{matrix} x, & y, & z & v \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \end{matrix} \right\| = 0$$

have been taken, by anticipation, in accordance with this convention. They are given by

$$\mathrm{a} = \frac{m_1 n_2 - n_1 m_2}{\sin \omega}, \quad \mathrm{b} = \frac{n_1 l_2 - l_1 n_2}{\sin \omega}, \quad \mathrm{c} = \frac{l_1 m_2 - m_1 l_2}{\sin \omega},$$

$$\mathrm{f} = \frac{l_1 k_2 - k_1 l_2}{\sin \omega}, \quad \mathrm{g} = \frac{m_1 k_2 - k_1 m_2}{\sin \omega}, \quad \mathrm{h} = \frac{n_1 k_2 - k_1 n_2}{\sin \omega},$$

where ω denotes the angle between the two guiding directions of the plane: they are subject to the two universal relations

$$\mathrm{af} + \mathrm{bg} + \mathrm{ch} = 0, \quad \mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2 + \mathrm{f}^2 + \mathrm{g}^2 + \mathrm{h}^2 = 1.$$

Interchange of the guiding lines, estimated analytically, is equivalent to a reversal of rotational direction within the plane and that reversal is met by the accordant change of sign in each of the six orientation-cosines. For calculations concerned with external relations of the plane, we have to deal with the ratios of the orientation-cosines when these occur; so that a hypothetical internal interchange of guiding lines leaves the calculations unaffected. Moreover, these conventions give the following tableau of orientation-cosines characteristic of the six planes of reference:

	a	b	c	f	g	h
$Y\vec{O}Z$	1		.	.	.	.
$Z\vec{O}X$	.	1	.	.	.	.
$X\vec{O}Y$	.	.	1	.	.	.
$X\vec{O}V$			.	1	.	.
$Y\vec{O}V$	.	.	.	.	1	.
$Z\vec{O}V$	.	.	.	.	.	1

(The arrow, above the lettered title of the plane, denotes the conventionally positive direction of rotation within the plane.) The tableau shews that the convention accords with the requirement of making each plane, with the assumed positive direction of rotation within the plane, coincide in sense with itself.

Orientation coordinates of a flat.

113. Conventions as to rotations of a flat have been adopted implicitly, so far as these are concerned with internal rotations; for the customary convention of three-dimensional space, with OX, OY, OZ, as axes of reference, has been adopted. When it is necessary to consider simultaneous movements of the whole flat, without compression or distortion, as by rotation round a normal to the flat, we adopt, as a convention, the assumption that, in any displacement, the three axes of reference are in a configuration which is a continuous displacement of their original configuration so that, without any reversal of sense on the part of any one axis alone, the original configuration can be restored.

As regards orientation coordinates of a flat, determined by three guiding lines implied in the equation

$$\begin{vmatrix} x, & y, & z, & v \\ l, & m, & n, & k \\ l', & m', & n', & k' \\ l'', & m'', & n'', & k'' \end{vmatrix} = 0,$$

we have taken, as magnitudes sufficient for purposes of direction, the direction-cosines of a normal to the flat, which are proportional to the four determinants

$$\begin{vmatrix} m, & n, & k \\ m', & n', & k' \\ m'', & n'', & k'' \end{vmatrix}, \quad -\begin{vmatrix} n, & k, & l \\ n', & k', & l' \\ n'', & k'', & l'' \end{vmatrix}, \quad \begin{vmatrix} k, & l, & m \\ k', & l', & m' \\ k'', & l'', & m'' \end{vmatrix}, \quad -\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix}.$$

The four sets of direction-cosines for the four axes are

$$\begin{array}{lllll} 1, & 0, & 0, & 0, & \text{for } OX, \\ 0, & 1, & 0, & 0, & \ldots\ OY, \\ 0, & 0, & 1, & 0, & \ldots\ OZ, \\ 0, & 0, & 0, & 1, & \ldots\ OV. \end{array}$$

When these are arranged in a determinant of four rows and columns, the same convention gives the direction-cosines of the normal to a flat containing three of the directions as proportional to the minors representing the fourth direction. Thus for the flat $OYZV$, the direction-cosines of the normal are proportional to

$$\left\| \begin{matrix} 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{matrix} \right\|,$$

that is, they are 1, 0, 0, 0: for the flat $OZVX$, they are proportional to

$$-\left\| \begin{matrix} 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \\ 1, & 0, & 0, & 0 \end{matrix} \right\|,$$

that is, they are 0, 1, 0, 0: for the flat $OVXY$, they are proportional to

$$\left\| \begin{matrix} 0, & 0, & 0, & 1 \\ 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \end{matrix} \right\|,$$

that is, they are 0, 0, 1, 0: and for the flat $OXYZ$, they are proportional to

$$-\left\| \begin{matrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \end{matrix} \right\|,$$

that is, they are 0, 0, 0, 1. All these results accord with the convention.

Projection of volumes.

114. The most frequent use of conventions, as regards directions, occurs in connection with line-distances; projections are a conspicuous example. But the conventions are used in connection with magnitudes of different orders, such as areas and volumes, when a sense-direction has significance. Again, the projection of an area is a conspicuous example.

Thus as regards the projection of an area, consider a triangle of which the angular points are: first, the origin; second, a point distant r_1 from the origin along a direction l_1, m_1, n_1, k_1; third, a point distant r_2 from the origin along a direction l_2, m_2, n_2, k_2. The area of the triangle is $\frac{1}{2} r_1 r_2 \sin \omega$, with the former significance of ω. The orientation-cosines of the plane, in which the triangle lies, are a, b, c, f, g, h. The area of projection of the triangle, upon the plane XOY, is

$$c \, . \, \tfrac{1}{2} r_1 r_2 \sin \omega,$$

that is,

$$\tfrac{1}{2} r_1 r_2 (l_1 m_2 - m_1 l_2).$$

But, in that plane XOY, the projected triangle has $r_1 l_1$, $r_1 m_1$; $r_2 l_2$, $r_2 m_2$; and 0, 0; for its angular points, so that its area is

$$\tfrac{1}{2} (r_1 l_1 \, . \, r_2 m_2 - r_1 m_1 \, . \, r_2 l_2).$$

The two results agree. So for the other planes: the requirements are met by the convention.

Similarly, as regards the projections of volumes: a volume can be projected from one flat into another flat: the ratio of the projection to the original volume can be taken as the cosine of the inclination of the two flats, that is, as the cosine of the inclination of their respective normals. Thus consider a parallelepiped of which three adjacent sides are: first, a distance r_1 from the origin along the direction l_1, m_1, n_1, k_1; second, a distance r_2 from the origin along the direction l_2, m_2, n_2, k_2; and third, a distance r_3 from the origin along the direction l_3, m_3, n_3, k_3. This parallelepiped lies in the flat

$$\left| \begin{matrix} x, & y\,, & z\,, & v \\ l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \end{matrix} \right| = 0,$$

and the direction-cosines of the normal to this flat are L, M, N, K, where

$$\Gamma L = \begin{vmatrix} m_1, & n_1, & k_1 \\ m_2, & n_2, & k_2 \\ m_3, & n_3, & k_3 \end{vmatrix}, \quad \Gamma M = -\begin{vmatrix} n_1, & k_1, & l_1 \\ n_2, & k_2, & l_2 \\ n_3, & k_3, & l_3 \end{vmatrix},$$

$$\Gamma N = \begin{vmatrix} k_1, & l_1, & m_1 \\ k_2, & l_2, & m_2 \\ k_3, & l_3, & m_3 \end{vmatrix}, \quad \Gamma K = -\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix},$$

and

$$\Gamma^2 = 1 - \cos^2 \theta_{12} - \cos^2 \theta_{23} - \cos^2 \theta_{13} + 2 \cos \theta_{12} \cos \theta_{23} \cos \theta_{13}.$$

The area of the face of the parallelepiped through the first and the second of the sides is

$$r_1 r_2 \sin \theta_{12}.$$

The perpendicular from the extremity of the third side upon this face is $r_3 \sin \theta$, where θ is the inclination of that side to the face, that is, the inclination of the direction l_3, m_3, n_3, k_3, to the plane with l_1, m_1, n_1, k_1, and l_2, m_2, n_2, k_2, as guiding lines: thus (§ 69) θ is given by

$$\sin \theta \sin \theta_{12} = \Gamma.$$

Hence the volume of the parallelepiped is

$$r_1 r_2 \sin \theta_{12} . r_3 \sin \theta,$$

that is, it is

$$r_1 r_2 r_3 \Gamma.$$

Let this parallelepiped be projected from the flat, in which it lies, into the flat $v = 0$; as the inclination of the flats is the supplement of the inclination of the normals to the flats, the volume of the projection of the parallelepiped is

$$(-k)\, r_1 r_2 r_3 \Gamma,$$

that is, it is

$$r_1 r_2 r_3 \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}.$$

But in the projection, the origin remains unaltered; the projection of the side r_1 is the line joining the origin to the point $r_1 l_1, r_1 m_1, r_1 n_1$; the projection of the side r_2 is the line joining the origin to the point $r_2 l_2, r_2 m_2, r_2 n_2$; and the projection of the side r_3 is the line joining the origin to the point $r_3 l_3, r_3 m_3, r_3 n_3$. Thus the volume of the projection is

$$\begin{vmatrix} r_1 l_1, & r_1 m_1, & r_1 n_1 \\ r_2 l_2, & r_2 m_2, & r_2 n_2 \\ r_3 l_3, & r_3 m_3, & r_3 n_3 \end{vmatrix},$$

agreeing with the former result. Similarly for projections into the other three principal flats of reference.

The agreement of the two results can also be interpreted as giving the orientation coordinates $-L$, $-M$, $-N$, $-K$, of the flat.

Also, the sides r_1, r_2, r_3, with their assigned directions can be projected into another flat

$$\begin{vmatrix} x, & y, & z, & v \\ l_1', & m_1', & n_1', & k_1' \\ l_2', & m_2', & n_2', & k_2' \\ l_3', & m_3', & n_3', & k_3' \end{vmatrix} = 0,$$

after the results of §§ 64, 110. The consequent derivation of the expression

$$\cos^{-1}(LL' + MM' + NN' + KK')$$

as the inclination of the flats is left as an exercise.

Rotations in quadruple space.

115. As with spherical representation in triple space, so with globular representation in quadruple space, the representation of the displacement of an orthogonal frame is facilitated.

Let the orthogonal frame of axes in either of the figures (§§ 9, 111) be displaced to positions OX', OY', OZ', OV', so that the new coordinates of a point x', y', z', v', are connected with the old coordinates x, y, z, v, by relations

$$x', y', z', v' = \left(\begin{array}{cccc} l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \\ l_4, & m_4, & n_4, & k_4 \end{array} \right\rgroup x, y, z, v),$$

the frame being rigid in the displacement. Then the sets of direction-cosines are subject to the ten independent relations

$$\Sigma l_r^2 = 1, \text{ for } \quad r = 1, 2, 3, 4,$$

$$\Sigma l_r l_s = 0, \text{ for } r, s, = 1, 2, 3, 4,$$

the summation being taken over l, m, n, k, in each instance. Thus there are six parameters effectively in any transformation. These six parameters can be associated with six rotations, as explained later (§§ 116, 117).

In quadruple space, there are three significant types of rotation.

A rotation can leave a point unaltered while, around it, there is a general displacement without deformation: it may be described as *rotation round a point*. Thus a rigid globe, with centre O, can be rotated round its centre.

A rotation can leave a line unaltered while, around it, there is a general displacement without deformation; it may be called *rotation round a line* or an axial rotation. The general displacement under rotation round a line is more restricted than the general displacement under rotation round a point; it is of the same character as the general three-dimensional displacement round a point. Thus in the rigid globe, we could have a rotation round the axis OV, the line whose equations are $x=0$, $y=0$, $z=0$; and in the diagram of § 111, the most general displacement, under rotation round the axis OV, is the most general displacement in the flat $v=0$, a triple space. In the displacement, the movement of every point is perpendicular to the axis of rotation: for the instance adduced, every point is displaced within the flat $v=0$.

A rotation can leave a plane unaltered while, around it, that is, in every orthogonal plane, there is a general displacement without deformation: it may be called *rotation round a plane*, or a planar rotation. The general displacement under rotation round a plane is still more restricted than the general displacements in the preceding instances: it is of the same character as the orthogonal displacement of the axes in a plane—such plane being, in this instance, orthogonal to the plane round which the rotation is effected. Thus we can have a rotation round the plane ZOV; it is a rotation of the axes within the plane XOY. Similarly we can have a rotation round the plane XOY; it is a rotation of the axes within the plane ZOV. Manifestly, these two rotations are independent of one another; and, generally, when there is a rotation round a plane P, yielding displacement in an orthogonal plane Q, there can be completely independent rotations round the plane Q, yielding displacements in the plane P.

There are no rotations, in quadruple space, which leave a region (in particular, a flat) unaltered. A displacement in quadruple space, leaving a flat unaltered, is merely a translation along a normal to the flat: it is not a rotation.

Displacement of an orthogonal frame by planar rotations.

116. It is a well-known theorem in the geometry of triple space that an orthogonal frame of three axes, the origin being fixed, can be changed from one position to another by a suitable rotation round a duly chosen axis. The corresponding theorem in the geometry of quadruple space is that *an orthogonal frame of four axes, the origin being fixed, can be changed from any position to any other position by two suitable rotations round two orthogonal planes, the rotation round either plane being independent of the rotation round the other.* To the establishment of this proposition we now proceed.

Rotations[*] round a plane lead to a rotation of the axes of reference in the orthogonal plane. Thus simultaneous independent rotations round two orthogonal planes result in simultaneous independent rotations of the axes of reference in the two planes: and consequently it is convenient to consider simultaneously these independent changes in the positions of the axes in two orthogonal planes. Accordingly, as regards the six coordinate planes of reference when these are taken in orthogonal pairs, there are three distinct operations, each operation consisting of a rotation round one plane of a magnitude that can be chosen at will and of a simultaneous rotation round the orthogonal plane also of a magnitude that can be chosen at will, the two choices being completely independent of one another. These three operations are as follows:

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[*] For rotations in n-fold space, the memoir by C. Jordan (quoted § 124, *post*) should be consulted.

I. A rotation round the plane YOZ through an angle α' and a simultaneous rotation round the plane XOV through an angle α, where α' and α are independent of one another. A point x, y, z, v, in the undisturbed configuration is thus displaced so as to have coordinates X_1, Y_1, Z_1, V_1, where

$$\left.\begin{array}{l} X_1 = \quad x\cos\alpha' + v\sin\alpha' \\ Y_1 = \quad y\cos\alpha + z\sin\alpha \\ Z_1 = -y\sin\alpha + z\cos\alpha \\ V_1 = -x\sin\alpha' + v\cos\alpha' \end{array}\right\} \ldots\ldots\ldots\ldots (\mathrm{I}).$$

II. A rotation round the plane ZOX through an angle β' and a simultaneous rotation round the plane YOV through an angle β, where β' and β are independent of one another. A point x, y, z, v, in the undisturbed configuration is thus displaced so as to have coordinates X_2, Y_2, Z_2, V_2, where

$$\left.\begin{array}{l} X_2 = \quad x\cos\beta - z\sin\beta \\ Y_2 = \quad y\cos\beta' + v\sin\beta' \\ Z_2 = \quad x\sin\beta + z\cos\beta \\ V_2 = -y\sin\beta' + v\cos\beta' \end{array}\right\} \ldots\ldots\ldots\ldots (\mathrm{II}).$$

III. A rotation round the plane XOY through an angle γ' and a simultaneous rotation round the plane ZOV through an angle γ where γ' and γ are independent of one another. A point x, y, z, v, in the undisturbed configuration is thus displaced so as to have coordinates X_3, Y_3, Z_3, V_3, where

$$\left.\begin{array}{l} X_3 = \quad x\cos\gamma + y\sin\gamma \\ Y_3 = -x\sin\gamma + y\cos\gamma \\ Z_3 = \quad z\cos\gamma' + v\sin\gamma' \\ V_3 = -z\sin\gamma' + v\cos\gamma' \end{array}\right\} \ldots\ldots\ldots\ldots (\mathrm{III}).$$

117. Let these operations (or substitutions) be denoted by O_1, O_2, O_3, respectively. Then we proceed to shew that, if x', y', z', v', represent an assigned displaced position of x, y, z, v, owing to a displacement of the quadruple orthogonal system, given by

$$\left.\begin{aligned} x' &= l_1 x + m_1 y + n_1 z + k_1 v \\ y' &= l_2 x + m_2 y + n_2 z + k_2 v \\ z' &= l_3 x + m_3 y + n_3 z + k_3 v \\ v' &= l_4 x + m_4 y + n_4 z + k_4 v \end{aligned}\right\},$$

it is always possible to determine rotations α' and α, round a selected pair of orthogonal planes, the selection being made by operations II and III, and the effect of the rotations α' and α being such that

$$(x', y', z', v') = (O_3^{-1} O_2^{-1} O_1 O_2 O_3 \between x, y, z, v).$$

The effect of the operations, O_3 and then O_2, is to select two suitable orthogonal planes; round these two orthogonal planes, the rotations α' and α are effected, independently of one another, after the rotations have been effected, the effect of the operations O_2^{-1} and O_3^{-1} is to bring back the displaced configuration to the original frame of reference, so as to express x', y', z', v', in terms of x, y, z, v. The sixteen constants, in the relations connecting the original and the final positions, are connected by the ten permanent relations already stated, so that they imply six independent constants. In the sequence of operations indicated, six constants occur that can be selected at will independently of one another, being α', α; β', β; γ', γ. The theorem will be established by shewing that these six constants suffice for the construction of the sixteen related constants; and conversely, it will be shewn that the six constants can be expressed explicitly in terms of the sixteen related constants.

It is convenient to denote the various stages in the transition from x, y, z, v, to x', y', z', v', by means of coordinate symbols at each stage. Accordingly, we write

$$\left.\begin{aligned} (\xi, \eta, \zeta, v) &= (O_3 \between x, y, z, v) \\ (X', Y', Z', V') &= (O_2 \between \xi, \eta, \zeta, v) \\ (X'', Y'', Z'', V'') &= (O_1 \between X', Y', Z', V') \\ (\xi', \eta', \zeta', v') &= (O_2^{-1} \between X'', Y'', Z'', V'') \\ (x', y', z', v') &= (O_3^{-1} \between \xi', \eta', \zeta', v') \end{aligned}\right\}.$$

We have

$$\left.\begin{aligned} X' &= \xi\cos\beta - \zeta\sin\beta, & Y' &= \eta\cos\beta' + v\sin\beta' \\ Z' &= \xi\sin\beta + \zeta\cos\beta, & V' &= -\eta\sin\beta' + v\cos\beta' \end{aligned}\right\},$$

$$\left.\begin{aligned} X'' &= X'\cos\alpha' + V'\sin\alpha', & Y'' &= Y'\cos\alpha + Z'\sin\alpha \\ V'' &= -X'\sin\alpha' + V'\cos\alpha', & Z'' &= -Y'\sin\alpha + Z'\cos\alpha \end{aligned}\right\};$$

$$\left.\begin{aligned} \xi' &= X''\cos\beta + Z''\sin\beta, & \eta' &= Y''\cos\beta' - V''\sin\beta' \\ \zeta' &= -X''\sin\beta + Z''\cos\beta, & v' &= Y''\sin\beta' + V''\cos\beta' \end{aligned}\right\}.$$

When these successive substitutions leading from $\xi, \eta, \zeta, \upsilon$, to $\xi', \eta', \zeta', \upsilon'$, are effected, the results are

$$\begin{aligned}
\xi' &= L_1\xi + M_1\eta + N_1\zeta + K_1\upsilon,\\
\eta' &= L_2\xi + M_2\eta + N_2\zeta + K_2\upsilon,\\
\zeta' &= L_3\xi + M_3\eta + N_3\zeta + K_3\upsilon,\\
\upsilon' &= L_4\xi + M_4\eta + N_4\zeta + K_4\upsilon,
\end{aligned}$$

where

$$\left.\begin{aligned}
L_1 &= \cos\alpha'\cos^2\beta + \cos\alpha\sin^2\beta\\
M_1 &= -\sin\alpha'\cos\beta\sin\beta' - \sin\alpha\sin\beta\cos\beta'\\
N_1 &= -\cos\alpha'\sin\beta\cos\beta + \cos\alpha\sin\beta\cos\beta\\
K_1 &= \sin\alpha'\cos\beta\cos\beta' - \sin\alpha\sin\beta\sin\beta'
\end{aligned}\right\}$$

$$\left.\begin{aligned}
-M_1 = L_2 &= \sin\alpha'\cos\beta\sin\beta' + \sin\alpha\sin\beta\cos\beta'\\
M_2 &= \cos\alpha'\sin^2\beta' + \cos\alpha\cos^2\beta'\\
N_2 &= -\sin\alpha'\sin\beta\sin\beta' + \sin\alpha\cos\beta\cos\beta'\\
K_2 &= -\cos\alpha'\sin\beta'\cos\beta' + \cos\alpha\sin\beta'\cos\beta'
\end{aligned}\right\},$$

$$\left.\begin{aligned}
N_1 = L_3 &= -\cos\alpha'\sin\beta\cos\beta + \cos\alpha\sin\beta\cos\beta\\
-N_2 = M_3 &= \sin\alpha'\sin\beta\sin\beta' - \sin\alpha\cos\beta\cos\beta'\\
N_3 &= \cos\alpha'\sin^2\beta + \cos\alpha\cos^2\beta\\
K_3 &= -\sin\alpha'\sin\beta\cos\beta' - \sin\alpha\cos\beta\sin\beta'
\end{aligned}\right\},$$

$$\left.\begin{aligned}
-K_1 = L_4 &= -\sin\alpha'\cos\beta\cos\beta' + \sin\alpha\sin\beta\sin\beta'\\
K_2 = M_4 &= -\cos\alpha'\sin\beta'\cos\beta' + \cos\alpha\sin\beta'\cos\beta'\\
-K_3 = N_4 &= \sin\alpha'\sin\beta\cos\beta' + \sin\alpha\cos\beta\sin\beta'\\
K_4 &= \cos\alpha'\cos^2\beta' + \cos\alpha\sin^2\beta'
\end{aligned}\right\}$$

We have, as the first stage in the whole transition,

$$\left.\begin{aligned}
\xi &= x\cos\gamma + y\sin\gamma, & \zeta &= z\cos\gamma' + v\sin\gamma'\\
\eta &= -x\sin\gamma + y\cos\gamma, & \upsilon &= -z\sin\gamma' + v\cos\gamma'
\end{aligned}\right\},$$

and, as the last stage in the whole transition,

$$\left.\begin{aligned}
x' &= \xi'\cos\gamma - \eta'\sin\gamma, & z' &= \zeta'\cos\gamma' - \upsilon'\sin\gamma'\\
y' &= \xi'\sin\gamma + \eta'\cos\gamma, & v' &= \zeta'\sin\gamma' + \upsilon'\cos\gamma'
\end{aligned}\right\}$$

In the last stage, substitute the values of $\xi', \eta', \zeta', \upsilon'$, obtained in terms of $\xi, \eta, \zeta, \upsilon$; and then substitute the values of $\xi, \eta, \zeta, \upsilon$, in terms of x, y, z, v, as given in the first stage. We find the values

$$\left.\begin{aligned}
x' &= l_1x + m_1y + n_1z + k_1v\\
y' &= l_2x + m_2y + n_2z + k_2v\\
z' &= l_3x + m_3y + n_3z + k_3v\\
v' &= l_4x + m_4y + n_4z + k_4v
\end{aligned}\right\},$$

where the respective coefficients l, m, n, k, are

$$\left\{\begin{array}{l} l_1 \;\,= L_1 \cos^2 \gamma + M_2 \sin^2 \gamma, \\ m_1 = (L_1 - M_2) \sin \gamma \cos \gamma + M_1, \\ n_1 \;= N_1 \cos \gamma' \cos \gamma - K_1 \sin \gamma' \cos \gamma - N_2 \cos \gamma' \sin \gamma + K_2 \sin \gamma' \sin \gamma, \\ k_1 \;= N_1 \sin \gamma' \cos \gamma + K_1 \cos \gamma' \cos \gamma - N_2 \sin \gamma' \sin \gamma - K_2 \cos \gamma' \sin \gamma, \end{array}\right.$$

$$\left\{\begin{array}{l} l_2 \;\,= (L_1 - M_2) \sin \gamma \cos \gamma - M_1, \\ m_2 = L_1 \sin^2 \gamma + M_2 \cos^2 \gamma, \\ n_2 \;= N_1 \cos \gamma' \sin \gamma - K_1 \sin \gamma' \sin \gamma + N_2 \cos \gamma' \cos \gamma - K_2 \sin \gamma' \cos \gamma, \\ k_2 \;= N_1 \sin \gamma' \sin \gamma + K_1 \cos \gamma' \sin \gamma + N_2 \sin \gamma' \cos \gamma + K_2 \cos \gamma' \cos \gamma; \end{array}\right.$$

$$\left\{\begin{array}{l} l_3 \;\,= N_1 \cos \gamma' \cos \gamma + K_1 \sin \gamma' \cos \gamma + N_2 \cos \gamma' \sin \gamma + K_2 \sin \gamma' \sin \gamma, \\ m_3 = N_1 \cos \gamma' \sin \gamma + K_1 \sin \gamma' \sin \gamma - N_2 \cos \gamma' \cos \gamma - K_2 \sin \gamma' \cos \gamma, \\ n_3 \;= N_3 \cos^2 \gamma' + K_4 \sin^2 \gamma', \\ k_3 \;= (N_3 - K_4) \sin \gamma' \cos \gamma' + K_3; \end{array}\right.$$

$$\left\{\begin{array}{l} l_4 \;\,= N_1 \sin \gamma' \cos \gamma - K_1 \cos \gamma' \cos \gamma + N_2 \sin \gamma' \sin \gamma - K_2 \cos \gamma' \sin \gamma, \\ m_4 = N_1 \sin \gamma' \sin \gamma - K_1 \cos \gamma' \sin \gamma - N_2 \sin \gamma' \cos \gamma + K_2 \cos \gamma' \cos \gamma, \\ n_4 \;= (N_3 - K_4) \sin \gamma' \cos \gamma' - K_3, \\ k_4 \;= N_3 \sin^2 \gamma' + K_4 \cos^2 \gamma'. \end{array}\right.$$

118. Thus the sixteen constants, being direction-cosines in the displaced position, arise in the final result. They are expressed in terms of the six constants α', α; β', β; γ', γ. It is easy to verify that the relations

$$\begin{array}{l} l_r^2 + m_r^2 + n_r^2 + k_r^2 = 1, \\ l_r l_t + m_r m_t + n_r n_t + k_r k_t = 0, \end{array}$$

for $r, t, = 1, 2, 3, 4$, are satisfied, being the necessary ten relations affecting the sixteen constants. Thus the combined operations lead to a general displacement in the quadruple space with a conservation of origin.

Conversely, given a general displacement the sixteen constants of which satisfy the ten relations, we can obtain the two quantities α' and α, and the four quantities β', β, γ', γ, which specify the independent rotations round the two orthogonal planes. The quantities α' and α measure the actual rotations; the two orthogonal planes are $Y'OZ'$ and $X'OV'$ in the intermediate stage, given in terms of β, β', γ, γ', by the equations

$$\left.\begin{array}{l} X' = x \cos \beta \cos \gamma + y \cos \beta \sin \gamma - z \sin \beta \cos \gamma' - v \sin \beta \sin \gamma' \\ Y' = -x \cos \beta' \sin \gamma + y \cos \beta' \cos \gamma - z \sin \beta' \sin \gamma' + v \sin \beta' \cos \gamma' \\ Z' = x \sin \beta \cos \gamma + y \sin \beta \sin \gamma + z \cos \beta \cos \gamma' + v \cos \beta \sin \gamma' \\ V' = x \sin \beta' \sin \gamma - y \sin \beta' \cos \gamma - z \cos \beta' \sin \gamma' + v \cos \beta' \cos \gamma' \end{array}\right\},$$

where the equations of the plane $Y'OZ'$ are $X' = 0$, $V' = 0$, and those of the plane $X'OV'$ are $Y' = 0$, $Z' = 0$.

119. In § 117, the four sets of orthogonal direction-cosines are expressed in terms of six independent quantities $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$; and the ten relations of condition are satisfied by the expressions there given. We now proceed to obtain the converse expressions for the six quantities in terms of the sixteen direction-cosines. We write

$$\left.\begin{aligned}\cos\alpha' + \cos\alpha &= 2P, & \sin\alpha' + \sin\alpha &= 2R\\ \cos\alpha' - \cos\alpha &= 2Q, & \sin\alpha' - \sin\alpha &= 2S\end{aligned}\right\}; \quad \left.\begin{aligned}\beta + \beta' &= B\\ \beta - \beta' &= E\end{aligned}\right\}; \quad \left.\begin{aligned}\gamma + \gamma' &= C\\ \gamma - \gamma' &= F\end{aligned}\right\}.$$

In place of the former sixteen equations, which express the four sets of orthogonal direction-cosines, we take the following sixteen linearly independent equations, easily verified as their equivalent:

$$\tfrac{1}{4}(l_1 + m_2 + n_3 + k_4) = P \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

$$\tfrac{1}{4}(-l_1 - m_2 + n_3 + k_4) = Q\sin B\sin E \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

$$\left.\begin{aligned}\tfrac{1}{2}(m_1 + l_2) &= Q\cos B\cos E\sin 2\gamma\\ \tfrac{1}{2}(l_1 - m_2) &= Q\cos B\cos E\cos 2\gamma\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(3);$$

$$\left.\begin{aligned}-\tfrac{1}{2}(k_3 + n_4) &= Q\cos B\cos E\sin 2\gamma'\\ \tfrac{1}{2}(k_4 - n_3) &= Q\cos B\cos E\cos 2\gamma'\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(4);$$

$$\left.\begin{aligned}\tfrac{1}{4}(-l_2 + m_1 - n_4 + k_3) &= R\sin E\\ \tfrac{1}{4}(l_3 - m_4 - n_1 + k_2) &= R\cos E\sin C\\ \tfrac{1}{4}(-l_4 - m_3 + n_2 + k_1) &= R\cos E\cos C\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5);$$

$$\left.\begin{aligned}\tfrac{1}{4}(l_2 - m_1 - n_4 + k_3) &= S\sin B\\ \tfrac{1}{4}(-l_3 - m_4 + n_1 + k_2) &= S\cos B\sin F\\ \tfrac{1}{4}(-l_4 + m_3 - n_2 + k_1) &= S\cos B\cos F\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

$$\left.\begin{aligned}\tfrac{1}{4}(l_4 + m_3 + n_2 + k_1) &= Q\sin B\cos E\sin C\\ \tfrac{1}{4}(l_3 - m_4 + n_1 - k_2) &= Q\sin B\cos E\cos C\\ \tfrac{1}{4}(-l_4 + m_3 + n_2 - k_1) &= Q\cos B\sin E\sin F\\ \tfrac{1}{4}(l_3 + m_4 + n_1 + k_2) &= Q\cos B\sin E\cos F\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(7)$$

The set (5) of these equations determines R, E, C, and the set (6) determines S, B, F, in each instance merely by polar coordinates in three dimensions. The values of B and E determine β and β', and the values of C and F determine γ and γ': that is, the two orthogonal planes, round which the rotations are to be effected, are determinate. The values of R and S determine $\sin\alpha'$ and $\sin\alpha$. As B and E are now determined, the equation (2) determines Q; also the equation (1) determines P; thus $\cos\alpha'$ and $\cos\alpha$ are determinate. Consequently the rotations α' and α are determinate.

It therefore follows that *any displacement of a quadruple orthogonal frame can be obtained by two single independent rotations, taken independently of one another round two orthogonal planes.*

Further, we have

$$\tan C = \frac{l_4 + m_3 + n_2 + k_1}{l_3 - m_4 + n_1 - k_2}, \quad \cot E \sin C = \frac{l_3 - m_4 + n_1 - k_2}{-l_2 + m_1 - n_4 + k_3},$$

$$\tan F = \frac{-l_4 + m_3 + n_2 - k_1}{l_3 + m_4 + n_1 - k_2}, \quad \cot B \sin F = \frac{-l_3 - m_4 + n_1 + k_2}{l_2 - m_1 - n_4 + k_3},$$

so that B and E, C and F, are determinate: thus, within the frame, the orientations of the two orthogonal planes for rotations are obtained. Then from equations (1), (2), (5), (6), we have values of

$$P, \quad R \sin E, \quad S \sin B, \quad Q \sin B \sin E:$$

that is, values of P, Q, R, S, thus determining the necessary rotations.

It may be remarked that the sixteen equations, which give the values of the four orthogonal sets of direction-cosines, contain only six independent magnitudes; and therefore they must satisfy ten relations. These, in turn, whatever form they take, are equivalent to the ten relations of condition which must be satisfied by four orthogonal sets of direction-cosines; and it is not difficult to verify that these ten relations are actually satisfied by the values of the sixteen quantities l, m, n, k, as obtained in § 117.

Moreover, as will now be shewn, each of the operations in § 116, whether direct or inverse in action, substitutes one orthogonal frame for another; and therefore the gradually cumulative effect, at every stage, is a substitution of that character.

Characteristic property of rotation round a plane.

120. After the foregoing statements, it is not difficult to infer that, so far as concerns all rotations in quadruple space, the fundamental element can be made the rotation round a plane which compels a displacement of perpendicular axes in the orthogonal plane.

The rotation round a point, taken to be the origin, is composite, as will be proved later, the most general rotation of that kind effected without distortion of the space, thus leaving any orthogonal frame undeformed, can be compounded of rotations round planes.

When rotation round a line is effected, that line is unchanged: and the result of the rotation is to have the most completely free movement in the three-dimensional flat normal to the line, because every direction in the flat is perpendicular to the line but otherwise can be unrestricted. Now every such three-dimensional displacement, leaving the origin unchanged and conserving the orthogonality of an orthogonal three-dimensional frame, can be exhibited as a rotation round one axis. In such a rotation, the axis is fixed: the normal to the flat is fixed: and thus the plane in four-dimensional space, through the axis of the component rotation and the normal to the flat, is fixed during the rotation which, accordingly, is a rotation round a plane.

Thus, in quadruple space, every rotation round an axis can be resolved into rotations round planes.

One property of rotation round a plane, which is an essential in the maintenance of the rigidity of an orthogonal frame, can be inferred immediately from the globular representation of a displacement with a fixed origin. In the diagram (Fig. 10), OX, OY, OZ, OV, are a set of lines constituting an orthogonal frame. Let there be a rotation of this frame round the plane VOZ; it is a rotation in the plane XOY upon itself, displacing the perpendicular axes OX and OY to a position Ox and Oy, the angle xOy being a right angle. Thus the arcs Xx and Yy are equal, each of them being a measure of the rotation. Now Z is the pole of the great circle $XxYy$ on the spherical surface XYZ in the flat $V=0$; hence

$$Zx=\tfrac{1}{2}\pi,\quad Zy=\tfrac{1}{2}\pi.$$

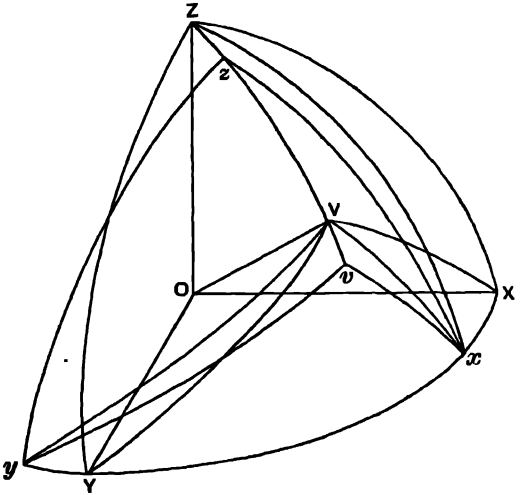

Fig. 10.

Also, V is the pole of the same great circle $XxYy$ on the spherical surface XYV in the flat $Z=0$; hence

$$Vx=\tfrac{1}{2}\pi,\quad Vy=\tfrac{1}{2}\pi.$$

Also $xy=\frac{1}{2}\pi$, $ZV=\frac{1}{2}\pi$; consequently in the frame $xyZV$, we have

$$xy=xZ=xV=yZ=yV=ZV=\tfrac{1}{2}\pi:$$

that is, the frame $xyZV$ is orthogonal, and it is the position of the frame $XYZV$ after a rotation (measured by Xx or Yy) round the plane of reference ZOV.

Next, take a displacement of the frame $xyZV$, round the plane xOy, that is, round the plane XOY which is orthogonal to ZOV the plane of the former rotation: and, by this new rotation, let the axes OZ and OV of the frame $xyZV$ be displaced to the positions Oz and Ov, the angle zOv being a right angle. The arcs Zz and Vv are equal, each of them being a measure of the new rotation. Then, as in the preceding rotation, y is the pole of the great circle $ZzVv$ in the spherical surface yZV, so that

$$yz = \tfrac{1}{2}\pi, \quad yv = \tfrac{1}{2}\pi.$$

Similarly x is the pole of the same great circle in the spherical surface xZV, so that

$$xz = \tfrac{1}{2}\pi, \quad xv = \tfrac{1}{2}\pi.$$

Also $xy = \tfrac{1}{2}\pi$, for it has been unaffected by the rotation; and $zv = \tfrac{1}{2}\pi$. Thus the displaced frame $xyzv$ is orthogonal.

Hence rotation round a plane, and successive rotations round any number of planes, leave the orthogonality of a frame unaffected. Further, rotation round a plane, and another rotation (independent of the first) round the orthogonal plane, can be combined into a single operation; and for such an operation, thus composed of two independent rotations round two orthogonal planes respectively, and displacing the frame from the position $XYZV$ to the position $xyzv$ without disturbance of the orthogonality of the frame, the arc Xx or the arc Yy measures the rotation round one plane, and the arc Zz or the arc Vv measures the rotation round the orthogonal plane. But, as the rotations are independent of one another, the measure of Xx and Yy is unrelated to the measure of Zz and Vv.

Representation of rotations causing a general displacement of a frame

121. The various rotations which in pairs constitute the different operations are represented in the globular illustration (p. 197). The initial configuration is $xyzv$. By the O_3 operation, x and y are displaced to ξ and η along xy, while z and v are displaced to ζ and υ along zv: the result is the configuration $\xi\eta\zeta\upsilon$. By the O_2 operation, ξ and ζ are displaced to X' and Z' along $\xi\zeta$, while η and υ are displaced to Y' and V': the result is the configuration $X'Y'Z'V'$, the arcs of displacement of ξ, η, ζ, υ, alone being shewn, but not the arcs connecting X', Y', Z', V', in pairs. By the O_1 operation, X' and V' are displaced to X'' and V'' along $X'V'$, while Y' and Z' are displaced to Y'' and Z'' along $Y'Z'$: the result is the configuration $X''Y''Z''V''$. At this stage in the analysis, the coordinates of any point are referred to OX'', OY'', OZ'', OV'', as axes. In order to obtain the analytical effect upon the position of a point due to the rotation in the O_1 operation, we reverse the operations which brought the axes to the configuration $X'Y'Z'V'$. Accordingly, to the configuration $X''Y''Z''V''$ we apply, firstly, the O_2^{-1} operation (being the inverse of O_2); and, secondly, the O_3^{-1} operation (being the inverse of O_3). By the O_2^{-1} operation, X'' and Z'' are displaced to ξ' and ζ' along $X''Z''$ backwards along an arc-distance equal to $\xi X'$ and $\zeta Z'$, the forward arc-distance in the corresponding rotation in the O_2 operation; while Y'' and V'' are displaced to η' and υ' along $Y''V''$ backwards along an arc-distance equal to $\eta Y'$ and $\upsilon V'$, the forward arc-distance in the other corresponding rotation in the O_2 operation: the result is the configuration $\xi'\eta'\zeta'\upsilon'$. Finally, by the O_3^{-1} operation, ξ' and η' are displaced to x' and y' along $\xi'\eta'$ backwards along an arc-distance equal to $x\xi$ and $y\eta$, the forward arc-distance in the corresponding rotation in the O_3 operation; while ζ' and υ' are displaced to z' and v' along $\zeta'\upsilon'$ backwards along an arc-distance equal to $z\zeta$ and $v\upsilon$, the forward arc-distance in the other corresponding rotation in the O_3 operation. The final configuration is $x'y'z'v'$, where the arcs are shewn joining the four points in pairs. That configuration results from the displacement of the axes Ox, Oy, Oz, Ov, due to the rotation α' round the plane $Y'OZ'$ and the concomitant rotation α round the orthogonal plane $X'OV'$.

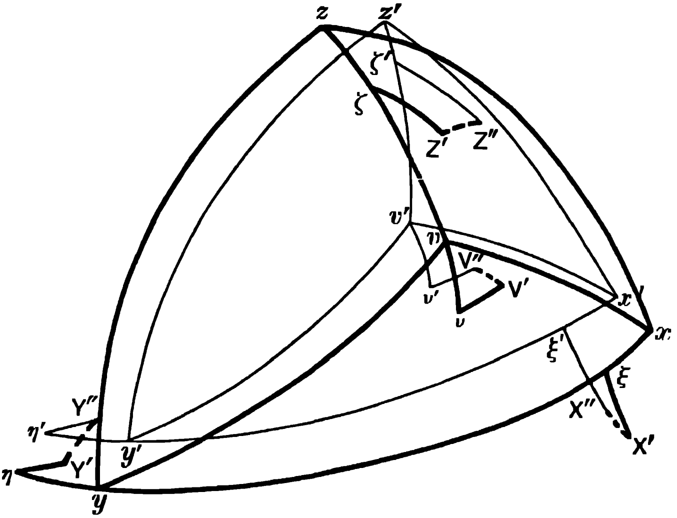

Fig 11.

As regards the various arcs in the figure, which represent the respective rotations, we have

$$X'X'' = V'V'' = \alpha', \quad Y'Y'' = Z'Z'' = \alpha,$$

$$\left.\begin{aligned} \xi X' = \zeta Z' = \zeta' Z'' = \xi' X'' = \beta \\ \eta Y' = \upsilon V' = \upsilon' V'' = \eta' Y'' = \beta' \end{aligned}\right\},$$

$$\left.\begin{aligned} x\xi &= y\eta = x'\xi' = y'\eta' = \gamma \\ z\zeta &= v\upsilon = z'\zeta' = v'\upsilon' = \gamma' \end{aligned}\right\}.$$

The quantities α' and α are the magnitudes of the rotations; if the displacement is infinitesimal, α' and α are small; otherwise they are finite.

The angles β' and β are usually finite, even for the general infinitesimal displacement; and, similarly, the angles γ' and γ are usually finite, even for the general infinitesimal displacement. Thus the only small quantities that can occur in the operations are α' and α; and they are small only for an infinitesimal displacement.

Six modes of generation of a displacement.

122. As regards the succession of rotations in any one operation, their order can be changed without affecting the result: consequently, they have been taken simultaneously for the operation. But a full operation is not commutative with another full operation. Thus the effect of $O_2\{O_3(xyzv)\}$ has been given as the configuration $X'Y'Z'V'$. The effect of $O_3\{O_2(xyzv)\}$, arising from the alteration of the succession of the two operations O_2 and O_3, is to give

$$\left.\begin{aligned}
\overline{X}' &= x\cos\beta\cos\gamma + y\cos\beta'\sin\gamma - z\sin\beta\cos\gamma + v\sin\beta'\sin\gamma\\
\overline{Y}' &= -x\cos\beta\sin\gamma + y\cos\beta'\cos\gamma + z\sin\beta\sin\gamma + v\sin\beta'\cos\gamma\\
\overline{Z}' &= x\sin\beta\cos\gamma' - y\sin\beta'\sin\gamma' + z\cos\beta\cos\gamma' + v\cos\beta'\sin\gamma'\\
\overline{V}' &= -x\sin\beta\sin\gamma' - y\sin\beta'\cos\gamma' - z\cos\beta\sin\gamma' + v\cos\beta'\cos\gamma'
\end{aligned}\right\},$$

which manifestly is different from the configuration $X'Y'Z'V'$.

Now let the operation O_1 be applied to this configuration, so that we have a rotation α' round the plane $\overline{Y}'O\overline{Z}'$ and a simultaneous rotation α round the orthogonal plane $\overline{X}'O\overline{V}'$; and afterwards apply, firstly, the inverse operation O_3^{-1} and, secondly, the inverse operation O_2^{-1}. When the final configuration is taken to be $x'y'z'v'$, represented analytically as before, the parametric quantities β and β', γ and γ', α and α', are involved in the sets of direction-cosines in the forms

$$\left\{\begin{aligned}
l_1 &= L_1\cos^2\beta + N_3\sin^2\beta,\\
m_1 &= M_1\cos\beta'\cos\beta - K_1\sin\beta'\cos\beta - N_2\cos\beta'\sin\beta - K_3\sin\beta'\sin\beta,\\
n_1 &= -(L_1 - N_3)\cos\beta\sin\beta + N_1,\\
k_1 &= M_1\sin\beta'\cos\beta + K_1\cos\beta'\cos\beta - N_2\sin\beta'\sin\beta + K_3\cos\beta'\sin\beta;
\end{aligned}\right.$$

$$\left\{\begin{aligned}
l_2 &= M_1\cos\beta'\cos\beta + K_1\sin\beta'\cos\beta + N_2\cos\beta'\sin\beta - K_3\sin\beta'\sin\beta,\\
m_2 &= M_2\cos^2\beta' + K_4\sin^2\beta',\\
n_2 &= -M_1\cos\beta'\sin\beta - K_1\sin\beta'\sin\beta + N_2\cos\beta'\cos\beta - K_3\sin\beta'\cos\beta,\\
k_2 &= (M_2 - K_4)\cos\beta'\sin\beta' + K_2;
\end{aligned}\right.$$

$$\left\{\begin{aligned}
l_3 &= -(L_1 - N_3)\cos\beta\sin\beta - N_1,\\
m_3 &= -M_1\cos\beta'\sin\beta + K_1\sin\beta'\sin\beta - N_2\cos\beta'\cos\beta - K_3\sin\beta'\cos\beta,\\
n_3 &= L_1\sin^2\beta + N_3\cos^2\beta,\\
k_3 &= -M_1\sin\beta'\sin\beta - K_1\cos\beta'\sin\beta - N_2\sin\beta'\cos\beta + K_3\cos\beta'\cos\beta;
\end{aligned}\right.$$

$$\begin{cases} l_4 = M_1 \sin\beta' \cos\beta - K_1 \cos\beta' \cos\beta + N_2 \sin\beta' \sin\beta + K_3 \cos\beta' \sin\beta, \\ m_4 = (M_2 - K_4)\cos\beta' \sin\beta' - K_2, \\ n_4 = -M_1 \sin\beta' \sin\beta + K_1 \cos\beta' \sin\beta + N_2 \sin\beta' \cos\beta + K_3 \cos\beta' \cos\beta, \\ k_4 = M_2 \sin^2\beta' + K_4 \cos^2\beta', \end{cases}$$

where, with the same signification for P, Q; R, S; B, E; C, F, as before,

$$\begin{array}{llll} L_1 = P + Q\cos 2\gamma & , & M_1 = Q\sin 2\gamma & , \\ N_1 = -R\sin C + S\sin F, & & K_1 = R\cos C + S\cos F, & \\ L_2 = M_1 & , & M_2 = P - Q\cos 2\gamma & , \\ N_2 = R\cos C - S\cos F, & & K_2 = R\sin C + S\sin F, & \\ L_3 = -N_1 & , & M_3 = -N_2 & , \\ N_3 = P - Q\cos 2\gamma' & , & K_3 = -Q\sin 2\gamma' & , \\ L_4 = -K_1 & , & M_4 = -K_2 & , \\ N_4 = K_3 & , & K_4 = P + Q\cos 2\gamma' & . \end{array}$$

The final relations, connecting the sixteen direction-cosines for the displaced configuration with the six parameters (B, E, C, F, and the two parameters α, α', involved in P, Q, R, S), are as follows:

$$\tfrac{1}{4}(l_1 + m_2 + n_3 + k_4) = P \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)';$$

$$\tfrac{1}{4}(-l_1 + m_2 - n_3 + k_4) = Q\sin C\sin F \;\ldots\ldots\ldots\ldots\ldots\ldots (2)';$$

$$\left.\begin{array}{l} \tfrac{1}{2}(l_3 + n_1) = Q\cos C\cos F\sin 2\beta \\ \tfrac{1}{2}(l_1 - n_3) = Q\cos C\cos F\cos 2\beta \end{array}\right\} \ldots\ldots\ldots\ldots (3)';$$

$$\left.\begin{array}{l} -\tfrac{1}{2}(m_4 + k_2) = Q\cos C\cos F\sin 2\beta' \\ \tfrac{1}{2}(m_2 - k_4) = Q\cos C\cos F\cos 2\beta' \end{array}\right\} \ldots\ldots\ldots\ldots (4)';$$

$$\left.\begin{array}{l} \tfrac{1}{4}(l_3 - m_4 - n_1 + k_2) = R\sin C \\ \tfrac{1}{4}(-l_4 - m_3 + n_2 + k_1) = R\cos C\cos E \\ \tfrac{1}{4}(-l_2 + m_1 - n_4 + k_3) = R\cos C\sin E \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (5)';$$

$$\left.\begin{array}{l} \tfrac{1}{4}(-l_3 - m_4 + n_1 + k_2) = S\sin F \\ \tfrac{1}{4}(-l_4 + m_3 - n_2 + k_1) = S\cos F\cos B \\ \tfrac{1}{4}(l_2 - m_1 - n_4 + k_3) = S\cos F\sin B \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (6)';$$

$$\left.\begin{array}{l} \tfrac{1}{4}(l_4 + m_3 + n_2 + k_1) = Q\sin C\sin F\sin B \\ \tfrac{1}{4}(l_2 + m_1 - n_4 - k_3) = Q\sin C\cos F\cos B \\ \tfrac{1}{4}(-l_4 + m_3 + n_2 - k_1) = Q\cos C\sin F\sin E \\ \tfrac{1}{4}(l_2 + m_1 + n_4 + k_3) = Q\cos C\cos F\cos E \end{array}\right\} \ldots\ldots\ldots\ldots (7)'.$$

It thus appears that, given a displacement of the orthogonal frame, there are six different ways in which it can be composed out of the operations which consist, each of them, of simultaneous rotations round two orthogonal planes: for the full transformation leading to the result can be expressed in the form

$$O_r^{-1} O_q^{-1} O_p O_q O_r (x, y, z, v),$$

where the integers p, q, r, are the three integers 1, 2, 3, in any order, associated with the three operations O in §§ 116, 117.

Infinitesimal displacement of a frame.

123. Among the most interesting displacements of an orthogonal frame are those usually called 'small' or 'infinitesimal.' In an infinitesimal displacement, the two effective rotations round the pair of orthogonal planes are of small magnitude; and they remain as independent of one another as for displacements of finite magnitude

Because an infinitesimal displacement, under retention of a fixed origin, means only a small change in the coordinates expressing a position, the smallness of the change can be made evident in the equations of transformation. Accordingly, the equations of general transformation will be appropriately modified so as to exhibit the smallness of the change. Thus, in a transformation which is to be infinitesimal, and initially represented by the equations

$$\left.\begin{aligned} x' &= l_1 x + m_1 y + n_1 z + k_1 v \\ y' &= l_2 x + m_2 y + n_2 z + k_2 v \\ z' &= l_3 x + m_3 y + n_3 z + k_3 v \\ v' &= l_4 x + m_4 y + n_4 z + k_4 v \end{aligned}\right\},$$

these equations must be made to represent each of the magnitudes $x' - x$, $y' - y$, $z' - z$, $v' - v$, as a quantity which is small on account of the constant coefficients. Thus l_1, m_2, n_3, k_4, must, each of them, be nearly equal to unity, let them be $1 - \epsilon_1$, $1 - \epsilon_2$, $1 - \epsilon_3$, $1 - \epsilon_4$, respectively, where ϵ_1, ϵ_2, ϵ_3, ϵ_4, are small quantities. The twelve remaining constants in the transformation must be small quantities. Now

$$l_1^2 + m_1^2 + n_1^2 + k_1^2 = 1,$$

that is,

$$-2\epsilon_1 + \epsilon_1^2 + m_1^2 + n_1^2 + k_1^2 = 0.$$

Thus ϵ_1 is a small quantity of order higher than that of m_1, n_1, k_1, and therefore when we decide to include, in the expression of a small displacement, small quantities of no order higher than the first, we can take ϵ_1 as zero in that expression. Similarly, and under that same decision, we can make $\epsilon_2 = 0$, $\epsilon_3 = 0$, $\epsilon_4 = 0$: that is, for our small orthogonal displacement, we can take

$$l_1 = 1, \quad m_2 = 1, \quad n_3 = 1, \quad k_4 = 1.$$

Again, among the conditions of the orthogonality which is to be maintained in the deformation, there is a condition

$$l_1 l_2 + m_1 m_2 + n_1 n_2 + k_1 k_2 = 0.$$

Now $n_1 n_2$ and $k_1 k_2$ are of the second order of small quantities; when these are neglected in comparison with surviving terms of the first order, this equation gives

$$l_2 + m_1 = 0.$$

Similarly, the other five conditions of orthogonality, which are

$$\Sigma l_1 l_3 = 0, \quad \Sigma l_1 l_4 = 0, \quad \Sigma l_2 l_3 = 0, \quad \Sigma l_2 l_4 = 0, \quad \Sigma l_3 l_4 = 0,$$

for the general transformation, simplify to the respective five conditions

$$l_3 + n_1 = 0, \quad l_4 + k_1 = 0, \quad m_3 + n_2 = 0, \quad m_4 + k_2 = 0, \quad n_4 + k_3 = 0,$$

when the transformation is made infinitesimal. When these relations are satisfied, all the conditions of orthogonality in the displaced transformation are met. Let

$$\begin{aligned} m_1 &= -l_2 = a, & n_1 &= -l_3 = h, & k_1 &= -l_4 = g, \\ n_2 &= -m_3 = b, & k_2 &= -m_4 = f, & k_3 &= -n_4 = c, \end{aligned}$$

so that a, b, c, f, g, h, are small quantities, taken to be of the first order; and, being six in number, they are sufficient to represent a general transformation. Thus the most general infinitesimal transformation can be represented by the equations

$$\left.\begin{aligned} x' &= \phantom{-ax+{}} x + ay + hz + gv \\ y' &= -ax + y + bz + fv \\ z' &= -hx - by + z + cv \\ v' &= -gx - fy - cz + v \end{aligned}\right\},$$

where the conditions of orthogonality are satisfied to the first-order, and the six small constants are independent of one another.

Jordan's theorem on the generation of a small displacement.

124. In a general finite displacement of an orthogonal frame, the transition from the one position to the other can be effected by two rotations round a couple of orthogonal planes, these rotations being finite. In a general small displacement of an orthogonal frame, the transition from the initial position to the final position can be effected also by two rotations round a couple of orthogonal planes; but, now, the two rotations themselves are small.

The analysis, necessary to express the magnitude of the small rotations and the positions of the two orthogonal planes (which are not approximately coincident with coordinate planes of reference), can be investigated from the beginning, by merely making the magnitudes α' and α, that define the two rotations, small quantities initially. It is, however, unnecessary to perform the various operations in detail; the results can be obtained from the results already given, by requiring the quantities α' and α to be small in those results, and still leaving them otherwise independent. We therefore take

$$\left.\begin{aligned} \cos\alpha' &= 1 \\ \sin\alpha' &= \alpha' \end{aligned}\right\}, \quad \left.\begin{aligned} \cos\alpha &= 1 \\ \sin\alpha &= \alpha \end{aligned}\right\}.$$

Then the following values are consequently obtained, for the respective intermediate magnitudes which occur in the investigation of the general transition:

$$\left.\begin{array}{l} P=1 \\ Q=0 \end{array}\right\}; \quad \left.\begin{array}{l} 2R=\alpha'+\alpha \\ 2S=\alpha'-\alpha \end{array}\right\};$$

$$\begin{array}{llll} L_1=1, & M_1= R\sin E-S\sin B, & N_1=0, & K_1=R\cos E+S\cos B, \\ L_2=-R\sin E+S\sin B, & M_2=1, & N_2= R\cos E-S\cos B, & K_2=0; \\ L_3=0, & M_3=-R\cos E+S\cos B, & N_3=1, & K_3=R\sin E+S\sin B; \\ L_4=-R\cos E-S\cos B, & M_4=0, & N_4=-R\sin E-S\sin B, & K_4=1, \end{array}$$

with the relations

$$L_2=-M_1, \quad M_3=-N_2, \quad L_4=-K_1, \quad N_4=-K_3.$$

The final quantities, which express the infinitesimal displacement, are found to be

$$l_1=1, \quad m_2=1, \quad n_3=1, \quad k_4=1,$$

$$\begin{array}{rl} m_1=-l_2 = & R\sin E \qquad\qquad -S\sin B \qquad\qquad =a, \\ n_1=-l_3 = & -R\cos E\sin C+S\cos B\sin F=h, \\ k_1=-l_4 = & R\cos E\cos C+S\cos B\cos F=g, \\ n_2=-m_3 = & R\cos E\cos C-S\cos B\cos F=b, \\ k_2=-m_4 = & R\cos E\sin C+S\cos B\sin F=f, \\ k_3=-n_4 = & R\sin E \qquad\qquad +S\sin B \qquad\qquad =c, \end{array}$$

where all the magnitudes a, b, c, f, g, h, are small, of the same order as the small rotations α' and α. Hence *there is obtained a general small transformation, conforming to the required type.*

Conversely, when an infinitesimal transformation is assigned, the small rotations and the two orthogonal planes which can generate the transformation are derivable from these relations. We have

$$\left.\begin{array}{r} R\cos E\cos C=\tfrac{1}{2}(g+b) \\ R\cos E\sin C=\tfrac{1}{2}(f-h) \\ R\sin E=\tfrac{1}{2}(c+a) \end{array}\right\}, \quad \left.\begin{array}{r} S\cos B\cos F=\tfrac{1}{2}(g-b) \\ S\cos B\sin F=\tfrac{1}{2}(f+h) \\ S\sin B=\tfrac{1}{2}(c-a) \end{array}\right\},$$

which lead to the values of B and E, of C and F, determining the orthogonal planes round which the rotations are effected, and to the values of R and of S, determining the magnitudes of the (small) rotations*.

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* This theorem, as regards the generation of a small displacement of a quadruple orthogonal frame, was first obtained by Camille Jordan, "Essai sur la géométrie à n dimensions," *Bull. Soc. Math. de France*, t. iii (1875), pp. 103–174; see, in particular, pp. 152–171.

Successive small displacements of the orthogonal frame of a skew curve along the curve.

125. The simplest (and, at the same time, the most immediate) application of this result occurs when f, g, h, vanish separately: there then remain three infinitesimal elements in the expressions that define the displacement. As will appear in the section relating to skew curves, this type of displacement arises when the orthogonal frame of the curve moves along the curve to a consecutive point and there becomes the orthogonal frame at that point. The Frenet equations (§ 164) characteristic of a general curve can be written in a form

$$\left.\begin{aligned} x' &= x + y\,d\epsilon \\ y' &= -x\,d\epsilon + y + z\,d\eta \\ z' &= -y\,d\eta + z + v\,d\omega \\ v' &= -z\,d\omega + v \end{aligned}\right\}.$$

The quantities x, y, z, v, now represent the inclinations of any one of the axes OX, OY, OZ, OV, to the principal lines of the frame of reference of the curve at the point (the tangent, the principal normal, the binormal, and the trinormal), so that

$$a = d\epsilon, \quad b = d\eta, \quad c = d\omega, \quad f = g = h = 0.$$

The quantity $d\epsilon$ is the angle of contingence of the curve at the point, $d\eta$ is the angle of torsion, and $d\omega$ is the angle of tilt.

Without pursuing the application of this particular interpretation at this stage, we shall deal only with the actual form of the preceding infinitesimal transformation. As f, g, h, vanish, the equations for the determination of the small rotations α' and α, and for the specification of the two orthogonal planes round which these small rotations are effected, become

$$\left.\begin{aligned} R\cos E\cos C &= \tfrac{1}{2}b \\ R\cos E\sin C &= 0 \\ R\sin E &= \tfrac{1}{2}(c+a) \end{aligned}\right\}, \quad \left.\begin{aligned} S\cos B\cos F &= -\tfrac{1}{2}b \\ S\cos B\sin F &= 0 \\ S\sin B &= \tfrac{1}{2}(c-a) \end{aligned}\right\}.$$

Thus we have $C = 0$, $F = 0$: that is,

$$\gamma = 0, \quad \gamma' = 0.$$

Consequently, the operation O_3 and its inverse O_3^{-1} are not required.

The equations now become

$$\left.\begin{aligned} R\cos E &= \tfrac{1}{2}b \\ R\sin E &= \tfrac{1}{2}(c+a) \end{aligned}\right\}, \quad \left.\begin{aligned} S\cos B &= -\tfrac{1}{2}b \\ S\sin B &= \tfrac{1}{2}(c-a) \end{aligned}\right\}.$$

Hence

$$\tan(\beta - \beta') = \tan E = \frac{a+c}{b},$$

$$\tan(\beta + \beta') = \tan B = \frac{a-c}{b};$$

and therefore

$$\tan 2\beta = \frac{2ab}{c^2 + b^2 - a^2}, \quad \tan 2\beta' = \frac{2cb}{c^2 - b^2 - a^2}.$$

Thus we have the angle β settling the guiding lines in the plane XOZ, and the angle β' settling the guiding lines in the plane YOV; and we thus obtain the planes round which the rotations are effected.

For the magnitudes of the rotations, we have

$$2(\alpha' + \alpha) = 2R = \{b^2 + (a + c)^2\}^{\frac{1}{2}},$$
$$2(\alpha' - \alpha) = 2S = \{b^2 + (a - c)^2\}^{\frac{1}{2}};$$

and therefore

$$\left.\begin{aligned} 4\alpha' &= \{b^2 + (a + c)^2\}^{\frac{1}{2}} + \{b^2 + (a - c)^2\}^{\frac{1}{2}} \\ 4\alpha &= \{b^2 + (a + c)^2\}^{\frac{1}{2}} - \{b^2 + (a - c)^2\}^{\frac{1}{2}} \end{aligned}\right\},$$

the rotation α' being effected round the plane given by axes OY' and OZ', and the rotation α being effected round the orthogonal plane given by axes OX' and OV', where

$$\left.\begin{aligned} X' &= x\cos\beta + z\sin\beta, & Y' &= y\cos\beta' + v\sin\beta' \\ Z' &= -x\sin\beta + z\cos\beta, & V' &= -y\sin\beta' + v\cos\beta' \end{aligned}\right\}.$$

NOTE. The following expressions for an orthogonal transformation are due to Cayley* and can easily be verified:—

Let constants a_{rs}, for $r, s, = 1, 2, 3, 4$, be such that $a_{rr} = 1$, $a_{sr} = -a_{rs}$ when r and s are different, let Δ denote their determinant; and let α_{rs} denote the minors of Δ, according to the formal conventions

$$\alpha_{rr} = \frac{\partial\Delta}{\partial a_{rr}}; \quad \alpha_{rs} = \frac{1}{2}\frac{\partial\Delta}{\partial a_{rs}}.$$

Then an orthogonal transformation between variables x_1, x_2, x_3, x_4, and X_1, X_2, X_3, X_4, is

$$\Delta X_r = 2\sum_s^r \alpha_{sr} x_s + (2\alpha_{rr} - \Delta) x_r, \quad \Delta x_s = 2\sum_r^s \alpha_{sr} X_r + (2\alpha_{ss} - \Delta) X_s,$$

where $\sum\limits_s^r$ denotes summation for all values of s except r, and $\sum\limits_r^s$ denotes summation for all values of r except s.

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* "Sur quelques propriétés des déterminants gauches," (1846), *Coll. Math. Papers*, vol. i, pp. 332—336; see also Salmon's *Modern Higher Algebra*, (1885), p 43.

CHAPTER VIII.

Curves. Principal Lines.

Skew curves: their analytical representation.

126. When we pass from the amplitudes in quadruple space which are characterised, in ultimate resolution, by development from a line, itself with the characteristic of uniformity of direction, and pass to amplitudes of corresponding similar dimensions devoid of the characteristic which may be described vaguely as general evenness, the equations, by which the various types of amplitudes are represented, will no longer be linear: that is to say, not all the equations representing an amplitude will be linear. We can, of course, have a two-dimensional surface which is not a plane and which can, in all its extent, lie in a flat; and the equations of such a surface would be capable of transformation to a shape

$$\phi(x, y, z, v) = 0, \quad ax + by + cz + dv = e,$$

where the function ϕ is not linear in all its arguments. We have had an example (§ 103) in the section of a globe by a flat. Now such a surface would relatively not be of the most general type; for by a transformation of (orthogonal) axes of reference such that in the new system

$$ax + by + cz + dv - e = V,$$

the new forms of the equations would be

$$\Phi(X, Y, Z, V) = 0, \quad V = 0.$$

We should therefore have a surface $\Phi(X, Y, Z, 0) = 0$ in the flat three-dimensional space $V = 0$, the discussion of the geometry of such a surface is that of a surface in a customary three-dimensional space.

We shall assume generally that, where more than one equation is required for the analytical expression of an amplitude, there is no such linear equation either existent in, or derivable from, the analytical expression of the amplitude.

An amplitude of one dimension is called a *curve*, the simplest curve being a straight line or (briefly) a line. There are diverse modes of representing a curve analytically. Thus we might have three independent equations

$$F(x, y, z, v) = 0, \quad G(x, y, z, v) = 0, \quad H(x, y, z, v) = 0,$$

not all of which may be linear; but there is the disadvantage that, unlike the corresponding representation of a line, the three equations may represent two or more curves, which geometrically are discrete though analytically they are the same. Or the coordinates of a point in quadruple space might be represented in terms of three parameters, say

$$x = x(p, q, r), \quad y = y(p, q, r), \quad z = z(p, q, r), \quad v = v(p, q, r),$$

in effect, determining a region within the quadruple space: and then two relations

$$\psi(p, q, r) = 0, \quad \chi(p, q, r) = 0,$$

would determine a curve, lying in the region. Or the coordinates of a point in the quadruple space might be represented in terms of two parameters, say

$$x = x(p, q), \quad y = y(p, q), \quad z = z(p, q), \quad v = v(p, q),$$

in effect, determining a surface, within the quadruple space but otherwise unrestricted: and then one relation

$$\varpi(p, q) = 0$$

would determine a curve, lying on the surface.

The analytical implication of all the different modes of expression is that, throughout the configuration, there subsists only a single independent variable, whatever be the geometrical significance of that variable. As ultimately there subsists one independent variable in the analytical expression of a curve, that variable must be capable of transformation into some other equally independent variable, if such change be found convenient. Our concern, except for the measurement of circular curvature, will seldom be occupied by topographical relations to surrounding space: it will mainly be devoted to intrinsic relations and properties which, even when related to surrounding space, remain intrinsic to the curve. Accordingly, there is an almost overwhelming advantage in using one special intrinsic variable from the beginning of the investigations: we select, for our use, the length of the arc of the curve, measured along the curve from some point of reference. It is a comparatively rare occurrence that this initial point of reference has any importance, so far as concerns the curve itself: circumstances of course change when we have to deal with the properties of a curve, upon a surface or within a region, in connection with that surface or that region. And, also in the comparatively rare instances where the length of the arc happens not to be the appropriate variable, a new variable can be adopted: then it will be found desirable, and usually necessary, to determine the arc of the curve in terms of such a variable.

As a rule, we denote the length of the arc of a curve, measured along the curve from a fixed point on the curve, by s; and then the coordinates of a point on the curve, usually denoted by x, y, z, v, are taken to be functions of this parametric variable s.

Selected magnitudes involving arc-derivatives of the point-coordinates.

127. In the analytical geometry of the curve, we shall have to deal with derivatives of x, y, z, v, with respect to s, of various orders; we shall have to deal also with certain combinations, either pure symmetric or skew symmetric, of these derivatives of various orders. Accordingly, there is a convenience in obtaining initially some algebraical results as affecting these combinations of derivatives, with a tacit restriction to such results as prove useful in the analytical developments of the geometry.

We denote the successive derivatives of x, y, z, v, with respect to s, by the customary notation, and write

$$\frac{dx}{ds}=x', \quad \frac{d^2x}{ds^2}=x'', \quad \frac{d^3x}{ds^3}=x''', \quad \frac{d^4x}{ds^4}=x^{\text{iv}},$$

and so for derivatives of y, z, v. For the most part, derivatives of order higher than four will not be required; if we were dealing with a curve in a space of n dimensions, derivatives of order higher than n would be of infrequent occurrence. For general representation, we write

$$\begin{aligned} s_{mn} &= \frac{d^mx}{ds^m}\frac{d^nx}{ds^n}+\frac{d^my}{ds^m}\frac{d^ny}{ds^n}+\frac{d^mz}{ds^m}\frac{d^nz}{ds^n}+\frac{d^mv}{ds^m}\frac{d^nv}{ds^n} \\ &= \Sigma\frac{d^mx}{ds^m}\frac{d^nx}{ds^n}, \end{aligned}$$

for positive integer values of m and n; and the summation sign, Σ, will regularly be used, without specific mention at each occurrence, to denote a symmetric summation over all the four variables x, y, z, v, to whatever combination as a typical term that sign of summation may be prefixed. Occasionally, other symbols are used for s_{mn}, in connection with the values $m, n, \geqslant 1$ and $\leqslant 4$, and the geometrical significance of some of these combinations is stated, dogmatically at this stage, from their occurrence in the geometry, so that ρ, σ, R, and the like, temporarily remain mere symbols. We write ρ' for $\frac{d\rho}{ds}$, ρ'' for $\frac{d^2\rho}{ds^2}$, and similarly for the arc-derivatives of other quantities. There is, moreover, the fundamental relation

$$ds^2=dx^2+dy^2+dz^2+dv^2,$$

characteristic of homaloidal quadruple space: and this relation is

$$x'^2+y'^2+z'^2+v'^2=1.$$

The full tale of these useful symbols is as follows:

$$\begin{aligned} A &= s_{11}=x'^2+y'^2+z'^2+v'^2, \\ H &= s_{12}=x'x''+y'y''+z'z''+v'v'', \\ B &= s_{22}=x''^2+y''^2+z''^2+v''^2, \\ G &= s_{13}=x'x'''+y'y'''+z'z'''+v'v''', \end{aligned}$$

$$\begin{aligned}
F &= s_{23} = x''x''' + y''y''' + z''z''' + v''v''',\\
C &= s_{33} = x'''^2 + y'''^2 + z'''^2 + v'''^2,\\
L &= s_{14} = x'x^{\text{iv}} + y'y^{\text{iv}} + z'z^{\text{iv}} + v'v^{\text{iv}},\\
M &= s_{24} = x''x^{\text{iv}} + y''y^{\text{iv}} + z''z^{\text{iv}} + v''v^{\text{iv}},\\
N &= s_{34} = x'''x^{\text{iv}} + y'''y^{\text{iv}} + z'''z^{\text{iv}} + v'''v^{\text{iv}},\\
D &= s_{44} = x^{\text{iv}\,2} + y^{\text{iv}\,2} + z^{\text{iv}\,2} + v^{\text{iv}\,2}.
\end{aligned}$$

As already explained, we have

$$A = 1\,;$$

consequently

$$H = \tfrac{1}{2}\frac{dA}{ds} = 0.$$

We write

$$B = x''^2 + y''^2 + z''^2 + v''^2 = \frac{1}{\rho^2}.$$

Because $H = 0$, it follows that

$$\Sigma\,(x'x''' + x''^2) = 0,$$

so that

$$\begin{aligned}
G = s_{13} &= \Sigma x'x'''\\
&= -\Sigma x''^2 = -\frac{1}{\rho^2}.
\end{aligned}$$

Obviously

$$F = \Sigma x''x''' = -\frac{\rho'}{\rho^3}.$$

Again,

$$\begin{aligned}
\frac{dG}{ds} &= \Sigma\,(x'x^{\text{iv}} + x''x''')\\
&= s_{14} + s_{23}\,;
\end{aligned}$$

and therefore

$$\begin{aligned}
L = s_{14} &= G' - s_{23}\\
&= 2\frac{\rho'}{\rho^3} + \frac{\rho'}{\rho^3} = 3\frac{\rho'}{\rho^3}.
\end{aligned}$$

In connection with C, $= s_{33}$, two related new symbols σ and R are introduced, each with a subsequent geometrical significance; we write

$$\begin{aligned}
C = s_{33} &= x'''^2 + y'''^2 + z'''^2 + v'''^2\\
&= \frac{1}{\sigma^2\rho^2} + \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4}\\
&= \frac{R^2 + \sigma^2}{\sigma^2\rho^4},
\end{aligned}$$

where R and σ are connected by the equation

$$R^2 = \rho^2 + \sigma^2\rho'^2.$$

Again,

$$\frac{dF}{ds} = \Sigma\left(x''x^{\text{iv}} + x'''^2\right)$$

$$= s_{24} + s_{33};$$

and therefore

$$\begin{aligned} M &= s_{24} \\ &= \frac{dF}{ds} - C \\ &= -\frac{\rho''}{\rho^3} + 2\frac{\rho'^2}{\rho^4} - \frac{1}{\rho^4} - \frac{1}{\sigma^2\rho^2}. \end{aligned}$$

Finally,

$$\begin{aligned} N &= s_{34} \\ &= \tfrac{1}{2}\frac{dC}{ds} \\ &= \frac{\rho'\rho''}{\rho^4} - 2\frac{\rho'^3}{\rho^5} - 2\frac{\rho'}{\rho^5} - \frac{\rho'}{\sigma^2\rho^3} - \frac{\sigma'}{\sigma^3\rho^2}. \end{aligned}$$

One combination, which frequently recurs, may be noted: it is easy to shew that

$$N + \frac{\rho'}{\rho}M + \frac{1}{\rho^2}L = -\frac{1}{\rho^2\sigma^2}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right).$$

We retain D as the symbol for s_{44}. It will be found convenient, later, to have a value for the determinant Ω defined by

$$\Omega = \begin{vmatrix} x', & y', & z', & v' \\ x'', & y'', & z'', & v'' \\ x''', & y''', & z''', & v''' \\ x^{\text{iv}}, & y^{\text{iv}}, & z^{\text{iv}}, & v^{\text{iv}} \end{vmatrix}.$$

By the customary rule for the square of a determinant, we have

$$\Omega^2 = \begin{vmatrix} s_{11}, & s_{12}, & s_{13}, & s_{14} \\ s_{12}, & s_{22}, & s_{23}, & s_{24} \\ s_{13}, & s_{23}, & s_{33}, & s_{34} \\ s_{14}, & s_{24}, & s_{34}, & s_{44} \end{vmatrix}$$

$$= \begin{vmatrix} 1, & 0, & -\frac{1}{\rho^2}, & L \\ 0, & \frac{1}{\rho^2}, & -\frac{\rho'}{\rho^3}, & M \\ -\frac{1}{\rho^2}, & -\frac{\rho'}{\rho^3}, & \frac{1}{\sigma^2\rho^2} + \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4}, & N \\ L, & M, & N, & D \end{vmatrix}$$

$$= \begin{vmatrix} 1, & 0, & -\dfrac{1}{\rho^2} & , & L \\ 0, & \dfrac{1}{\rho^2}, & -\dfrac{\rho'}{\rho^3} & , & M \\ 0, & 0, & \dfrac{1}{\sigma^2\rho^2} & , & N+\dfrac{\rho'}{\rho}M+\dfrac{1}{\rho^2}L \\ L, & M, & N & , & D \end{vmatrix}$$

$$= \begin{vmatrix} 1, & 0, & 0 & , & L \\ 0, & \dfrac{1}{\rho^2}, & 0 & , & M \\ 0, & 0, & \dfrac{1}{\sigma^2\rho^2} & , & N+\dfrac{\rho'}{\rho}M+\dfrac{1}{\rho^2}L \\ L. & M, & N+\dfrac{\rho'}{\rho}M+\dfrac{1}{\rho^2}L, & & D \end{vmatrix}$$

Consequently

$$\Omega^2 = \frac{D}{\sigma^2\rho^4} - \frac{M^2}{\sigma^2\rho^2} - \frac{L^2}{\sigma^2\rho^4} - \frac{1}{\rho^2}\left(N + \frac{\rho'}{\rho}M + \frac{1}{\rho^2}L\right)^2$$
$$= \frac{D}{\sigma^2\rho^4} - \frac{M^2}{\sigma^2\rho^2} - \frac{L^2}{\sigma^2\rho^4} - \frac{1}{\sigma^4\rho^6}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)^2,$$

a result to be used hereafter.

Tangent line: normal flat.

128. We now come to the geometry of a curve and to the determination of the measures of its deviations, from a straight line and from associated planes and flats along its course.

A point P is taken on the curve, with coordinates x, y, z, v: a contiguous point P' has coordinates $x+dx$, $y+dy$, $z+dz$, $v+dv$, where the variations of the coordinates of P' from those of P are of the first order of small quantities. The length of the arc, ds, being estimated in homaloïdal space, is given by

$$ds^2 = \{(x+dx)-x\}^2 + \{(y+dy)-y\}^2 + \{(z+dz)-z\}^2 + \{(v+dv)-v\}^2$$
$$= dx^2 + dy^2 + dz^2 + dv^2;$$

and, consequently, there is the permanent relation

$$x'^2 + y'^2 + z'^2 + v'^2 = 1.$$

We denote the space-coordinates of a point, current in an amplitude, by $\bar{x}, \bar{y}, \bar{z}, \bar{v}$. The equations of the line PP' are

$$\frac{\bar{x}-x}{(x+dx)-x} = \frac{\bar{y}-y}{(y+dy)-y} = \frac{\bar{z}-z}{(z+dz)-z} = \frac{\bar{v}-v}{(v+dv)-v};$$

and therefore the equations of the *tangent* to the curve at P, being the limit of the line PP' as P' approaches to coincidence with P, are

$$\frac{\bar{x}-x}{x'} = \frac{\bar{y}-y}{y'} = \frac{\bar{z}-z}{z'} = \frac{\bar{v}-v}{v'}.$$

As $x'^2+y'^2+z'^2+v'^2=1$, manifestly x', y', z', v', are the direction-cosines of the tangent. In the accompanying figure, PT is the tangent at P.

The flat, through the point P and normal to the tangent at P, is called the *normal flat* of the curve. It is unique, as there is only one flat through a point to which a given direction is normal. The equation of the normal flat is

$$(\bar{x}-x)\,x'+(\bar{y}-y)\,y'+(\bar{z}-z)\,z'+(\bar{v}-v)\,v'=0.$$

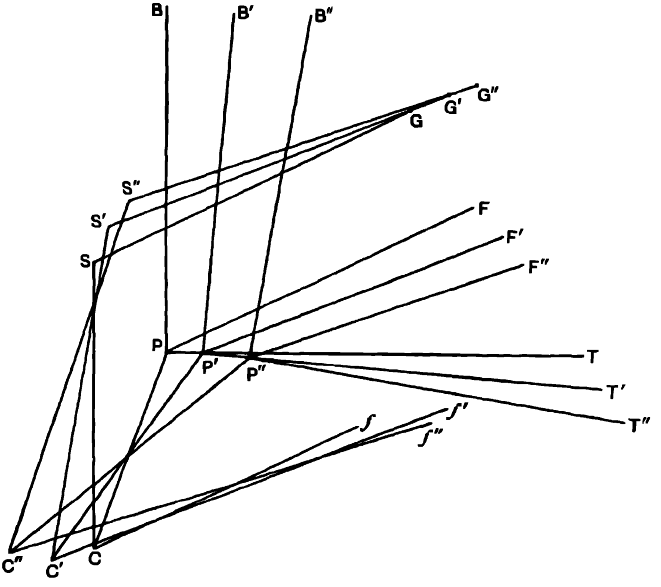

Fig. 12.

Every direction α, β, γ, δ, in the flat is perpendicular to the normal to the flat, so that the equation

$$\alpha x'+\beta y'+\gamma z'+\delta v'=0$$

must be satisfied. In the figure, the normal flat is $PCBF$.

Osculating plane, and an orthogonal plane.

129. A plane, drawn through the tangent at P and containing any assigned direction λ, μ, ν, κ, is given by the equations

$$\left\| \begin{matrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ x', & y', & z', & v' \\ \lambda, & \mu, & \nu, & \kappa \end{matrix} \right\| = 0.$$

Among such planes, let that plane be chosen which, already passing through the consecutive point P' on the tangent at P, contains the direction of the tangent at that point P'. The direction-cosines of the tangent at P', distant ds from P, are

$$x' + x''ds, \quad y' + y''ds, \quad z' + z''ds, \quad v' + v''ds,$$

to the first order of small quantities; hence when these are taken as the values of λ, μ, ν, κ, and the equations are slightly modified, the limiting form of the equations is

$$\left\| \begin{matrix} \bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\ x' \ , & y' \ , & z' \ , & v' \\ x'' \ , & y'' \ , & z'' \ , & v'' \end{matrix} \right\| = 0.$$

This plane, containing the tangent at P, contains a point P' on that tangent consecutive to P; and, containing the tangent at P', contains a point P'' on that tangent consecutive to P'. Thus the plane passes through three points on the curve, viz. P, and two successively consecutive points, P' and P''. As the greatest number of arbitrarily assigned points determining a plane is three, it follows that the selected plane passes through as many points as can be assigned or are required to determine any plane: and as P' and P'' are consecutive to P, this plane lies as close to the curve through P as is possible by selection of points It is therefore called the *osculating plane* of the curve at P.

The equations of the osculating plane can be obtained by making the osculating plane the limiting position of a plane through three consecutive points P, P', P'', on the curve. If the arc $PP' = \sigma_1$, and $PP'' = \sigma_2$, we have

$$x_{P'} = x + \sigma_1 x' + \tfrac{1}{2}\sigma_1^2 x'' + \ldots,$$
$$x_{P''} = x + \sigma_2 x' + \tfrac{1}{2}\sigma_2^2 x'' + \ldots,$$

and so for the other coordinates. The equations of the plane are

$$\left\| \begin{matrix} \bar{x} \ - x, & \bar{y} \ - y, & \bar{z} \ - z, & \bar{v} \ - v \\ x_{P'} - x, & y_{P'} - y, & z_{P'} - z, & v_{P'} - v \\ x_{P''} - x, & y_{P''} - y, & z_{P''} - z, & v_{P''} - v \end{matrix} \right\| = 0;$$

the form of these equations, when we pass to the limit as σ_1 and σ_2 tend to zero independently, is again

$$\left\| \begin{matrix} \bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\ x' \ , & y' \ , & z' \ , & v' \\ x'' \ , & y'' \ , & z'' \ , & v'' \end{matrix} \right\| = 0.$$

In the accompanying figure (p. 211), the osculating plane at P is the limiting position of the plane $TP'PC$.

Further, the equation of the normal flat at P is

$$\Sigma (\bar{x} - x) x' = (\bar{x} - x) x' + (\bar{y} - y) y' + (\bar{z} - z) z' + (\bar{v} - v) v' = 0.$$

The equation of the normal flat at the consecutive point P' is

$$\Sigma (\bar{x} - x - x'ds)(x' + x''ds) = 0,$$

that is,

$$\Sigma (\bar{x} - x) x' + [\{\Sigma (\bar{x} - x) x''\} - 1] ds = 0.$$

By the property of any two flats, the two normal flats at P and P' intersect in a plane: in the limit, the equations of this plane of intersection are

$$\left.\begin{aligned} \Sigma (\bar{x} - x) x' &= 0 \\ \Sigma (\bar{x} - x) x'' &= 1 \end{aligned}\right\}.$$

In the first place, this plane is orthogonal (and not merely perpendicular) to the osculating plane; for every direction α, β, γ, δ, lying in this plane, satisfies the equations

$$\alpha x' + \beta y' + \gamma z' + \delta v' = 0,$$

$$\alpha x'' + \beta y'' + \gamma z'' + \delta v'' = 0,$$

and therefore

$$\alpha (px' + qx'') + \beta (py' + qy'') + \gamma (pz' + qz'') + \delta (pv' + qv'') = 0.$$

But $px' + qx''$, $py' + qy''$, $pz' + qz''$, $pv' + qv''$, are direction-cosines of any direction in the osculating plane. Thus every direction in the new plane is perpendicular to any direction in the osculating plane. The two planes are therefore orthogonal.

Principal normal: circular (prime) curvature.

130. In the next place, this new plane does not pass through the point P; for x, y, z, v, the coordinates of P, do not satisfy the second of its equations. But it must intersect the osculating plane in a point. To find this point of intersection, we take any point

$$\bar{x} = x + px' + qx'', \quad \bar{y} = y + py' + qy'', \quad \bar{z} = z + pz' + qz'', \quad \bar{v} = v + pv' + qv'',$$

in the osculating plane: in order that it may lie in the new plane, we must have

$$\Sigma \{(x + px' + qx'') - x\} x' = 0,$$

$$\Sigma \{(x + px' + qx'') - x\} x'' = 1$$

The former equation is

$$p\Sigma x'^2 + q\Sigma x'x'' = 0.$$

Now $\Sigma x'^2 = 1$ and therefore $\Sigma x'x'' = 0$, hence we have

$$p = 0.$$

The latter equation is

$$p\Sigma x'x'' + q\Sigma x''^2 = 1.$$

We write, as anticipated in § 127,

$$\Sigma x''^2 = x''^2 + y''^2 + z''^2 + v''^2 = \frac{1}{\rho^2},$$

necessarily a positive quantity; and now the latter equation is

$$q = \rho^2.$$

Hence the two planes intersect in a point, C, having coordinates

$$x + \rho^2 x'', \quad y + \rho^2 y'', \quad z + \rho^2 z'', \quad v + \rho^2 v''.$$

For the magnitude of PC, we have

$$PC^2 = \Sigma \{(x + \rho^2 x'') - x\}^2$$
$$= \rho^4 \Sigma x''^2 = \rho^2;$$

and therefore denoting the positive square root of ρ^2 by ρ, we take ρ as the magnitude of PC. Also as

$$x + \rho^2 x'' - x = \rho . \rho x'', \quad y + \rho^2 y'' - y = \rho . \rho y'', \quad z + \rho^2 z'' - z = \rho . \rho z'', \quad v + \rho^2 v'' - v = \rho . \rho v'',$$

and as ρ is the magnitude of PC, it follows that the *direction-cosines of PC, measured from P towards C*, are

$$\rho x'', \; \rho y'', \; \rho z'', \; \rho v''.$$

The point C lies in the new plane (the orthogonal plane), but not the point P: the line PC does not lie in that plane. The point C lies in the osculating plane, and so does the point P: the line PC lies in the osculating plane, passing through P. The direction-cosines of the tangent PT are x', y', z', v', those of the line PC are $\rho x''$, $\rho y''$, $\rho z''$, $\rho v''$; and

$$\rho x'' . x' + \rho y'' . y' + \rho z'' . z' + \rho v'' . v' = 0.$$

Hence PC is perpendicular to PT, and it lies in the osculating plane of the curve; as it is perpendicular to the tangent, it is a normal to the curve.

Again, in the osculating plane, take the line $P'C$, passing through the point P', which is $x + x'ds$, $y + y'ds$, $z + z'ds$, $v + v'ds$, and through the point C, which is $x + \rho^2 x''$, $y + \rho^2 y''$, $z + \rho^2 z''$, $v + \rho^2 v''$; its direction-cosines are proportional to

$$\rho^2 x'' - x'ds, \quad \rho^2 y'' - y'ds, \quad \rho^2 z'' - z'ds, \quad \rho^2 v'' - v'ds.$$

The direction-cosines of the tangent at P', which also lies in the osculating plane, are $x' + x''ds$, $y' + y''ds$, $z' + z''ds$, $v' + v''ds$, up to the first order of small quantities; and

$$\Sigma (\rho^2 x'' - x'ds)(x' + x''ds) = 0,$$

that is, $P'C$ is perpendicular to the tangent at P'. Thus in the plane $PP'RC$, being the osculating plane which contains three consecutive points P, P', P'', on the curve, PC is perpendicular to the chord $P'P$, and $P'C$ is perpendicular to the consecutive chord $P'P''$; and in the plane, a circle can be drawn through the three points P, P', P''. In the limiting position, as P, P', P'', approach to coincidence, we have a circle in the plane, PT is the tangent to the circle; PC and $P'C$ are consecutive perpendiculars in the plane to consecutive tangents; thus C is the centre of the circle.

This circle is therefore called the *circle of prime curvature*, sometimes the *circle of plane curvature*, sometimes merely the *circle of curvature*. Its centre C is called the *centre of prime curvature*, sometimes the *centre of plane curvature*, sometimes the *centre of circular curvature*; and its radius ρ is called the *radius of prime curvature*, or the *radius of plane curvature*, or the *radius of circular curvature*. The line PC, being a normal to the curve and being the direction along which the radius of prime curvature lies, is called the *principal normal* of the curve.

The equations of the principal normal, which has direction-cosines $\rho x''$, $\rho y''$, $\rho z''$, $\rho v''$, are

$$\frac{\bar{x} - x}{x''} = \frac{\bar{y} - y}{y''} = \frac{\bar{z} - z}{z''} = \frac{\bar{v} - v}{v''};$$

the principal normal therefore lies in the osculating plane

$$\left\| \begin{matrix} \bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\ x', & y', & z', & v' \\ x'', & y'', & z'', & v'' \end{matrix} \right\| = 0.$$

It also lies in the normal flat: thus the principal normal of the curve is the intersection of the normal flat by the osculating plane.

To sum up as regards circular (or prime, or plane) curvature, the direction-cosines of the radius, measured from the curve towards the centre, are

$$\rho x'', \ \rho y'', \ \rho z'', \ \rho v'';$$

the coordinates of the centre are

$$\xi_1 = x + \rho^2 x'', \quad \eta_1 = y + \rho^2 y'', \quad \zeta_1 = z + \rho^2 z'', \quad v_1 = v + \rho^2 v'';$$

and the magnitude, ρ, of the radius is the positive square root of ρ^2, where

$$\frac{1}{\rho^2} = x''^2 + y''^2 + z''^2 + v''^2.$$

Angle of contingence.

131. The *angle of contingence* at any point P is the angle between the tangent at P and the tangent at the consecutive point P', this second tangent lying in the osculating plane at P: it is *the angle between two consecutive tangents*. Thus it is the angle between the lines, the respective direction-cosines of which are x', y', z', v', and $x' + x''ds$, $y' + y''ds$, $z' + z''ds$, $v' + v''ds$. When the magnitude of the angle of contingence is denoted by $d\epsilon$, we have

$$\begin{aligned} \sin^2 d\epsilon &= \Sigma \{x'(y' + y''ds) - y'(x' + x''ds)\}^2 \\ &= \Sigma (x'y'' - y'x'')^2 . ds^2 \\ &= \{\Sigma x'^2 \Sigma x''^2 - (\Sigma x'x'')^2\} ds^2 \\ &= \frac{1}{\rho^2} ds^2; \end{aligned}$$

and therefore

$$\frac{1}{\rho}=\frac{d\epsilon}{ds},$$

a result that can be inferred immediately from the geometry of the circle of curvature in the osculating plane.

Two consecutive principal normals do not meet: but their inclination may be noted. If $d\iota$ be the *inclination of two consecutive radii of plane curvature*, we have

$$\sin^2 d\iota = \Sigma\,(lm' - l'm)^2,$$

where

$$l,\ m,\ n,\ k = \rho x'',\ \rho y'',\ \rho z'',\ \rho v'',$$

$$l',\ m',\ n',\ k' = \rho x'' + \frac{d}{ds}(\rho x'')ds,\ \rho y'' + \frac{d}{ds}(\rho y'')ds,\ \rho z'' + \frac{d}{ds}(\rho z'')ds,\ \rho v'' + \frac{d}{ds}(\rho v'')ds.$$

Thus

$$lm' - l'm = \rho^2\,(x''y''' - y''x''')\,ds,$$

and so for the other terms in $\sin^2 d\iota$; consequently

$$\sin^2 d\iota = \rho^4\,\{\Sigma\,(x''y''' - y''x''')^2\}\,ds^2,$$

and therefore

$$\begin{aligned}\left(\frac{d\iota}{ds}\right)^2 &= \rho^4\,\Sigma\,(x''y''' - y''x''')^2\\ &= \rho^4\,\{(\Sigma x''^2)(\Sigma x'''^2) - (\Sigma x''x''')^2\}\\ &= \frac{1}{\sigma^2}+\frac{1}{\rho^2},\end{aligned}$$

on reduction.

The quantity $\dfrac{d\iota}{ds}$ is called the *curvature of screw*; and the angle $d\iota$ is sometimes called the *angle of screw*. The curvature thus named has no centre, has no line for its radius, and is merely a magnitude; as will be seen later, it is merely a combination of two essential curvatures of the curve, and has no independent geometrical significance beyond that already stated.

Equations of the circle of curvature.

132. The equations of the osculating plane are

$$\begin{Vmatrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ x', & y', & z', & v' \\ x'', & y'', & z'', & v'' \end{Vmatrix} = 0.$$

Any direction in that plane is given by direction-cosines

$$\xi x' + \eta x'',\quad \xi y' + \eta y'',\quad \xi z' + \eta z'',\quad \xi v' + \eta v'',$$

provided

$$\Sigma (\xi x' + \eta x'')^2 = 1 ;$$

hence

$$\xi^2 + \frac{1}{\rho^2}\eta^2 = 1,$$

and therefore any direction in the osculating plane has direction-cosines

$$-x' \sin\theta + \rho x'' \cos\theta, \quad -y' \sin\theta + \rho y'' \cos\theta,$$
$$-z' \sin\theta + \rho z'' \cos\theta, \quad -v' \sin\theta + \rho v'' \cos\theta,$$

where θ is a parametric quantity.

Any point in the osculating plane has coordinates

$$x + tx' + n\rho x'', \quad y + ty' + n\rho y'',$$
$$z + tz' + n\rho z'', \quad v + tv' + n\rho v'',$$

where t is a length measured from P along the tangent PT, and n is a length measured from P along the principal normal towards C.

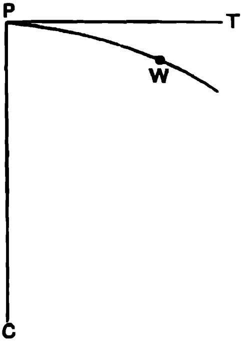

Fig. 13.

When such a point W lies on the circle of plane curvature, we have

$$\Sigma \{x + tx' + n\rho x'' - (x + \rho^2 x'')\}^2 = \rho^2,$$

and therefore

$$t^2 + (n - \rho)^2 = \rho^2,$$

so that this may be taken as an equation of the circle of (plane) curvature in the osculating plane. We must have

$$0 \leqslant n \leqslant 2\rho,$$

while

$$t = (2\rho n - n^2)^{\frac{1}{2}}.$$

In the accompanying figure, the preceding direction is that of WC, where θ is the angle PCW.

Torsion: angle of torsion.

133. We require the small *angle between two consecutive osculating planes*, as at the points P and P'. Because these two planes intersect in a line and not in a point only (their intersection being the tangent at P'), we can, by § 89, adopt the method of three-dimensional geometry and draw lines in the respective planes perpendicular to the common intersection: the angle between the perpendicular lines thus drawn is the inclination of the planes.

The common line, to which these perpendiculars in the respective osculating planes are to be drawn, is the tangent $P'T'$ of which the direction-cosines are

$$x' + x''ds, \quad y' + y''ds, \quad z' + z''ds, \quad v' + v''ds,$$

up to the first order of small quantities. Any direction in the osculating plane at P has direction-cosines

$$\alpha x' + \beta\rho x'', \quad \alpha y' + \beta\rho y'', \quad \alpha z' + \beta\rho z'', \quad \alpha v' + \beta\rho v'',$$

where $\alpha^2 + \beta^2 = 1$. If this direction, lying in the osculating plane at P, is perpendicular to $P'T'$, the condition

$$\Sigma\,(x' + x''ds)\,(\alpha x' + \beta\rho x'') = 0$$

must be satisfied: that is,

$$\alpha + \frac{\beta}{\rho}\,ds = 0.$$

As $\alpha^2 + \beta^2 = 1$, and as we are retaining only the lowest power of ds, these equations give $\beta = 1$ and $\alpha = -\frac{ds}{\rho}$. Consequently the line, with direction-cosines

$$\rho x'' - x'\frac{ds}{\rho}, \quad \rho y'' - y'\frac{ds}{\rho}, \quad \rho z'' - z'\frac{ds}{\rho}, \quad \rho v'' - v'\frac{ds}{\rho},$$

is the perpendicular in the osculating plane at P to the tangent $P'T'$.

The perpendicular in the osculating plane at P' to the tangent $P'T'$ is the principal normal at P', and it has direction-cosines

$$\rho x'' + (\rho x''' + \rho' x'')\,ds, \quad \rho y'' + (\rho y''' + \rho' y'')\,ds,$$
$$\rho z'' + (\rho z''' + \rho' z'')\,ds, \quad \rho v'' + (\rho v''' + \rho' v'')\,ds.$$

Now the angle between two lines with direction-cosines l, m, n, k, and l', m', n', k', is given by the relation (§ 19)

$$\sin^2\chi = \Sigma\,(lm' - l'm)^2,$$

where the summation is taken over the six terms, arising from the six combinations of cosines in pairs. In the present instance,

$$l = \rho x'' - x'\frac{ds}{\rho}, \quad l' = \rho x'' + (\rho x''' + \rho' x'')\,ds,$$

$$m = \rho y'' - y'\frac{ds}{\rho}, \quad m' = \rho y'' + (\rho y''' + \rho' y'')\,ds,$$

and therefore

$$\begin{aligned} lm' - l'm &= \rho^2 (x''y''' - y''x''')\,ds - (x'y'' - y'x'')\,ds \\ &= \{x''(\rho^2 y''' + y') - y''(\rho^2 x''' + x')\}\,ds. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\sin^2\chi}{ds^2} &= \Sigma\,\{x''(\rho^2 y''' + y') - y''(\rho^2 x''' + x')\}^2 \\ &= (\Sigma x''^2)\,\{\Sigma\,(\rho^2 x''' + x')^2\} - \{\Sigma x''(\rho^2 x''' + x')\}^2 \\ &= \frac{1}{\rho^2}(\rho^4 \Sigma x'''^2 + 2\rho^2 \Sigma x' x''' + 1) - \rho^4\frac{\rho'^2}{\rho^6} \\ &= \frac{1}{\sigma^2}, \end{aligned}$$

on reduction, and using the symbols of § 127.

If, then, the *angle between two consecutive osculating planes* be denoted by $d\eta$, we have

$$\frac{d\eta}{ds} = \frac{1}{\sigma}.$$

This quantity $\frac{1}{\sigma}$ may be called the *torsion* of the curve at the point; it represents the arc-rate of turning of the osculating plane round the tangent. But there is no point which can be called the centre of torsion; and there is no line, the direction of which can be called the radius of torsion. Often σ is called the radius of torsion: but, in that use, it is only a magnitude and not a vector.

The angle $d\eta$ is sometimes called the *angle of torsion.*

Orthogonal plane: binormal: trinormal.

134. The equations of the osculating plane, taken in a form to shew that it passes through the centre of plane curvature, are

$$\begin{vmatrix} \bar{x}-x-\rho^2x'', & \bar{y}-y-\rho^2y'', & \bar{z}-z-\rho^2z'', & \bar{v}-v-\rho^2v'' \\ x', & y', & z', & v' \\ x'', & y'', & z'', & v'' \end{vmatrix} = 0.$$

The equations of the plane, orthogonal to the osculating plane at the centre of plane curvature, are

$$\left.\begin{aligned} (\bar{x}-x-\rho^2x'')\,x' + (\bar{y}-y-\rho^2y'')\,y' + (\bar{z}-z-\rho^2z'')\,z' + (\bar{v}-v-\rho^2v'')\,v' &= 0 \\ (\bar{x}-x-\rho^2x'')\,x'' + (\bar{y}-y-\rho^2y'')\,y'' + (\bar{z}-z-\rho^2z'')\,z'' + (\bar{v}-v-\rho^2v'')\,v'' &= 0 \end{aligned}\right\}.$$

When any direction, given by direction-cosines λ, μ, ν, κ, lies in this plane, the relations

$$\lambda x' + \mu y' + \nu z' + \kappa v' = 0,$$

$$\lambda x'' + \mu y'' + \nu z'' + \kappa v'' = 0,$$

must be satisfied. When two distinct sets of values of λ, μ, ν, κ, are obtained, the equations of the plane can be exhibited by means of these directions as guiding lines. Two distinct sets of values, satisfying both relations and perpendicular to one another, are as follows, suggested by later investigations connected with the curve. One such set is given by

$$\left.\begin{aligned} \lambda &= \alpha x' + \beta\rho x'' + \gamma\rho^2 x''' \\ \mu &= \alpha y' + \beta\rho y'' + \gamma\rho^2 y''' \\ \nu &= \alpha z' + \beta\rho z'' + \gamma\rho^2 z''' \\ \kappa &= \alpha v' + \beta\rho v'' + \gamma\rho^2 v''' \end{aligned}\right\},$$

provided α, β, γ, satisfy the relations

$$\alpha\Sigma x'^2 + \beta\rho\Sigma x'x'' + \gamma\rho^2\Sigma x'x''' = 0,$$
$$\alpha\Sigma x'x'' + \beta\rho\Sigma x''^2 + \gamma\rho^2\Sigma x''x''' = 0.$$

The former relation gives

$$\alpha - \gamma = 0;$$

the latter relation gives

$$\beta - \gamma\rho' = 0;$$

and therefore, dropping the factor γ as unnecessary for the immediate purpose though its value will be necessary later, one direction in the plane through C, which is orthogonal to the osculating plane, is given by direction-cosines proportional to

$$x' + \rho\rho' x'' + \rho^2 x''', \quad y' + \rho\rho' y'' + \rho^2 y''', \quad z' + \rho\rho' z'' + \rho^2 z''', \quad v' + \rho\rho' v'' + \rho^2 v'''.$$

Next, a direction L, M, N, K, lying in this orthogonal plane, and perpendicular to λ, μ, ν, κ, in the plane as well as perpendicular to the tangent and the radius of prime curvature that lie out of the plane, satisfies the relations

$$Lx' + My' + Nz' + Kv' = 0,$$
$$Lx'' + My'' + Nz'' + Kv'' = 0,$$
$$L\lambda + M\mu + N\nu + K\kappa = 0.$$

On account of the values of λ, μ, ν, κ, and having regard to the first two equations, we can substitute an equation

$$Lx''' + My''' + Nz''' + Kv''' = 0$$

to be associated with the first two equations, in place of the third. Let

$$J_x = \begin{vmatrix} y', & z', & v' \\ y'', & z'', & v'' \\ y''', & z''', & v''' \end{vmatrix}, \quad J_y = -\begin{vmatrix} z', & v', & x' \\ z'', & v'', & x'' \\ z''', & v''', & x''' \end{vmatrix},$$

$$J_z = \begin{vmatrix} v', & x', & y' \\ v'', & x'', & y'' \\ v''', & x''', & y''' \end{vmatrix}, \quad J_v = -\begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix},$$

then we can take

$$\Theta L = J_x, \quad \Theta M = J_y, \quad \Theta N = J_z, \quad \Theta K = J_v,$$

where

$$\Theta^2 = J_x^2 + J_y^2 + J_z^2 + J_v^2,$$

the value of Θ being unnecessary for the moment. Thus another direction in the plane through C orthogonal to the osculating plane is given by direction-cosines proportional to

$$J_x, \quad J_y, \quad J_z, \quad J_v.$$

Consequently, when λ, μ, ν, κ; and L, M, N, K; are taken as guiding directions, the equations of the plane through the centre of plane curvature orthogonal to the osculating plane are

$$\left.\begin{matrix} \bar{x} - x - \rho^2 x'' , & \bar{y} - y - \rho^2 y'' , & \bar{z} - z - \rho^2 z'' , & \bar{v} - v - \rho^2 v'' \\ x' + \rho\rho' x'' + \rho^2 x''', & y' + \rho\rho' y'' + \rho^2 y''', & z' + \rho\rho' z'' + \rho^2 z''', & v' + \rho\rho' v'' + \rho^2 v''' \\ J_x , & J_y , & J_z , & J_v \end{matrix}\right\| = 0.$$

The direction through the point P on the curve, parallel to the line whose direction-cosines are λ, μ, ν, κ, determined earlier, is called the *binormal* to the curve at the point P; it is the line PB in the figure on p. 211. A direction through P, parallel to the line whose direction-cosines are proportional to J_x, J_y, J_z, J_v, is called the *trinormal* to the curve at the point P; it is the line PF in the same figure. If therefore a plane be drawn through P, containing the binormal at P and the trinormal at P, its equations are

$$\left.\begin{matrix} \bar{x} - x , & \bar{y} - y , & \bar{z} - z , & \bar{v} - v \\ x' + \rho\rho' x'' + \rho^2 x''', & y' + \rho\rho' y'' + \rho^2 y''', & z' + \rho\rho' z'' + \rho^2 z''', & v' + \rho\rho' v'' + \rho^2 v''' \\ J_x , & J_y , & J_z , & J_v \end{matrix}\right\| = 0.$$

This plane is called the *orthogonal plane* at P: and it is orthogonal at P to the osculating plane of the curve at P, being of course parallel to the plane through C orthogonal to the osculating plane. It is the plane BPF in the figure on p. 211.

Orthogonal frame at any point.

135. It is to be noted that, with the point P as vertex, we now have a quadruply orthogonal frame, constituted by the four directions

(i) the tangent, PT,

(ii) the radius of plane curvature, PC,

(iii) the binormal, PB,

(iv) the trinormal, PF,

these four directions being perpendicular in pairs. The first two directions lie in the osculating plane; the last two directions lie in the orthogonal plane.

Osculating flat.

136. The osculating plane has been defined as a plane through two consecutive tangents; and an equivalent property is that it is the limiting position of a plane through three consecutive points.

It is a fundamental property of a flat that it can be made uniquely definite by the assignment of three guiding directions which do not, all of them, lie in one plane. By an equivalent property it can be made uniquely definite through the assignment of four points which do not, all of them, lie in one plane Again, it has been proved that, if two planes intersect in a line, they lie in one flat.

Now consider the osculating plane at P; it contains the tangent at P, and the tangent at a consecutive point P'. Consider the osculating plane at P'; it contains the tangent at P', and the tangent at a consecutive point P''. Thus the osculating plane at P and the osculating plane at P' intersect in the tangent at P': hence there is one flat in which they both lie. As this flat contains the osculating plane at P, it contains the tangent at P' as well as the tangent at P; and as it contains the osculating plane at P', it contains the tangent at P'' as well as the tangent at P'. Consequently the flat contains the tangent at P, the tangent at P', the tangent at P'', three directions which do not, all of them, lie in one plane: it is determined by those three directions. Further, the tangent at P passes through the consecutive point P'; the tangent at P' passes through the consecutive point P''; the tangent at P'' passes through the consecutive point P''': that is, the flat passes through four consecutive points P, P', P'', P''', on the curve, while these four points do not, all of them, lie in one plane: the flat is determinable by those four consecutive points on the curve.

Any flat through the point x, y, z, v, is represented by the equation

$$A(\bar{x}-x)+B(\bar{y}-y)+C(\bar{z}-z)+D(\bar{v}-v)=0;$$

the existence of a direction $\lambda, \mu, \nu, \kappa$, in the flat requires that the condition

$$A\lambda+B\mu+C\nu+D\kappa=0$$

shall be satisfied. Now the tangent at P is given by the direction

$$x', \quad y', \quad z', \quad v';$$

and therefore a relation

$$Ax'+By'+Cz'+Dv'=0$$

is satisfied. The tangent at the consecutive point P' is given by the direction

$$x'+x''ds+\tfrac{1}{2}x'''ds^2+\ldots, \quad y'+y''ds+\tfrac{1}{2}y'''ds^2+\ldots,$$
$$z'+z''ds+\tfrac{1}{2}z'''ds^2+\ldots, \quad v'+v''ds+\tfrac{1}{2}v'''ds^2+\ldots;$$

and therefore a relation

$$\Sigma A(x'+x''ds+\tfrac{1}{2}x'''ds^2+\ldots)=0$$

is satisfied. When account is taken of the former relation, we can modify this equation into

$$\Sigma Ax''+\tfrac{1}{2}(\Sigma Ax''')\,ds+\tfrac{1}{6}(\Sigma Ax^{\text{iv}})\,ds^2+\ldots=0.$$

The tangent at the next consecutive point P'', given by a further increment of arc $d\sigma$ along the curve, is given by the direction

$$x'+x''(ds+d\sigma)+\tfrac{1}{2}x'''(ds+d\sigma)^2+\ldots;$$

and therefore a relation

$$\Sigma A\,\{x' + x''(ds + d\sigma) + \tfrac{1}{2}x'''(ds + d\sigma)^2 + \ldots\} = 0$$

is satisfied. When account is taken of the relation first established, this equation can be modified to the form

$$\Sigma Ax'' + \tfrac{1}{2}(\Sigma Ax''')(ds + d\sigma) + \tfrac{1}{6}\Sigma Ax^{\text{iv}}(ds + d\sigma)^2 + \ldots = 0\,;$$

and when account is taken of the second relation, this equation can be modified to

$$\tfrac{1}{2}\Sigma Ax''' + \tfrac{1}{6}\Sigma Ax^{\text{iv}}(d\sigma + 2ds) + \tfrac{1}{24}\Sigma Ax^{\text{v}}(d\sigma^2 + 3d\sigma\,ds + 3ds^2) + \ldots = 0.$$

But a fourth consecutive tangent cannot lie in the flat, and therefore no further condition is to be imposed upon the constants in the equation of the flat.

We now pass to the limit by making the arcs ds and $d\sigma$ decrease independently towards the length zero. The first condition

$$Ax' + By' + Cz' + Dv' = 0$$

is unchanged; the second condition becomes

$$Ax'' + By'' + Cz'' + Dv'' = 0\,;$$

and the third condition becomes

$$Ax''' + By''' + Cz''' + Dv''' = 0.$$

Consequently the equation of the flat is

$$\begin{vmatrix} \bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\ x', & y', & z', & v' \\ x'', & y'', & z'', & v'' \\ x''', & y''', & z''', & v''' \end{vmatrix} = 0.$$

As the flat contains three consecutive tangents, the greatest possible number of non-complanar directions through which a flat can pass, it has the closest possible association at P which can be obtained by any flat through P; accordingly, it is called the *osculating flat* at the point P of the curve.

137. The same result follows by requiring the flat

$$A(\bar{x} - x) + B(\bar{y} - y) + C(\bar{z} - z) + D(\bar{v} - v) = 0,$$

which already passes through the point P, to pass through three points consecutive to P. For any point on the curve near P, the coordinates are

$$\begin{aligned} \bar{x} &= x + x'ds + \tfrac{1}{2}x''ds^2 + \tfrac{1}{6}x'''ds^3 + \tfrac{1}{24}x^{\text{iv}}ds^4 + \ldots, \\ \bar{y} &= y + y'ds + \tfrac{1}{2}y''ds^2 + \tfrac{1}{6}y'''ds^3 + \tfrac{1}{24}y^{\text{iv}}ds^4 + \ldots, \\ \bar{z} &= z + z'ds + \tfrac{1}{2}z''ds^2 + \tfrac{1}{6}z'''ds^3 + \tfrac{1}{24}z^{\text{iv}}ds^4 + \ldots, \\ \bar{v} &= v + v'ds + \tfrac{1}{2}v''ds^2 + \tfrac{1}{6}v'''ds^3 + \tfrac{1}{24}v^{\text{iv}}ds^4 + \ldots. \end{aligned}$$

When these values are substituted, the requirement as to passing through three consecutive points is satisfied by making the coefficients of the first three powers of ds, viz. the powers ds, ds^2, ds^3, vanish, and by excluding the necessary evanescence of higher powers of ds: thus we have

$$\begin{aligned} Ax' + By' + Cz' + Dv' &= 0, \\ Ax'' + By'' + Cz'' + Dv'' &= 0, \\ Ax''' + By''' + Cz''' + Dv''' &= 0, \end{aligned}$$

while

$$Ax^{\text{iv}} + By^{\text{iv}} + Cz^{\text{iv}} + Dv^{\text{iv}}$$

does not vanish.

The three conditions lead to the same equation as before for the osculating flat.

The exclusion of the magnitude ΣAx^{iv} from evanescence implies that the quantity Ω (of § 127) does not vanish. As will be seen later (§ 146), the vanishing of Ω at the point P would mean that P is a point of zero tilt; the vanishing of Ω everywhere on the curve would mean that the curve is a three-dimensional curve, existing (that is to say) wholly in one flat.

138. Finally, as regards the construction of the equation of the flat, the same result follows from determining the flat in which the osculating plane at P and the osculating plane at the consecutive point P' lie. The equations of the osculating plane at P, which of course contains the consecutive point P', can be taken

$$\left\| \begin{matrix} \bar{x} - x - x'ds, & \bar{y} - y - y'ds, & \bar{z} - z - z'ds, & \bar{v} - v - v'ds \\ x' \quad , & y' \quad , & z' \quad , & v' \\ x'' \quad , & y'' \quad , & z'' \quad , & v'' \end{matrix} \right\| = 0,$$

so that every point in the plane is given by expressions

$$\left.\begin{aligned} \bar{x} - x &= x'ds + \lambda x' + \mu x'' \\ \bar{y} - y &= y'ds + \lambda y' + \mu y'' \\ \bar{z} - z &= z'ds + \lambda z' + \mu z'' \\ \bar{v} - v &= v'ds + \lambda v' + \mu v'' \end{aligned}\right\}.$$

The osculating plane at P', containing the tangents at P' and at P'', has equations

$$\left\| \begin{matrix} \bar{x} - x - x'ds, & \bar{y} - y - y'ds, & \bar{z} - z - z'ds, & \bar{v} - v - v'ds \\ x' + x''ds \quad , & y' + y''ds \quad , & z' + z''ds \quad , & v' + v''ds \\ x'' + x'''ds \quad , & y'' + y'''ds \quad , & z'' + z'''ds \quad , & v'' + v'''ds \end{matrix} \right\| = 0,$$

so that every point in this plane is given by expressions

$$\left.\begin{aligned} \bar{x} - x &= x'ds + \epsilon(x' + x''ds) + \eta(x'' + x'''ds) \\ \bar{y} - y &= y'ds + \epsilon(y' + y''ds) + \eta(y'' + y'''ds) \\ \bar{z} - z &= z'ds + \epsilon(z' + z''ds) + \eta(z'' + z'''ds) \\ \bar{v} - v &= v'ds + \epsilon(v' + v''ds) + \eta(v'' + v'''ds) \end{aligned}\right\}.$$

In the first double infinitude of points, λ and μ are parameters; in the second double infinitude of points, ϵ and η are parameters.

Both double infinitudes are included in the aggregate

$$\left.\begin{aligned} \bar{x} - x &= \alpha x' + \beta x'' + \gamma x''' \\ \bar{y} - y &= \alpha y' + \beta y'' + \gamma y''' \\ \bar{z} - z &= \alpha z' + \beta z'' + \gamma z''' \\ \bar{v} - v &= \alpha v' + \beta v'' + \gamma v''' \end{aligned}\right\},$$

where α, β, γ, are parameters. The first infinitude arises for the values

$$\alpha = \lambda + ds, \quad \beta = \mu, \quad \gamma = 0;$$

the second infinitude arises for the values

$$\alpha = \epsilon + ds, \quad \beta = \eta + \epsilon ds, \quad \gamma = \eta ds.$$

Hence all points in both the planes, that is, both the planes themselves, lie in a flat

$$\begin{vmatrix} \bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\ x', & y', & z', & v' \\ x'', & y'', & z'', & v'' \\ x''', & y''', & z''', & v''' \end{vmatrix} = 0,$$

which accordingly is the osculating flat at the point P of the curve.

The osculating flat contains the tangent, the principal normal, and the binormal.

139. Moreover, the last set of equations, expressing $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{v}$, in terms of the three parameters α, β, γ, give the coordinates of any point within the flat.

Further, any direction l, m, n, k, lying in the flat, is given by the equations

$$\left.\begin{aligned} l &= \lambda x' + \mu x'' + \nu x''' \\ m &= \lambda y' + \mu y'' + \nu y''' \\ n &= \lambda z' + \mu z'' + \nu z''' \\ k &= \lambda v' + \mu v'' + \nu v''' \end{aligned}\right\},$$

where λ, μ, ν, are parameters; and as $l^2 + m^2 + n^2 + k^2 = 1$, these parameters are subject to the relation

$$\Sigma\,(\lambda x' + \mu x'' + \nu x''')^2 = 1,$$

that is,

$$\lambda^2 + \frac{1}{\rho^2}(\mu^2 - 2\lambda\nu) - 2\frac{\rho'}{\rho^3}\mu\nu + \left(\frac{1}{\sigma^2\rho^2} + \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4}\right)\nu^2 = 1.$$

When the equation of the flat is taken in the form

$$L(\bar{x}-x)+M(\bar{y}-y)+N(\bar{z}-z)+K(\bar{v}-v)=0,$$

the direction-cosines of the normal to the flat are proportional to L, M, N, K; or they actually are L, M, N, K, when the relation

$$L^2+M^2+N^2+K^2=1$$

holds. As in § 134, we define

$$J_x=\begin{vmatrix} y', & z', & v' \\ y'', & z'', & v'' \\ y''', & z''', & v''' \end{vmatrix}, \quad J_y=-\begin{vmatrix} z', & v', & x' \\ z'', & v'', & x'' \\ z''', & v''', & x''' \end{vmatrix},$$

$$J_z=\begin{vmatrix} v', & x', & y' \\ v'', & x'', & y'' \\ v''', & x''', & y''' \end{vmatrix}, \quad J_v=-\begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix};$$

and then, if

$$L=\theta J_x, \quad M=\theta J_y, \quad N=\theta J_z, \quad K=\theta J_v,$$

the value of θ is given by

$$\frac{1}{\theta^2}=J_x^2+J_y^2+J_z^2+J_v^2=\begin{vmatrix} s_{11}, & s_{12}, & s_{13} \\ s_{12}, & s_{22}, & s_{23} \\ s_{13}, & s_{23}, & s_{33} \end{vmatrix}$$

$$=\begin{vmatrix} 1, & 0, & -\frac{1}{\rho^2} \\ 0, & \frac{1}{\rho^2}, & -\frac{\rho'}{\rho^3} \\ -\frac{1}{\rho^2}, & -\frac{\rho'}{\rho^3}, & \frac{1}{\sigma^2\rho^2}+\frac{1}{\rho^4}+\frac{\rho'^2}{\rho^4} \end{vmatrix}$$

$$=\frac{1}{\sigma^2\rho^4}.$$

We may take the positive root $\sigma\rho^2$ as the value of θ; and then the direction-cosines of the normal to the flat are

$$\sigma\rho^2 J_x, \quad \sigma\rho^2 J_y, \quad \sigma\rho^2 J_z, \quad \sigma\rho^2 J_v.$$

This line is the trinormal (§§ 134, 141).

In the equation of the flat as given, three guiding lines are indicated. One of these has direction-cosines x', y', z', v'; it is the tangent PT at P to the curve. The second has direction-cosines $\rho x'', \rho y'', \rho z'', \rho v''$; it is the principal normal PC at P to the curve, lying in its osculating plane. The third has direction-cosines proportional to x''', y''', z''', v'''. But while the first two directions are perpendicular to one another, the third direction is not perpendicular to either of the first two; and it is convenient, sometimes to substitute for this third direction, and usually to associate with the first two directions, a new third direction lying in the flat and perpendicular to the first two. Let this third direction be given by the direction-cosines l, m, n, k, as on p. 225. We are to have

$$\begin{aligned} lx' + my' + nz' + kv' &= 0, \\ lx'' + my'' + nz'' + kv'' &= 0. \end{aligned}$$

From the former, we have

$$\lambda - \frac{\nu}{\rho^2} = 0;$$

and from the latter, we have

$$\frac{\mu}{\rho^2} - \nu\frac{\rho'}{\rho^3} = 0.$$

Accordingly, we substitute these values of λ and μ in the relation connecting λ, μ, ν, expressing the property that $l^2 + m^2 + n^2 + k^2 = 1$; and we find

$$\nu^2 = \sigma^2\rho^2.$$

When the positive square root is taken as the value of ν, the direction-cosines of the new direction in the osculating flat, perpendicular to the tangent and the principal normal, are

$$\frac{\sigma}{\rho}(x' + \rho\rho' x'' + \rho^2 x'''), \quad \frac{\sigma}{\rho}(y' + \rho\rho' y'' + \rho^2 y'''),$$
$$\frac{\sigma}{\rho}(z' + \rho\rho' z'' + \rho^2 z'''), \quad \frac{\sigma}{\rho}(v' + \rho\rho' v'' + \rho^2 v''').$$

If it is desired to shew this direction in the equation of the flat, the equation is

$$\begin{vmatrix} \bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\ x', & y', & z', & v' \\ x'', & y'', & z'', & v'' \\ x' + \rho\rho' x'' + \rho^2 x''', & y' + \rho\rho' y'' + \rho^2 y''', & z' + \rho\rho' z'' + \rho^2 z''', & v' + \rho\rho' v'' + \rho^2 v''' \end{vmatrix} = 0;$$

and, now, the three guiding lines of the flat, as indicated in this equation, are perpendicular to one another. The third guiding line in the equation is the binormal, as already (§ 134) defined.

Normal plane.

140. The equation of the normal flat at P, that is, the flat perpendicular to the tangent at P, is

$$(\bar{x} - x)x' + (\bar{y} - y)y' + (\bar{z} - z)z' + (\bar{v} - v)v' = 0.$$

Any point in the osculating flat at P is given by

$$\begin{aligned} \bar{x} - x &= \alpha x' + \beta x'' + \gamma x''', \\ \bar{y} - y &= \alpha y' + \beta y'' + \gamma y''', \\ \bar{z} - z &= \alpha z' + \beta z'' + \gamma z''', \\ \bar{v} - v &= \alpha v' + \beta v'' + \gamma v'''. \end{aligned}$$

When such a point lies in the normal flat, α, β, γ, satisfy the relation

$$\alpha\Sigma x'^2 + \beta\Sigma x'x'' + \gamma\Sigma x'x''' = 0,$$

that is,

$$\alpha - \gamma\frac{1}{\rho^2} = 0.$$

Consequently the intersection, of the normal flat at P and the osculating flat at P, is the aggregate of points given by

$$\begin{aligned}
\bar{x} - x &= \alpha(x' + \rho^2 x''') + \beta x'' = \alpha(x' + \rho\rho' x'' + \rho^2 x''') + (\beta - \alpha\rho\rho')x'',\\
\bar{y} - y &= \alpha(y' + \rho^2 y''') + \beta y'' = \alpha(y' + \rho\rho' y'' + \rho^2 y''') + (\beta - \alpha\rho\rho')y'',\\
\bar{z} - z &= \alpha(z' + \rho^2 z''') + \beta z'' = \alpha(z' + \rho\rho' z'' + \rho^2 z''') + (\beta - \alpha\rho\rho')z'',\\
\bar{v} - v &= \alpha(v' + \rho^2 v''') + \beta v'' = \alpha(v' + \rho\rho' v'' + \rho^2 v''') + (\beta - \alpha\rho\rho')v'';
\end{aligned}$$

that is, the specified intersection is the plane

$$\left\| \begin{array}{cccc}
\bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\
x'', & y'', & z'', & v'' \\
x' + \rho\rho' x'' + \rho^2 x''', & y' + \rho\rho' y'' + \rho^2 y''', & z' + \rho\rho' z'' + \rho^2 z''', & v' + \rho\rho' v'' + \rho^2 v'''
\end{array} \right\| = 0.$$

This plane passes through the line whose direction-cosines are proportional to

$$x' + \rho\rho' x'' + \rho^2 x''',\quad y' + \rho\rho' y'' + \rho^2 y''',\quad z' + \rho\rho' z'' + \rho^2 z''',\quad v' + \rho\rho' v'' + \rho^2 v''';$$

and this last line, the binormal, is perpendicular to every line in the osculating plane. The plane, passing through the line, therefore either is perpendicular or is orthogonal to the osculating plane. It cannot be orthogonal to that plane, because both planes contain the line PC, the radius of circular curvature, it therefore is perpendicular to the osculating plane at P. We call it the *normal plane* at P.

Again, the binormal lies in the orthogonal plane at P, so that the orthogonal plane at P and the normal plane at P intersect in this line. The binormal thus lies in the normal plane, and it is perpendicular to every direction in the osculating plane: but it is, of course, only one of the infinitude of directions (all lying in the orthogonal plane) which are perpendicular to every direction in the osculating plane.

The trinormal: it is perpendicular to the osculating flat.

141. The normal to the osculating flat has

$$\sigma\rho^2 J_x,\quad \sigma\rho^2 J_y,\quad \sigma\rho^2 J_z,\quad \sigma\rho^2 J_v,$$

for its direction-cosines. It lies in the orthogonal plane at P. It is perpendicular to the tangent, it is perpendicular to the principal normal, and it is perpendicular to the binormal; accordingly, it is called the *trinormal*, as well as the normal to the osculating flat.

In the figure 12 in § 128, the tangent at P is PT; the principal normal at P is PC; the binormal at P is PB; and the trinormal at P is PF. We have seen that these four lines PT, PC, PB, PF, constitute a quadruply orthogonal frame at P. The plane through PT and PC is the osculating plane at P; the plane through PC and PB is the normal plane at P; and the plane through PB and PF is the orthogonal plane at P. The flat $PTCB$ is the osculating flat at P; and PF is the normal at P to this flat, being the trinormal.

Centre of second (spherical) curvature, within the osculating flat.

142. The intersection of the normal flat at P and the normal flat at the consecutive point P' is a plane through C, the centre of circular curvature: this plane is orthogonal to the osculating plane. Next, the intersection of the normal flat at P' and the normal flat at the consecutive point P'' is another plane, orthogonal at C' to the osculating plane at P', where C' is the centre of plane curvature at P'. The two planes, orthogonal to the two osculating planes at P and P', intersect in a line, for they lie, both of them, in the normal flat at P', the middle one of the three consecutive points: and manifestly this new line is the intersection of the three consecutive normal flats.

Now the equations of the plane, which is the intersection of the normal flat at P and the normal flat at P', are (§ 129)

$$\Sigma(\bar{x}-x)x' = 0, \quad \Sigma(\bar{x}-x)x'' = 1.$$

The equations of the plane, which is the intersection of the normal flat at P' and the normal flat at the next consecutive point P'', the arc $P'P''$ being $d\sigma$, are

$$\Sigma(\bar{x}-x)x' + \{\Sigma(\bar{x}-x)x'' - \Sigma x'^2\}\,d\sigma = 0,$$

$$\Sigma(\bar{x}-x)x'' + \{\Sigma(\bar{x}-x)x''' - \Sigma x'x''\}\,d\sigma = 1;$$

and therefore the intersection of the two planes, which meet in a line because they lie in one flat—or, what is the same thing, the intersection of the three consecutive normal flats—is, in the limit, given by the three equations

$$\Sigma(\bar{x}-x)x' = 0, \quad \Sigma(\bar{x}-x)x'' = 1, \quad \Sigma(\bar{x}-x)x''' = 0.$$

The intersection of the osculating plane (through two consecutive tangents) by the plane, which itself is the intersection of two consecutive normal flats, gave the centre of circular curvature in the osculating plane. The intersection of the osculating flat (through three consecutive tangents) by the foregoing line, which itself is the intersection of three consecutive normal flats, gives a centre of curvature in the osculating flat. It could be called the centre of flat curvature, or the centre of second curvature: manifestly it is distinct from the centre of plane curvature (or prime curvature). Let it be the point S: the coordinates of S are determinable as the intersection of the flat

$$\begin{vmatrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ x' \;, & y' \;, & z' \;, & v' \\ x'' \;, & y'' \;, & z'' \;, & v'' \\ x''' \;, & y''' \;, & z''' \;, & v''' \end{vmatrix} = 0$$

by the line, whose equations are

$$\Sigma(\bar{x}-x)x' = 0, \quad \Sigma(\bar{x}-x)x'' = 1, \quad \Sigma(\bar{x}-x)x''' = 0.$$

To find these coordinates, we take a current point in the flat

$$\begin{aligned} \bar{x}-x &= \alpha x' + \beta x'' + \gamma x''', \\ \bar{y}-y &= \alpha y' + \beta y'' + \gamma y''', \\ \bar{z}-z &= \alpha z' + \beta z'' + \gamma z''', \\ \bar{v}-v &= \alpha v' + \beta v'' + \gamma v''', \end{aligned}$$

and make the parameters α, β, γ, of these coordinates satisfy the three equations of the line. Thus, from the first equation,

$$\alpha\Sigma x'^2 + \beta\Sigma x'x'' + \gamma\Sigma x'x''' = 0,$$

that is,

$$\alpha - \gamma\frac{1}{\rho^2} = 0;$$

from the second equation,

$$\alpha\Sigma x'x'' + \beta\Sigma x''^2 + \gamma\Sigma x''x''' = 1,$$

that is,

$$\beta\frac{1}{\rho^2} - \gamma\frac{\rho'}{\rho^3} = 1;$$

and from the third equation,

$$\alpha\Sigma x'x''' + \beta\Sigma x''x''' + \gamma\Sigma x'''^2 = 0,$$

that is,

$$-\alpha\frac{1}{\rho^2} - \beta\frac{\rho'}{\rho^3} + \gamma\left(\frac{1}{\sigma^2\rho^2} + \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4}\right) = 0.$$

Hence

$$\alpha = \sigma^2\frac{\rho'}{\rho}, \quad \beta = \rho^2 + \sigma^2\rho'^2 = R^2, \quad \gamma = \sigma^2\rho\rho';$$

and therefore *the coordinates* ξ_2, η_2, ζ_2, v_2, *of* S, *the centre of second curvature*, are

$$\left.\begin{aligned} \xi_2 &= x + \sigma^2\frac{\rho'}{\rho}x' + (\rho^2+\sigma^2\rho'^2)x'' + \sigma^2\rho\rho'x''' \\ \eta_2 &= y + \sigma^2\frac{\rho'}{\rho}y' + (\rho^2+\sigma^2\rho'^2)y'' + \sigma^2\rho\rho'y''' \\ \zeta_2 &= z + \sigma^2\frac{\rho'}{\rho}z' + (\rho^2+\sigma^2\rho'^2)z'' + \sigma^2\rho\rho'z''' \\ v_2 &= v + \sigma^2\frac{\rho'}{\rho}v' + (\rho^2+\sigma^2\rho'^2)v'' + \sigma^2\rho \end{aligned}\right\}.$$

The equations of the line which, by its intersection with the osculating flat, determines the centre of second curvature, can now be expressed in a clearer form. They have been given as

$$\Sigma(\bar{x}-x)x'=0,\quad \Sigma(\bar{x}-x)x''=1,\quad \Sigma(\bar{x}-x)x'''=0.$$

The point S, being $\xi_2, \eta_2, \zeta_2, v_2$, lies on this line, so that

$$\Sigma(\xi_2-x)x'=0,\quad \Sigma(\xi_2-x)x''=1,\quad \Sigma(\xi_2-x)x'''=0,$$

and therefore the equations of the line can be taken

$$\Sigma(\bar{x}-\xi_2)x'=0,\quad \Sigma(\bar{x}-\xi_2)x''=0,\quad \Sigma(\bar{x}-\xi_2)x'''=0.$$

These are three equations, homogeneous and linear in the four quantities $\bar{x}-\xi_2$, $\bar{y}-\eta_2$, $\bar{z}-\zeta_2$, $\bar{v}-v_2$; when they are resolved, they give

$$\frac{\bar{x}-\xi_2}{J_x}=\frac{\bar{y}-\eta_2}{J_y}=\frac{\bar{z}-\zeta_2}{J_z}=\frac{\bar{v}-v_2}{J_v},$$

which accordingly can be taken as the *equations of the line which is the limit of the intersection of three consecutive normal flats.* The line is manifestly parallel to the trinormal at P (or what is the same thing) it is parallel to the normal at P to the osculating flat. It passes through the point S; it is the line SG, parallel to the line PF, in the figure on p. 211.

Spherical curvature: its centre and radius.

143. The coordinates of C, the centre of prime curvature, are

$$\xi_1=x+\rho^2x'',\quad \eta_1=y+\rho^2y'',\quad \zeta_1=z+\rho^2z'',\quad v_1=v+\rho^2v'';$$

consequently

$$\xi_2-\xi_1=\sigma^2\frac{\rho'}{\rho}(x'+\rho\rho'x''+\rho^2x'''),$$

$$\eta_2-\eta_1=\sigma^2\frac{\rho'}{\rho}(y'+\rho\rho'y''+\rho^2y'''),$$

$$\zeta_2-\zeta_1=\sigma^2\frac{\rho'}{\rho}(z'+\rho\rho'z''+\rho^2z'''),$$

$$v_2-v_1=\sigma^2\frac{\rho'}{\rho}(v'+\rho\rho'v''+\rho^2v''').$$

Hence the line SC, joining the centre of prime curvature to the centre of second curvature and measured from C towards S, is parallel to the binormal to the curve at P. Also we have seen (§139) that the four quantities of the type

$$\frac{\sigma}{\rho}(x'+\rho\rho'x''+\rho^2x''')$$

are actually direction-cosines of a line (of the binormal, and therefore of any parallel direction); hence the length of CS is $\sigma\rho'$. Now

$$PC=\rho,\quad CS=\sigma\rho',\quad PCS=\tfrac{1}{2}\pi,$$

for CS is parallel to the binormal; consequently,

$$PS^2 = PC^2 + CS^2 = \rho^2 + \sigma^2\rho'^2 = R^2.$$

Thus *the quantity denoted by R is the radius of second curvature*; and it is *given by the relation*

$$\frac{\rho^2\rho'^2}{R^2-\rho^2} = \Sigma\,(x' + \rho\rho' x'' + \rho^2 x''')^2,$$

where

$$\frac{1}{\rho^2} = \Sigma x''^2.$$

Also, it will be convenient to retain the relation

$$\frac{1}{\sigma^2} = \frac{1}{\rho^2}\Sigma\,(x' + \rho\rho' x'' + \rho^2 x''')^2;$$

and it is easy to verify, directly, that

$$(\xi_2 - x)^2 + (\eta_2 - y)^2 + (\zeta_2 - z)^2 + (v_2 - v)^2 = R^2.$$

The *direction-cosines of the radius of second curvature* are

$$\frac{\xi_2 - x}{R},\quad \frac{\eta_2 - y}{R},\quad \frac{\zeta_2 - z}{R},\quad \frac{v_2 - v}{R},$$

directed from the point P of the curve towards the centre S of second curvature; and these direction-cosines are proportional to

$$\left.\begin{array}{l}
\sigma^2\dfrac{\rho'}{\rho}x' + R^2x'' + \sigma^2\rho\rho'x''' \\
\sigma^2\dfrac{\rho'}{\rho}y' + R^2y'' + \sigma^2\rho\rho'y''' \\
\sigma^2\dfrac{\rho'}{\rho}z' + R^2z'' + \sigma^2\rho\rho'z''' \\
\sigma^2\dfrac{\rho'}{\rho}v' + R^2v'' + \sigma^2\rho\rho'v'''
\end{array}\right\}.$$

But the point S does not lie in the osculating plane at P; it is at a perpendicular distance $\sigma\rho'$ from that plane.

The significance of the point S can be otherwise established, as follows. The point C, the centre of plane curvature at P, is equidistant from three consecutive points P, P', P'', lying in the osculating plane at P; and therefore any point in any line through C, perpendicular to the osculating plane at P, is equidistant from P, P', P'': that is, any point, in any line through C

lying in the plane which, at C, is orthogonal to the osculating plane at P, is equidistant from P, P', P''. This plane is the intersection of the normal flat at P and the normal flat at P'.

Let C' be the centre of plane curvature at P', so that C' is the intersection of the osculating plane at P' with an orthogonal plane, itself the intersection of the normal flat at P' and the normal flat at P''. (It may be noted, in passing, that CC' and PP' are therefore perpendicular to one another.) Thus C' is equidistant from three consecutive points P', P'', P''', lying in the osculating plane at P', and therefore any point in any line through C', perpendicular to the osculating plane at P', is equidistant from P', P'', P''': that is, any point, in any line through C' lying in the plane which, at C', is orthogonal to the osculating plane at P', is equidistant from P', P'', P'''. This plane is the intersection of the normal flat at P' with the normal flat at P''.

Now there is a line common to the three normal flats at P, at P', at P'', respectively, it is the line SG. As the line CS lies in the plane which is the intersection of the normal flat at P and the normal flat at P', the line CS is perpendicular to every line in the osculating plane at P, and the point S is equidistant from the three points P, P', P'', so that $SP = SP' = SP''$. As the line $C'S$ lies in the plane which is the intersection of the normal flat at P' and the normal flat at P'', the line $C'S$ is perpendicular to every line in the osculating plane at P', and the point S is equidistant from the three points P', P'', P''', so that $SP' = SP'' = SP'''$.

Thus $SP = SP' = SP'' = SP'''$; and all the points P, P', P'', P''', S, lie in the osculating flat at P. All the four points P, P', P'', P''', cannot lie in one plane. We therefore can draw, lying in the osculating flat at P, a three-dimensional sphere, with centre S, passing through the four consecutive points P, P', P'', P''', lying on the curve. Also, four is the greatest number of points through which a three-dimensional spherical surface can be drawn. The foregoing sphere is therefore the sphere which has the closest contact, possible for a sphere, with the curve at the point P. We therefore call the sphere, thus constructed in the homaloidal triple space constituted by the osculating flat, the *sphere of curvature* at the point P: the point S, the centre of second curvature, is also called *the centre of spherical curvature*; and the magnitude R, the radius of second curvature, is also called *the radius of spherical curvature*.

The equations of the sphere of curvature at P, passing (that is) through four consecutive points of the curve at P, are

$$\begin{vmatrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ x', & y', & z', & v' \\ x'', & y'', & z'', & v'' \\ x''', & y''', & z''', & v''' \end{vmatrix} = 0,$$

being the osculating flat in which the sphere lies, and

$$\Sigma (\bar{x} - \xi_2)^2 = R^2,$$

or, what is its equivalent,

$$\Sigma (\bar{x} - x)^2 = 2\Sigma \{(\xi_2 - x)(\bar{x} - x)\},$$

representing a globular region of which the section by the osculating flat is the surface of the sphere of curvature of the curve.

The line SG, which is the intersection of the plane, orthogonal at C to the osculating plane at P, and the plane orthogonal at C' to the osculating plane at P', is normal to the osculating flat at P. Now this normal to the flat at S is perpendicular to every direction in the flat, and therefore is perpendicular to SP, to SP', to SP'', to SP'''. It follows that, because

$$SP = SP' = SP'' = SP''',$$

every point on the line SG is equidistant from the four consecutive points P, P', P'', P'''.

Equation of the sphere of curvature in the osculating flat.

144. The equation of the sphere of curvature, referred to suitable axes in the osculating flat, can be obtained simply from the foregoing form, which represents the sphere by the two equations respectively representing a flat and a globe.

Any point in the osculating flat is given by coordinates

$$\bar{x} - x = \alpha x' + \beta x'' + \gamma x''',$$

$$\bar{y} - y = \alpha y' + \beta y'' + \gamma y''',$$

$$\bar{z} - z = \alpha z' + \beta z'' + \gamma z''',$$

$$\bar{v} - v = \alpha v' + \beta v'' + \gamma v''',$$

where α, β, γ, are parameters. If this point lies on the sphere of curvature, its coordinates must satisfy the equation

$$\Sigma (\bar{x} - x)^2 = 2\Sigma \{(\bar{x} - x)(\xi_2 - x)\}.$$

Now

$$\begin{aligned}\Sigma (\bar{x} - x)^2 &= \Sigma (\alpha x' + \beta x'' + \gamma x''')^2 \\ &= \alpha^2 - 2\alpha\gamma \frac{1}{\rho^2} + \beta^2 \frac{1}{\rho^2} - 2\beta\gamma \frac{\rho'}{\rho^3} + \gamma^2 \left(\frac{1}{\sigma^2\rho^2} + \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4}\right) \\ &= \left(\alpha - \gamma \frac{1}{\rho^2}\right)^2 + \frac{1}{\rho^2}\left(\beta - \gamma \frac{\rho'}{\rho}\right)^2 + \gamma^2 \frac{1}{\sigma^2\rho^2};\end{aligned}$$

and

$$\begin{aligned}\Sigma \{(\bar{x} - x)(\xi_2 - x)\} &= \Sigma \left[(\alpha x' + \beta x'' + \gamma x''') \left\{\sigma^2 \frac{\rho'}{\rho} x' + (\rho^2 + \sigma^2\rho'^2) x'' + \sigma^2 \rho\rho' x'''\right\}\right] \\ &= \beta,\end{aligned}$$

on reduction. Hence α, β, γ, must satisfy the relation

$$\left(\alpha-\gamma\frac{1}{\rho^2}\right)^2+\frac{1}{\rho^2}\left(\beta-\gamma\frac{\rho'}{\rho}\right)^2+\gamma^2\frac{1}{\sigma^2\rho^2}=2\beta.$$

In order to clarify the significance of this relation, let the variable point be obtained by taking

(i) a distance t along the tangent, with direction-cosines x', y', z', v';

(ii) a distance n along the principal normal, with direction-cosines $\rho x''$, $\rho y''$, $\rho z''$, $\rho v''$, and

(iii) a distance b along the binormal, with direction-cosines

$$\frac{\sigma}{\rho}(x'+\rho\rho'x''+\rho^2x'''),\quad \frac{\sigma}{\rho}(y'+\rho\rho'y''+\rho^2y'''),$$

$$\frac{\sigma}{\rho}(z'+\rho\rho'z''+\rho^2z'''),\quad \frac{\sigma}{\rho}(v'+\rho\rho'v''+\rho^2v''').$$

When the sphere is referred to these three lines, as perpendicular axes in the (three-dimensional) osculating flat, so that t, n, b, are the projections of PQ upon the axes, Q being the current point on the spherical surface, the coordinates of Q are

$$\bar{x}-x=x't+\rho x''n+\frac{\sigma}{\rho}(x'+\rho\rho'x''+\rho^2x''')\,b,$$

$$\bar{y}-y=y't+\rho y''n+\frac{\sigma}{\rho}(y'+\rho\rho'y''+\rho^2y''')\,b,$$

$$\bar{z}-z=z't+\rho z''n+\frac{\sigma}{\rho}(z'+\rho\rho'z''+\rho^2z''')\,b,$$

$$\bar{v}-v=v't+\rho v''n+\frac{\sigma}{\rho}(v'+\rho\rho'v''+\rho^2v''')\,b.$$

Comparing the two sets of expressions for the coordinates of Q, we find

$$\alpha=t+\frac{\sigma}{\rho}b,$$

$$\beta=\rho n+\sigma\rho'b,$$

$$\gamma=\sigma\rho b.$$

Thus the relation connecting α, β, γ, becomes

$$t^2+n^2+b^2=2(\rho n+\sigma\rho'b),$$

that is,

$$t^2+(n-\rho)^2+(b-\sigma\rho')^2=\rho^2+\sigma^2\rho'^2=R^2.$$

In this form, the relation is simply the equation of the sphere, referred to the selected axes in the osculating flat, all in three-dimensional (homaloidal) space: t, n, b, are the coordinates of a current point on the sphere; 0, ρ, $\sigma\rho'$, are the coordinates of its centre S, because S lies in the plane $t=0$ in that space, while $PC=\rho$, $CS=\sigma\rho'$; and R is the radius of the sphere.

Angle of tilt: curvature of tilt.

145. Expressions have been obtained for the angles between some consecutive configurations: in § 131, for the angle $d\epsilon$ between two consecutive tangents, and for the angle $d\iota$ between two consecutive principal normals: in § 133, for the angle $d\eta$ between two consecutive osculating planes. We now proceed to obtain an expression for *the angle $d\omega$ between two consecutive trinormals*: as the trinormal is the direction uniquely normal to the osculating flat, $d\omega$ also denotes the angle between two consecutive osculating flats, and it will be called *the angle of tilt* of the osculating flat, or, simply, the angle of tilt.

The direction-cosines of the normal to the osculating flat at P are l, m, n, k, and those of the normal to the osculating flat at the consecutive point P' are l', m', n', k', where

$$\begin{aligned}
l &= \sigma\rho^2 J_x, & l' &= \sigma\rho^2 J_x + \frac{d}{ds}(\sigma\rho^2 J_x)\,ds,\\
m &= \sigma\rho^2 J_y, & m' &= \sigma\rho^2 J_y + \frac{d}{ds}(\sigma\rho^2 J_y)\,ds,\\
n &= \sigma\rho^2 J_z, & n' &= \sigma\rho^2 J_z + \frac{d}{ds}(\sigma\rho^2 J_z)\,ds,\\
k &= \sigma\rho^2 J_v, & k' &= \sigma\rho^2 J_v + \frac{d}{ds}(\sigma\rho^2 J_v)\,ds.
\end{aligned}$$

The angle of tilt $d\omega$ is given by

$$\sin^2 d\omega = \Sigma\,(lm' - l'm)^2,$$

where the summation is taken for the six combinations of the direction-cosines in pairs. Now

$$lm' - l'm = \sigma^2\rho^4\left(J_x\frac{dJ_y}{ds} - J_y\frac{dJ_x}{ds}\right)ds,$$

and so for the others: hence

$$\left(\frac{1}{\sigma^2\rho^4}\frac{d\omega}{ds}\right)^2 = \Sigma\left(J_x\frac{dJ_y}{ds} - J_y\frac{dJ_x}{ds}\right)^2.$$

Now

$$J_x = \begin{vmatrix} y', & z', & v' \\ y'', & z'', & v'' \\ y''', & z''', & v''' \end{vmatrix}, \qquad \frac{dJ_x}{ds} = \begin{vmatrix} y', & z', & v' \\ y'', & z'', & v'' \\ y^{\text{iv}}, & z^{\text{iv}}, & v^{\text{iv}} \end{vmatrix};$$

$$-J_y = \begin{vmatrix} z', & v', & x' \\ z'', & v'', & x'' \\ z''', & v''', & x''' \end{vmatrix}, \qquad -\frac{dJ_y}{ds} = \begin{vmatrix} z', & v', & x' \\ z'', & v'', & x'' \\ z^{\text{iv}}, & v^{\text{iv}}, & x^{\text{iv}} \end{vmatrix};$$

and therefore

$$J_x \frac{dJ_y}{ds} - J_y \frac{dJ_x}{ds}$$

$$= - \begin{vmatrix} y', & z', & v' \\ y'', & z'', & v'' \\ y''', & z''', & v''' \end{vmatrix} \begin{vmatrix} z', & v', & x' \\ z'', & v'', & x'' \\ z^{\text{iv}}, & v^{\text{iv}}, & x^{\text{iv}} \end{vmatrix} + \begin{vmatrix} z', & v', & x' \\ z'', & v'', & x'' \\ z''', & v''', & x''' \end{vmatrix} \begin{vmatrix} y', & z', & v' \\ y'', & z'', & v'' \\ y^{\text{iv}}, & z^{\text{iv}}, & v^{\text{iv}} \end{vmatrix}.$$

The whole coefficient of x^{iv} on the right-hand side is

$$-(z'v'' - v'z'') \begin{vmatrix} y', & z', & v' \\ y'', & z'', & v'' \\ y''', & z''', & v''' \end{vmatrix};$$

the whole coefficient of y^{iv} is

$$(z'v'' - v'z'') \begin{vmatrix} z', & v', & x' \\ z'', & v'', & x'' \\ z''', & v''', & x''' \end{vmatrix};$$

the whole coefficient of z^{iv} is

$$-(v'x'' - x'v'') \begin{vmatrix} y', & z', & v' \\ y'', & z'', & v'' \\ y''', & z''', & v''' \end{vmatrix} - (y'v'' - v'y'') \begin{vmatrix} z', & v', & x' \\ z'', & v'', & x'' \\ z''', & v''', & x''' \end{vmatrix},$$

which is equal to

$$-(z'v'' - v'z'') \begin{vmatrix} v', & x', & y' \\ v'', & x'', & y'' \\ v''', & x''', & y''' \end{vmatrix};$$

and the whole coefficient of v^{iv} is

$$(z'x'' - x'z'') \begin{vmatrix} y', & z', & v' \\ y'', & z'', & v'' \\ y''', & z''', & v''' \end{vmatrix} + (y'z'' - z'y'') \begin{vmatrix} z', & v', & x' \\ z'', & v', & x'' \\ z''', & v''', & x''' \end{vmatrix},$$

which is equal to

$$(z'v'' - v'z'') \begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix}.$$

Hence*

$$J_x \frac{dJ_y}{ds} - J_y \frac{dJ_x}{ds} = (z'v'' - v'z'')\, \Omega,$$

where

$$\Omega = \begin{vmatrix} x', & y', & z', & v' \\ x'', & y'', & z'', & v'' \\ x''', & y''', & z''', & v''' \\ x^{\text{iv}}, & y^{\text{iv}}, & z^{\text{iv}}, & v^{\text{iv}} \end{vmatrix}.$$

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* The result can be derived at once from the property that $-J_x$, $-J_y$, $\frac{dJ_x}{ds}$, $\frac{dJ_y}{ds}$, are the minors of x^{iv}, y^{iv}, x''', y''', in Ω.

Similarly for the other terms in the summation $\Sigma(lm' - l'm)^2$.

We define the *curvature of tilt* to be $\frac{1}{\tau}$, according to the relation

$$\frac{d\omega}{ds} = \frac{1}{\tau},$$

so that τ is a linear magnitude. There is no centre of tilt; and there is no line whose direction can be regarded as a radius of tilt, though the expression *radius of tilt* is used to describe the magnitude τ. With this definition, we have

$$\begin{aligned}\left(\frac{1}{\sigma^2\rho^4}\frac{d\omega}{ds}\right)^2 &= \Omega^2\Sigma(z'v'' - v'z'')^2 \\ &= \Omega^2\left\{\Sigma x'^2\Sigma x''^2 - (\Sigma x'x'')^2\right\} \\ &= \Omega^2\frac{1}{\rho^2};\end{aligned}$$

and therefore, with a choice of the angle $d\omega$ as positive according to the convention of § 112, and an assumption of a sign for τ, we have

$$\rho^3\sigma^2\tau\Omega = -1.$$

Significance of zero curvature, or zero torsion, or zero tilt.

146. As a skew curve in four-dimensional space usually has plane curvature, usually has spherical curvature distinct from plane curvature so that there is torsion, and usually has curvature of tilt, the quantities ρ, σ, τ, usually are not infinite. The determinant Ω, accordingly, is usually different from zero.

For such curves as allow this determinant Ω to be zero everywhere, we can have ρ, or σ, or τ, or any two of them, or all three of them, infinite.

When ρ is infinite everywhere, so that there is no plane curvature along the curve, the curve is a straight line.

When σ is infinite everywhere, so that there is no torsion, the osculating plane of curvature is one and the same all along the curve, the curve is a plane curve.

When τ is infinite everywhere, so that there is no tilt, the curve is a three-dimensional curve; for then the osculating flat is one and the same all along the curve.

An analytical proof of these statements is simple

I. When the plane curvature is zero, we have

$$\frac{1}{\rho^2} = x''^2 + y''^2 + z''^2 + v''^2 = 0,$$

so that $x''=0$, $y''=0$, $z''=0$, $v''=0$, the curve being real. Then

$$x=a_1s+b_1,\quad y=a_2s+b_2,\quad z=a_3s+b_3,\quad v=a_4s+b_4;$$

and the curve is

$$\frac{x-b_1}{a_1}=\frac{y-b_2}{a_2}=\frac{z-b_3}{a_3}=\frac{v-b_4}{a_4};$$

manifestly a straight line.

II. When the plane curvature is not zero, the curve is not a straight line; and then, as

$$\frac{1}{\rho^2}=\Sigma(x'y''-y'x'')^2,$$

not all the six quantities of the type $x'y''-y'x''$ are zero. We assume that $x'y''-y'x''$ does not vanish. Now

$$\Sigma\begin{vmatrix} y', & z', & v' \\ y'', & z'', & v'' \\ y''', & z''', & v''' \end{vmatrix}^2=\begin{vmatrix} \Sigma x'^2, & \Sigma x'x'', & \Sigma x'x''' \\ \Sigma x'x'', & \Sigma x''^2, & \Sigma x''x''' \\ \Sigma x'x''', & \Sigma x''x''', & \Sigma x'''^2 \end{vmatrix}=\frac{1}{\rho^4\sigma^2}$$

When there is no torsion, $\frac{1}{\sigma}$ is zero; and then

$$\begin{vmatrix} y', & z', & v' \\ y'', & z'', & v'' \\ y''', & z''', & v''' \end{vmatrix}=0,\quad \begin{vmatrix} z', & v', & x' \\ z'', & v'', & x'' \\ z''', & v''', & x''' \end{vmatrix}=0,$$

$$\begin{vmatrix} v', & x', & y' \\ v'', & x'', & y'' \\ v''', & x''', & y''' \end{vmatrix}=0,\quad \begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix}=0,$$

really equivalent to two relations. Taking the third and the fourth as the two independent relations, we have quantities α and β, γ and δ, such that

$$\left.\begin{aligned} z' &=\alpha x'+\beta y' \\ z'' &=\alpha x''+\beta y'' \\ z''' &=\alpha x'''+\beta y''' \end{aligned}\right\},\quad \left.\begin{aligned} v' &=\gamma x'+\delta y' \\ v'' &=\gamma x''+\delta y'' \\ v''' &=\gamma x'''+\delta y''' \end{aligned}\right\}.$$

Now, from $z'=\alpha x'+\beta y'$, we have

$$z''=\alpha x''+\beta y''+\alpha'x'+\beta'y',$$

that is,

$$\alpha'x'+\beta'y'=0;$$

and from $z''=\alpha x''+\beta y''$, we have

$$z'''=\alpha x'''+\beta y'''+\alpha'x''+\beta'y'',$$

that is,

$$\alpha'x''+\beta'y''=0.$$

Hence as $x'y''-y'x''$ is not zero, we have

$$\alpha'=0,\quad \beta'=0,$$

that is, α and β are constants; and now $z' = \alpha x' + \beta y'$ leads to

$$z = \alpha x + \beta y + f,$$

where f is a constant.

Similarly, we prove γ and δ to be constants, and derive the equation

$$v = \gamma x + \delta y + g,$$

where g is a constant.

Thus the coordinates of every point on the curve satisfy the two equations

$$\left.\begin{aligned} z &= \alpha x + \beta y + f \\ v &= \gamma x + \delta y + g \end{aligned}\right\},$$

where α, β, γ, δ, f, g, are constants. The curve is a plane curve.

III. When the plane curvature is not zero, and when there is torsion, so that the curve is not a plane curve, we have

$$-\frac{1}{\rho^3\sigma^2\tau} = \begin{vmatrix} x', & y', & z', & v' \\ x'', & y'', & z'', & v'' \\ x''', & y''', & z''', & v''' \\ x^{\text{iv}}, & y^{\text{iv}}, & z^{\text{iv}}, & v^{\text{iv}} \end{vmatrix}.$$

As the torsion is not zero, not all the quantities J_x, J_y, J_z, J_v, vanish: we can suppose that J_v does not vanish, that is, the determinant

$$\begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix}$$

does not vanish.

If there is no tilt, the four-rowed determinant is zero; and thus there are quantities α, β, γ, such that

$$\begin{aligned} v' &= \alpha x' + \beta y' + \gamma z', \\ v'' &= \alpha x'' + \beta y'' + \gamma z'', \\ v''' &= \alpha x''' + \beta y''' + \gamma z''', \\ v^{\text{iv}} &= \alpha x^{\text{iv}} + \beta y^{\text{iv}} + \gamma z^{\text{iv}}. \end{aligned}$$

Differentiating the first of these, and using the second, we have

$$0 = \alpha' x' + \beta' y' + \gamma' z';$$

differentiating the second and using the third, we have

$$0 = \alpha' x'' + \beta' y'' + \gamma' z'';$$

and differentiating the third and using the fourth, we have

$$0 = \alpha' x''' + \beta' y''' + \gamma' z'''.$$

Consequently

$$\alpha' = 0, \quad \beta' = 0, \quad \gamma' = 0,$$

that is, α, β, γ, are constants: and then the equation $v' = \alpha x' + \beta y' + \gamma z'$ leads to a relation

$$v = \alpha x + \beta y + \gamma z + h,$$

(where h is a constant), which is satisfied by the coordinates of every point on the curve. The curve therefore lies in a flat: it is a three-dimensional curve.

Globular (third) curvature: its centre and radius.

147. The tangent, the osculating plane, and the osculating flat, of one dimension, two dimensions, and three dimensions, respectively, constitute the aggregate of homaloidal amplitudes which, each in its own kind, have the closest relations of contact with the curve; they pass through two consecutive points, three consecutive points, and four consecutive points, respectively. The only remaining type of homaloidal amplitude is the full space of four dimensions, in which of course the curve exists.

Now, in a space of four dimensions, a globe (defined as a region each point of which is at the same distance from a centre as every other point) can be made to pass through five points which do not, all of them, lie in one flat A globe is thus made determinate. for there are five disposable constants, the radius of the globe and the four coordinates of the centre, and these can be determined from the five conditions that the globe passes through the five points But as there are only five disposable constants, a globe cannot be made to pass through six arbitrarily assigned points. We therefore take five consecutive points on the curve $P, P', P'', P''', P^{\text{iv}}$: they do not all lie in one flat, for a flat can contain only four arbitrarily assigned points, and we have seen that the flat which contains the tangents PP', $P'P''$, $P''P'''$, that is, contains the four consecutive points P, P', P'', P''', does not contain a fifth consecutive point P^{iv}. Accordingly, we proceed to determine the globe which passes through these five consecutive points: as five is the greatest number of arbitrarily assigned points through which a globe can be made to pass, this determinate globe will have closer contact with the curve at P than any other globe passing through P. The globe is called the *globe of curvature*, its centre is called the *centre of* (*third*, or) *globular curvature*, and its radius is called the *radius of* (*third*, or) *globular curvature*.

Let ξ_3, η_3, ζ_3, v_3, be the coordinates of the centre of globular curvature G, and let Γ be the radius of globular curvature, at the point P of the curve; then the equation of the globe is

$$(\bar{x} - \xi_3)^2 + (\bar{y} - \eta_3)^2 + (\bar{z} - \zeta_3)^2 + (\bar{v} - v_3)^2 = \Gamma^2,$$

or, with the usual notation,

$$\Sigma (\bar{x} - \xi_3)^2 = \Gamma^2.$$

The quantities ξ_3, η_3, ζ_3, v_3, Γ, are to be determined from the conditions that the globe shall pass through the five consecutive points P, P', P'', P''', P^{iv}, on the curve.

In order that the globe may pass through P, the point x, y, z, v, the relation

$$\Sigma (x - \xi_3)^2 = \Gamma^2$$

must be satisfied. In order that it may also pass through P', the additional relation

$$\Sigma (x - \xi_3) x' = 0$$

must be satisfied. In order that it may further pass through P'', an additional relation

$$\Sigma \{(x - \xi_3) x'' + x'^2\} = 0,$$

that is, the additional relation

$$\Sigma (x - \xi_3) x'' = -1$$

must be satisfied. In order that it may now pass through P''', an additional relation

$$\Sigma \{(x - \xi_3) x''' + x' x''\} = 0,$$

that is, the additional relation

$$\Sigma (x - \xi_3) x''' = 0$$

must be satisfied. Finally, in order that it may pass through P^{iv}, the last of the five consecutive points, one additional relation

$$\Sigma \{(x - \xi_3) x^{\text{iv}} + x' x'''\} = 0,$$

that is, the additional relation

$$\Sigma (x - \xi_3) x^{\text{iv}} = \frac{1}{\rho^2}$$

must be satisfied.

Thus there are five relations for the determination of the five quantities $\xi_3, \eta_3, \zeta_3, \upsilon_3, \Gamma$. Of these, Γ occurs in the first alone and will be determinable when the other four are known. These four quantities, the coordinates of the centre G of globular curvature, satisfy the four linear equations

$$\left.\begin{aligned}
(x - \xi_3) x' + (y - \eta_3) y' + (z - \zeta_3) z' + (v - \upsilon_3) v' &= 0 \\
(x - \xi_3) x'' + (y - \eta_3) y'' + (z - \zeta_3) z'' + (v - \upsilon_3) v'' &= -1 \\
(x - \xi_3) x''' + (y - \eta_3) y''' + (z - \zeta_3) z''' + (v - \upsilon_3) v''' &= 0 \\
(x - \xi_3) x^{\text{iv}} + (y - \eta_3) y^{\text{iv}} + (z - \zeta_3) z^{\text{iv}} + (v - \upsilon_3) v^{\text{iv}} &= \frac{1}{\rho^2}
\end{aligned}\right\}.$$

The determinant of the coefficients of $x - \xi_3$, $y - \eta_3$, $z - \zeta_3$, $v - \upsilon_3$, on the left-hand sides is the determinant Ω of § 127, which helps to measure the tilt of the curve; and this determinant Ω does not vanish (§ 146) when the curve is not pent within a homaloidal space of fewer than four dimensions.

Centre of globular (third) curvature as the intersection of four normal flats.

148. One important property arises at once from these equations. The coordinates $\xi_2, \eta_2, \zeta_2, \upsilon_2$, of the centre S of spherical curvature satisfy the equations (§ 142)

$$\Sigma(\xi_2 - x)x' = 0, \quad \Sigma(\xi_2 - x)x'' = 1, \quad \Sigma(\xi_2 - x)x''' = 0.$$

When these are combined, by respective addition, with the first three of the set of four equations determining $\xi_3, \eta_3, \zeta_3, \upsilon_3$, we find

$$\begin{aligned}(\xi_3 - \xi_2)x' + (\eta_3 - \eta_2)y' + (\zeta_3 - \zeta_2)z' + (\upsilon_3 - \upsilon_2)v' &= 0,\\ (\xi_3 - \xi_2)x'' + (\eta_3 - \eta_2)y'' + (\zeta_3 - \zeta_2)z'' + (\upsilon_3 - \upsilon_2)v'' &= 0,\\ (\xi_3 - \xi_2)x''' + (\eta_3 - \eta_2)y''' + (\zeta_3 - \zeta_2)z''' + (\upsilon_3 - \upsilon_2)v''' &= 0;\end{aligned}$$

and therefore

$$\frac{\xi_3 - \xi_2}{J_x} = \frac{\eta_3 - \eta_2}{J_y} = \frac{\zeta_3 - \zeta_2}{J_z} = \frac{\upsilon_3 - \upsilon_2}{J_v}.$$

The equations of the line, through the centre S of spherical curvature normal to the osculating flat, are

$$\frac{\bar{x} - \xi_2}{J_x} = \frac{\bar{y} - \eta_2}{J_y} = \frac{\bar{z} - \zeta_2}{J_z} = \frac{\bar{v} - \upsilon_2}{J_v};$$

and therefore the centre G of globular curvature lies on the normal to the osculating flat through the centre S of spherical curvature.

But we have seen that this axial line is the intersection of the three normal flats at the consecutive points P, P', P'', and that any point on it is equidistant from the four points P, P', P'', P'''. Now take the osculating flat at P', the centre S' of spherical curvature at P', and the axial line through S' which is the intersection of the three normal flats at the consecutive points P', P'', P'''; any point on this line through S' is equidistant from P', P'', P''', P^{iv}. The intersection of the axial line through S with the normal flat through P''' is the one point common to the four normal flats through P, P', P'', P''', and the intersection of the axial line through S' with the normal flat through P is that same single point common to those four normal flats. Thus the axial line through S, the centre of spherical curvature at P, meets the axial line through S', the centre of spherical curvature at the consecutive point P'. The points P, P', P'', P''', are equidistant from this point, because it lies on the axial line through S; the points P', P'', P''', P^{iv}, are equidistant from that same point, because it lies on the axial line through S'. Thus the particular point, the intersection of the axial lines through S and S', is such that its distances from the five consecutive points P, P', P'', P''', P^{iv}, are equal to one another. The point therefore is G, the centre of globular curvature; and therefore *the centre of globular curvature is the intersection of the normal flats at four consecutive points of the curve.*

This result can be established by an apparently more analytical process to which, however, the foregoing argument is merely equivalent. The normal flat at P is

$$\Sigma (\bar{x} - x) x' = 0.$$

The plane, which is the intersection of this normal flat by the normal flat at the consecutive point P', is given by the association, with this equation, of the additional equation

$$\Sigma \{(\bar{x} - x) x'' - x'^2\} = 0,$$

that is, of the additional equation

$$\Sigma (\bar{x} - x) x'' = 1.$$

The line, which is the intersection of this plane by the normal flat at the next consecutive point P'', that is, which is the line of intersection of the normal flats at P, P', P'', is given by the association, with these two equations, of the additional equation

$$\Sigma \{(\bar{x} - x) x''' - x'x''\} = 0,$$

that is, of the additional equation

$$\Sigma (\bar{x} - x) x''' = 0.$$

The point, which is the intersection of this line with the normal flat at the fourth consecutive point P''', that is, which is the point of intersection of the normal flats at P, P', P'', P''', is given by the association, with the preceding three equations, of the additional equation

$$\Sigma \{(\bar{x} - x) x^{\text{iv}} - x'x'''\} = 0,$$

that is, of the additional equation

$$\Sigma (\bar{x} - x) x^{\text{iv}} = -\frac{1}{\rho^2}.$$

But these four equations, linear in $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{v}$, are precisely the four linear equations determining ξ_3, η_3, ζ_3, v_3; and therefore we infer, as before, that the centre G of globular curvature is the point of intersection of the normal flats at the four consecutive points P, P', P'', P''', of the curve.

149. In order to resolve the equations in § 147, which give the coordinates of the centre of globular curvature, we introduce four quantities α, β, γ, g, defined by the relations

$$\xi_3 - x = x'\alpha + \rho x''\beta + \frac{\sigma}{\rho}(x' + \rho\rho' x'' + \rho^2 x''')\gamma + \sigma\rho^2 J_x g,$$

$$\eta_3 - y = y'\alpha + \rho y''\beta + \frac{\sigma}{\rho}(y' + \rho\rho' y'' + \rho^2 y''')\gamma + \sigma\rho^2 J_y g,$$

$$\zeta_3 - z = z'\alpha + \rho z''\beta + \frac{\sigma}{\rho}(z' + \rho\rho' z'' + \rho^2 z''')\gamma + \sigma\rho^2 J_z g,$$

$$v_3 - v = v'\alpha + \rho v''\beta + \frac{\sigma}{\rho}(v' + \rho\rho' v'' + \rho^2 v''')\gamma + \sigma\rho^2 J_v g.$$

The form of these relations is dictated by the consideration of passing from P to G along a course, made up of a length α along PT (so that α may be expected to be zero), of a length β along PC to C (so that β may be expected to be equal to ρ), of a length γ along CS to S (so that γ may be expected to be equal to $\sigma\rho'$), and of a length g along SG to G (so that the value of g will give the length SG).

The determination of the four quantities α, β, γ, g, is effected by actual substitution in the equations to be resolved. Substituting in the equation

$$\Sigma(\xi_3 - x)x' = 0,$$

we find

$$\alpha = 0,$$

as expected. Substituting in the equation

$$\Sigma(\xi_3 - x)x'' = 1,$$

we find

$$\beta = \rho,$$

as expected. Substituting in the equation

$$\Sigma(\xi_3 - x)x''' = 0,$$

we have

$$0 = -\alpha\frac{1}{\rho^2} + \beta\rho\left(-\frac{\rho'}{\rho^3}\right) + \gamma\frac{\sigma}{\rho}(\Sigma x'x''' + \rho\rho'\Sigma x''x''' + \rho^2\Sigma x'''^2),$$

which, on substitution for α and β, and after reduction, gives

$$\gamma = \sigma\rho',$$

as expected. Substituting in the remaining equation

$$\Sigma(\xi_3 - x)x^{\text{iv}} = -\frac{1}{\rho^2},$$

we have

$$\alpha\Sigma x'x^{\text{iv}} + \beta\rho\Sigma x''x^{\text{iv}} + \gamma\frac{\sigma}{\rho}(\Sigma x'x^{\text{iv}} + \rho\rho'\Sigma x''x^{\text{iv}} + \rho^2\Sigma x'''x^{\text{iv}}) + g\sigma\rho^2\Sigma x^{\text{iv}}J_x = -\frac{1}{\rho^2}.$$

We insert the values of the various sums $\Sigma x'x^{\text{iv}}$, $\Sigma x''x^{\text{iv}}$, $\Sigma x'''x^{\text{iv}}$, from the results in § 127 , also, we have

$$\Sigma x^{\text{iv}}J_x = -\Omega,$$

where Ω is the determinant of § 127 ; and we find

$$g\sigma\rho^2\Omega = \frac{1}{\rho^2} + \rho^2 M + \frac{\sigma^2\rho'}{\rho}(L + \rho\rho' M + \rho^2 N).$$

Now

$$L + \rho\rho' M + \rho^2 N = -\frac{1}{\sigma^2}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right),$$

$$M = -\frac{\rho''}{\rho^3} + 2\frac{\rho'^2}{\rho^4} - \frac{1}{\rho^4} - \frac{1}{\sigma^2\rho^2};$$

and therefore

$$g\sigma\rho^2\Omega = -\frac{1}{\rho\sigma^2}(\rho + \sigma^2\rho'' + \sigma\rho'\sigma').$$

But

$$R^2 = \rho^2 + \sigma^2\rho'^2,$$

and therefore

$$RR' = \rho\rho' + \rho'(\sigma^2\rho'' + \sigma\rho'\sigma');$$

hence

$$g\sigma\rho^2\Omega = -\frac{1}{\rho\sigma^2}R\frac{dR}{d\rho}.$$

The determinant Ω is connected with the curvature of tilt by the relation (§ 145)

$$\rho^3\sigma^2\tau\Omega = -1;$$

and therefore we have

$$g = \frac{\tau}{\sigma}R\frac{dR}{d\rho} = R\frac{\tau R'}{\sigma\rho'}.$$

Consequently, the values of the quantities α, β, γ, g, in the expressions for ξ_3, η_3, ζ_3, v_3, are

$$\alpha = 0, \quad \beta = \rho, \quad \gamma = \sigma\rho', \quad g = \frac{\tau}{\sigma}R\frac{dR}{d\rho};$$

and thus *the coordinates of the centre of globular curvature* are given by the equations

$$\left.\begin{aligned}
\xi_3 - x &= \rho^2x'' + \sigma^2\frac{\rho'}{\rho}(x' + \rho\rho'x'' + \rho^2x''') + \rho^2\tau R\frac{dR}{d\rho}J_x \\
\eta_3 - y &= \rho^2y'' + \sigma^2\frac{\rho'}{\rho}(y' + \rho\rho'y'' + \rho^2y''') + \rho^2\tau R\frac{dR}{d\rho}J_y \\
\zeta_3 - z &= \rho^2z'' + \sigma^2\frac{\rho'}{\rho}(z' + \rho\rho'z'' + \rho^2z''') + \rho^2\tau R\frac{dR}{d\rho}J_z \\
v_3 - v &= \rho^2v'' + \sigma^2\frac{\rho'}{\rho}(v' + \rho\rho'v'' + \rho^2v''') + \rho^2\tau R\frac{dR}{d\rho}J_v
\end{aligned}\right\}.$$

The *radius of globular curvature* Γ is given by

$$\Sigma(\xi_3 - x)^2 = \Gamma^2.$$

Now

$$\begin{aligned}
\Sigma x''(x' + \rho\rho'x'' + \rho^2x''') &= 0, \\
\Sigma x''J_x &= 0, \\
\Sigma J_x(x' + \rho\rho'x'' + \rho^2x''') &= 0,
\end{aligned}$$

because the three directions PC, CS, SG, are perpendicular to one another in pairs; hence

$$\begin{aligned}
\Gamma^2 &= \rho^4\Sigma x''^2 + \frac{\sigma^4\rho'^2}{\rho^2}\Sigma(x' + \rho\rho'x'' + \rho^2x''')^2 + \left(\rho^2\tau R\frac{dR}{d\rho}\right)^2\Sigma J_x^2 \\
&= \rho^2 + \sigma^2\rho'^2 + \left(\frac{\tau}{\sigma}R\frac{dR}{d\rho}\right)^2 \\
&= R^2 + \left(\frac{\tau}{\sigma}R\frac{dR}{d\rho}\right)^2,
\end{aligned}$$

thus determining a relation between the globular curvature and the curvature of tilt, similar to the relation between the spherical curvature and the curvature of torsion.

Alternative expressions for the coordinates of the centre of globular curvature.

150. The resolution of the four equations for ξ_3, η_3, ζ_3, v_3, may be attained in another (of course, an equivalent) form.

Let four new variables θ, ϕ, ψ, χ, be introduced with the definitions

$$\begin{aligned}
\xi_3 - x &= \theta x' + \phi x'' + \psi (x' + \rho\rho' x'' + \rho^2 x''') + \chi x^{\text{iv}},\\
\eta_3 - y &= \theta y' + \phi y'' + \psi (y' + \rho\rho' y'' + \rho^2 y''') + \chi y^{\text{iv}},\\
\zeta_3 - z &= \theta z' + \phi z'' + \psi (z' + \rho\rho' z'' + \rho^2 z''') + \chi z^{\text{iv}},\\
v_3 - v &= \theta v' + \phi v'' + \psi (v' + \rho\rho' v'' + \rho^2 v''') + \chi v^{\text{iv}},
\end{aligned}$$

and let these values be substituted in the equations.

From the equation $\Sigma (\xi_3 - x) x' = 0$, we have

$$\theta + \chi L = 0,$$

using the symbols of § 127. From the equation $\Sigma (\xi_3 - x) x'' = 1$, we have

$$\frac{\phi}{\rho^2} + \chi M = 1.$$

From the equation $\Sigma (\xi_3 - x) x''' = 0$, we have

$$-\frac{\theta}{\rho^2} - \phi \frac{\rho'}{\rho^3} + \frac{\psi}{\sigma^2} + \chi N = 0,$$

which, on substitution for θ and for ϕ, leads to the relation

$$\frac{\psi}{\sigma^2} + \chi \left(N + \frac{\rho'}{\rho} M + \frac{L}{\rho^2} \right) = \frac{\rho'}{\rho}.$$

From the equation $\Sigma (\xi_3 - x) x^{\text{iv}} = -\dfrac{1}{\rho^2}$, we have

$$\theta L + \phi M + \psi (L + \rho\rho' M + \rho^2 N) + \chi D = -\frac{1}{\rho^2},$$

which, on substitution for θ and for ϕ, leads to the relation

$$\psi\rho^2 \left(N + \frac{\rho'}{\rho} M + \frac{L}{\rho^2} \right) + \chi (D - L^2 - \rho^2 M^2) = -\frac{1}{\rho^2} - \rho^2 M.$$

Eliminating ψ between the last two relations, we find

$$\begin{aligned}
&\left\{ \frac{1}{\sigma^2} (D - L^2 - \rho^2 M^2) - \rho^2 \left(N + \frac{\rho'}{\rho} M + \frac{L}{\rho^2} \right)^2 \right\} \chi \\
&\qquad = -\frac{1}{\rho^2 \sigma^2} (1 + \rho^4 M) - \rho' \rho \left(N + \frac{\rho'}{\rho} M + \frac{L}{\rho^2} \right).
\end{aligned}$$

Now

$$N+\frac{\rho'}{\rho}M+\frac{L}{\rho^2}=-\frac{1}{\rho^2\sigma^2}\left(2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right);$$

and thus the coefficient of χ on the left-hand side

$$=\frac{1}{\sigma^2}\left\{(D-L^2-\rho^2M^2)-\frac{1}{\rho^2\sigma^2}\left(2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right)^2\right\}$$

$$=\rho^4\Omega^2,$$

by the relation obtained in § 127; while the right-hand side

$$=-\frac{1}{\sigma^2\rho^2}(1+\rho^4M)+\frac{\rho'}{\rho\sigma^2}\left(2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right)$$

$$=\frac{1}{\sigma^4\rho}(\sigma^2\rho''+\rho+\sigma\rho'\sigma')=\frac{1}{\sigma^4\rho}R\frac{dR}{d\rho}.$$

Also $\rho^3\sigma^2\tau\Omega=-1$; consequently

$$\chi=\tau^2\rho R\frac{dR}{d\rho},$$

which gives the value of χ.

We now have

$$\theta=-L\chi=-3\frac{\rho'}{\rho^3}\chi;$$

$$\phi=\rho^2(1-M\chi);$$

$$\psi=\sigma^2\frac{\rho'}{\rho}+\frac{1}{\rho^2}\left(2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right)\chi,$$

from the third relation; and thus the expression for ξ_3-x becomes

$$\xi_3-x=\rho^2x''+\sigma^2\frac{\rho'}{\rho}(x'+\rho\rho'x''+\rho^2x''')$$

$$+\left\{x^{\text{iv}}-3\frac{\rho'}{\rho^3}x'-\rho^2x''M+\frac{1}{\rho^2}\left(2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right)(x'+\rho\rho'x''+\rho^2x''')\right\}\chi.$$

The coefficient of χ on the right-hand side can be changed to the form

$$\frac{1}{\rho^2}\left(\frac{\sigma'}{\sigma}-\frac{\rho'}{\rho}\right)x'+\left(\frac{1}{\rho^2}+\frac{R}{\sigma^2\rho}\frac{dR}{d\rho}\right)x''+\left(2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right)x'''+x^{\text{iv}};$$

and thus we have

$$\xi_3-x-\rho^2x''-\sigma^2\frac{\rho'}{\rho}(x'+\rho\rho'x''+\rho^2x''')$$

$$=\tau^2\rho R\frac{dR}{d\rho}\left\{\frac{1}{\rho^2}\left(\frac{\sigma'}{\sigma}-\frac{\rho'}{\rho}\right)x'+\left(\frac{1}{\rho^2}+\frac{R}{\sigma^2\rho}\frac{dR}{d\rho}\right)x''+\left(2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right)x'''+x^{\text{iv}}\right\}.$$

There are corresponding expressions for η_3-y, ζ_3-z, υ_3-v; and thus we have an alternative set of expressions for the coordinates of the centre of globular curvature.

Equations of the fourth order satisfied by the coordinates of any point on the curve.

151. A comparison of the two sets of expressions, which have been obtained for $\xi_3 - x$, $\eta_3 - y$, $\zeta_3 - z$, $v_3 - v$, yields the relations

$$\left.\begin{aligned}
x^{\text{iv}} &= \frac{\rho}{\tau} J_x - \frac{1}{\rho^2}\left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho}\right) x' - \left(\frac{1}{\rho^2} + \frac{R}{\sigma^2\rho}\frac{dR}{d\rho}\right) x'' - \left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right) x''' \\
y^{\text{iv}} &= \frac{\rho}{\tau} J_y - \frac{1}{\rho^2}\left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho}\right) y' - \left(\frac{1}{\rho^2} + \frac{R}{\sigma^2\rho}\frac{dR}{d\rho}\right) y'' - \left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right) y''' \\
z^{\text{iv}} &= \frac{\rho}{\tau} J_z - \frac{1}{\rho^2}\left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho}\right) z' - \left(\frac{1}{\rho^2} + \frac{R}{\sigma^2\rho}\frac{dR}{d\rho}\right) z'' - \left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right) z''' \\
v^{\text{iv}} &= \frac{\rho}{\tau} J_v - \frac{1}{\rho^2}\left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho}\right) v' - \left(\frac{1}{\rho^2} + \frac{R}{\sigma^2\rho}\frac{dR}{d\rho}\right) v'' - \left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right) v'''
\end{aligned}\right\}.$$

From these values, by squaring both sides and adding, we obtain an expression for $\Sigma x^{\text{iv}\,2}$, that is, for D, which is found to agree with the former expression (§ 127) by using the relations already obtained.

These values of the fourth derivatives can also be used in another connection. The equations of the orthogonal plane through P, being the plane BPF which is orthogonal to the osculating plane

$$\left\|\begin{matrix} \bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\ x', & y', & z', & v' \\ x'', & y'', & z'', & v'' \end{matrix}\right\| = 0,$$

are

$$\left.\begin{aligned}
(\bar{x} - x)x' + (\bar{y} - y)y' + (\bar{z} - z)z' + (\bar{v} - v)v' &= 0 \\
(\bar{x} - x)x'' + (\bar{y} - y)y'' + (\bar{z} - z)z'' + (\bar{v} - v)v'' &= 0
\end{aligned}\right\}.$$

This orthogonal plane will contain a direction with direction-cosines

$$\left.\begin{aligned}
\lambda &= \alpha x' + \beta\rho x'' + \gamma\rho^2 x''' + \delta x^{\text{iv}} \\
\mu &= \alpha y' + \beta\rho y'' + \gamma\rho^2 y''' + \delta y^{\text{iv}} \\
\nu &= \alpha z' + \beta\rho z'' + \gamma\rho^2 z''' + \delta z^{\text{iv}} \\
\kappa &= \alpha v' + \beta\rho v'' + \gamma\rho^2 v''' + \delta v^{\text{iv}}
\end{aligned}\right\},$$

provided the four quantities $\alpha, \beta, \gamma, \delta$, are chosen to satisfy the two equations

$$\Sigma\lambda x' = 0, \quad \Sigma\lambda x'' = 0.$$

From the former, we have

$$\alpha - \gamma + \delta s_{14} = 0;$$

from the latter, we have

$$\beta - \gamma\rho' + \delta\rho s_{24} = 0;$$

and therefore

$$\begin{aligned}
\lambda &= \gamma(x' + \rho\rho' x'' + \rho^2 x''') + \delta(x^{\text{iv}} - x' s_{14} - \rho^2 x'' s_{24}),\\
\mu &= \gamma(y' + \rho\rho' y'' + \rho^2 y''') + \delta(y^{\text{iv}} - y' s_{14} - \rho^2 y'' s_{24}),\\
\nu &= \gamma(z' + \rho\rho' z'' + \rho^2 z''') + \delta(z^{\text{iv}} - z' s_{14} - \rho^2 z'' s_{24}),\\
\kappa &= \gamma(v' + \rho\rho' v'' + \rho^2 v''') + \delta(v^{\text{iv}} - v' s_{14} - \rho^2 v'' s_{24}).
\end{aligned}$$

But the orthogonal plane contains the two lines *PB* and *PF*, which can be taken as guiding lines for the expression of its equations in the form

$$\left\| \begin{array}{cccc} \bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\ x' + \rho\rho' x'' + \rho^2 x''', & y' + \rho\rho' y'' + \rho^2 y''', & z' + \rho\rho' z'' + \rho^2 z''', & v' + \rho\rho' v'' + \rho^2 v''' \\ J_x, & J_y, & J_z, & J_v \end{array} \right\| = 0.$$

Manifestly one direction in the plane is provided by taking $\gamma = 0$ and δ different from zero; and so there must be quantities α' and β', such that

$$\begin{aligned}
x^{\text{iv}} - x' s_{14} - \rho^2 x'' s_{24} &= \alpha' J_x + \beta'(x' + \rho\rho' x'' + \rho^2 x'''),\\
y^{\text{iv}} - y' s_{14} - \rho^2 y'' s_{24} &= \alpha' J_y + \beta'(y' + \rho\rho' y'' + \rho^2 y'''),\\
z^{\text{iv}} - z' s_{14} - \rho^2 z'' s_{24} &= \alpha' J_z + \beta'(z' + \rho\rho' z'' + \rho^2 z'''),\\
v^{\text{iv}} - v' s_{14} - \rho^2 v'' s_{24} &= \alpha' J_v + \beta'(v' + \rho\rho' v'' + \rho^2 v''').
\end{aligned}$$

Multiplying by J_x, J_y, J_z, J_v, and adding, we have

$$\Sigma x^{\text{iv}} J_x = \alpha' \Sigma J_x^2,$$

that is,

$$-\Omega = \alpha' \frac{1}{\sigma^2 \rho^4},$$

and therefore

$$\alpha' = -\sigma^2 \rho^4 \Omega = \frac{\rho}{\tau}.$$

When we multiply by x', y', z', v', and add, an identity ensues; similarly, when we multiply by x'', y'', z'', v'', and add, another identity ensues. When we multiply by x''', y''', z''', v''', and add, then

$$N + \frac{\rho'}{\rho} M + \frac{L}{\rho^2} = \beta' \left\{ -\frac{1}{\rho^2} - \frac{\rho'^2}{\rho^4} + \left(\frac{1}{\sigma^2} + \frac{1}{\rho^2} + \frac{\rho'^2}{\rho^2} \right) \right\},$$

that is,

$$\beta' = -\frac{1}{\rho^2} \left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma} \right).$$

Hence the first equation becomes

$$x^{\text{iv}} - x' L - \rho^2 x'' M = \frac{\rho}{\tau} J_x - \frac{1}{\rho\sigma} \left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma} \right) (x' + \rho\rho' x'' + \rho^2 x''') \frac{\sigma}{\rho},$$

with three other relations, and these relations have already been established.

Consequently, any direction in the orthogonal plane at P has direction-cosines

$$\left(\alpha, \beta, \gamma, \delta \between \begin{array}{|cccc|} x', & x'', & x''', & x^{\text{iv}} \\ y', & y'', & y''', & y^{\text{iv}} \\ z', & z'', & z''', & z^{\text{iv}} \\ v', & v'', & v''', & v^{\text{iv}} \end{array}\right),$$

with appropriately determined values of $\alpha, \beta, \gamma, \delta$.

152. The magnitude of the radius, Γ, of globular curvature can also be derived from the equations of § 150, viz.

$$\begin{aligned} \xi_3 - x &= \theta x' + \phi x'' + \psi(x' + \rho\rho' x'' + \rho^2 x''') + \chi x^{\text{iv}}, \\ \eta_3 - y &= \theta y' + \phi y'' + \psi(y' + \rho\rho' y'' + \rho^2 y''') + \chi y^{\text{iv}}, \\ \zeta_3 - z &= \theta z' + \phi z'' + \psi(z' + \rho\rho' z'' + \rho^2 z''') + \chi z^{\text{iv}}, \\ \upsilon_3 - v &= \theta v' + \phi v'' + \psi(v' + \rho\rho' v'' + \rho^2 v''') + \chi v^{\text{iv}}, \end{aligned}$$

with the help of the equations

$$\Sigma(\xi_3 - x)x' = 0, \quad \Sigma(\xi_3 - x)x'' = 1, \quad \Sigma(\xi_3 - x)x''' = 0, \quad \Sigma(\xi_3 - x)x^{\text{iv}} = -\frac{1}{\rho^2},$$

in their original form (§ 147). Multiply the four equations by $\xi_3 - x$, $\eta_3 - y$, $\zeta_3 - z$, $\upsilon_3 - v$, respectively, and add: then

$$\begin{aligned} \Gamma^2 &= (\xi_3 - x)^2 + (\eta_3 - y)^2 + (\zeta_3 - z)^2 + (\upsilon_3 - v)^2 \\ &= \theta\{\Sigma(\xi_3 - x)x'\} + \phi\{\Sigma(\xi_3 - x)x''\} + \chi\{\Sigma(\xi_3 - x)x^{\text{iv}}\} \\ &\qquad + \psi[\{\Sigma(\xi_3 - x)x'\} + \rho\rho'\{\Sigma(\xi_3 - x)x''\} + \rho^2\{\Sigma(\xi_3 - x)x'''\}] \\ &= \phi + \psi\rho\rho' - \chi\frac{1}{\rho^2} \\ &= \rho^2(1 - \chi M) + \left\{\sigma^2\rho'^2 + \frac{\rho'}{\rho}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)\chi\right\} - \chi\frac{1}{\rho^2}, \end{aligned}$$

on substituting the values of ϕ and ψ. Thus

$$\begin{aligned} \Gamma^2 &= \rho^2 + \sigma^2\rho'^2 + \left\{2\frac{\rho'^2}{\rho^2} + \frac{\rho'\sigma'}{\rho\sigma} - \frac{1}{\rho^2} - \rho^2 M\right\}\chi \\ &= R^2 + \frac{1}{\rho\sigma^2}R\frac{dR}{d\rho}\chi \\ &= R^2 + \left(\frac{\tau}{\sigma}R\frac{dR}{d\rho}\right)^2, \end{aligned}$$

in agreement with the former value (§ 149).

153. An illustration of the curvatures is provided by a globular loxodrome.

Ex. Consider a curve in a globular region

$$x^2 + y^2 + z^2 + v^2 = 1,$$

defined by the property that it is to be inclined at constant angles to two perpendicular diameters of the globe. (Such a curve seems the natural extension, to quadruple space, of the three-dimensional loxodrome: as will be seen, it actually is that type of loxodrome.)

Let the two perpendicular diameters be taken as the axes of z and of v; and, on the definition of the curve, let

$$z' = \cos\alpha\sin\beta, \quad v' = \cos\alpha\cos\beta.$$

Thus

$$z \cos\beta - v \sin\beta = c,$$

where c is a constant, determinable from the values of any point on the curve Thus the curve lies in a flat, usually not through the centre of the globe; hence (§ 145) the tilt is zero. Also (§ 103) the curve lies on a sphere.

Let the axes be changed by a planar rotation through an angle β round the plane of x, y; new variables are

$$z \cos\beta - v \sin\beta = V, \quad z \sin\beta + v \cos\beta = Z.$$

The globe becomes

$$x^2 + y^2 + Z^2 + V^2 = 1;$$

the flat, containing the curve, becomes $V = c$, and therefore the curve lies in the three-dimensional spherical surface

$$x^2 + y^2 + Z^2 = 1 - c^2 = a^2.$$

We now have

$$\frac{dV}{ds} = 0, \quad \frac{dZ}{ds} = z' \sin\beta + v' \cos\beta = \cos\alpha;$$

and therefore

$$x'^2 + y'^2 = \sin^2\alpha,$$

the curve being a (spherical) loxodrome.

We take

$$Z = a \sin\theta, \quad x = a \cos\theta \cos\phi, \quad y = a \cos\theta \sin\phi$$

Then

$$a\theta' = \frac{\cos\alpha}{\cos\theta};$$

and, as

$$a^2 (\theta'^2 + \cos^2\theta \,.\, \phi'^2) = 1,$$

we have

$$a\phi' = \frac{1}{\cos^2\theta} (\sin^2\alpha - \sin^2\theta)^{\frac{1}{2}}.$$

With these values, we have

$$V'' = 0, \quad Z'' = 0;$$

and so the circular curvature is given by

$$\frac{1}{\rho^2} = x''^2 + y''^2$$

$$= \frac{1}{a^2} \frac{\sin^2\alpha}{\sin^2\alpha - \sin^2\theta},$$

on substitution and reduction.

Further, as the curve lies entirely on a sphere of radius a, its radius of sph curvature is a; thus

$$a^2 = \rho^2 + \sigma^2 \rho'^2;$$

and thus

$$\sigma = \rho \tan\alpha = a (\sin^2\alpha - \sin^2\theta)^{\frac{1}{2}} \sec\alpha.$$

The tilt has already been proved to be zero. Thus the three curvatures of the curve are known.

The circular curvature and the torsion can be expressed in the forms

$$\rho \sin\alpha = (r^2 - a^2 \cos^2\alpha)^{\frac{1}{2}}, \quad \sigma \cos\alpha = (r^2 - a^2 \cos^2\alpha)^{\frac{1}{2}},$$

where r is the distance of the current point of the curve from the plane of ZV, that is, from the plane of the two diameters with which the curve makes constant angles.

As the tilt is zero, and R is constant, the formal expression for the globular curvature ceases to have a determinate value. Though each of the loxodromes (for different values of α and of β) is three-dimensional, and globular curvature thus ceases to be significant, we can take $\Gamma = 1$, because all the curves lie in the initial unit globe.

CHAPTER IX.

ORGANIC FRAMEWORK OF A CURVE.

The moving orthogonal frame of a curve.

154. At this stage, it is convenient to summarise the conformation of the curve in relation to the quadruply orthogonal frame at any point, as illustrated by the diagram of § 128. In the diagram, this frame at the point P of the curve is constituted by the tangent PT, the radius of plane curvature PC, the binormal PB, and the trinormal PF. The diagram can be constructed, gradually, as follows.

I. The points $P, P', P'', \dots$ are successive points on the curve. The lines $PT, P'T', P''T'', \dots$ are the tangents at these points. The plane $T'P'T$ is the pre-limit position of CPT, the osculating plane of the curve at P containing two successive tangents $PP'T$ and $P'T'$.

The normal flat at P and the normal flat at P' intersect in a plane, to every direction in which PP' is perpendicular; this plane is orthogonal to the osculating plane TPC, which it meets in the single point C, the centre of plane curvature at P. This orthogonal plane is SCf in the diagram: the line CS is parallel to the binormal PB at P, and the line Cf is parallel to the trinormal PF at P, but the construction of these lines is not yet given.

The plane $T''P''T'$ is the pre-limit position of $C'P'T'$, the osculating plane of the curve at P' containing two successive tangents $P'P''T'$ and $P''T''$. The normal flat at P' and the normal flat at the next consecutive point P'' intersect in a plane: this plane is orthogonal to the osculating plane $T'P'C'$, which it meets at the single point C', the centre of plane curvature at P': and this orthogonal plane is $S'C'f'$, where $C'S'$ is parallel to the binormal $P'B'$ at P' and $C'f'$ is parallel to the trinormal $P'F'$ at P'.

Similarly C'' is the centre of plane curvature at P'': the osculating plane at P'' is $C''P''T''$; and the orthogonal plane through C'' is $S''C''f''$.

II. As a flat and a plane intersect in a line, the normal flat at P intersects the osculating plane at P in a line; the direction of this line is PC.

The osculating flat at P contains three consecutive tangents PT, $P'T'$, $P''T''$: that is, it contains the osculating plane at P determined by the tangents PT and $P'T'$, and the osculating plane at P' determined by the tangents $P'T'$ and $P''T''$. The normal flat at P intersects the osculating flat at P in a plane: the line PT is perpendicular to that plane, because it is perpendicular to every direction in the normal flat: the line PC lies in that plane. Within that plane, draw PB perpendicular to PC: and it is perpendicular to PT. We thus have a set of three perpendicular directions PT, PC, PB, in the osculating flat at P; the direction PB is the binormal at P.

Similarly, there are the three perpendicular directions $P'T'$, $P'C'$, $P'B'$, in the osculating flat at P'; the line $P'C'$ is the direction of the radius of plane curvature at P'; and the line $P'B'$, drawn perpendicular to $P'C'$ in the plane which is the intersection of the normal flat at P' and the osculating flat at P', is the binormal at P'. The three lines $P'T'$, $P'C'$, $P'B'$, are guiding lines for the osculating flat at P'.

Similarly for the osculating flat at P''; and so on, for successive points.

But the successive directions PC, $P'C'$, $P''C''$, ... do not meet: thus the locus of the centre of plane curvature is not an evolute of the curve.

III. The normal flat at P and the normal flat at P' intersect in a plane CSf; the normal flat at P' and the normal flat at P'' intersect in a plane $C'S'f'$; and these two planes CSf and $C'S'f'$, both lying in the same flat (viz the normal flat at P'), meet in a line. The line is the intersection of the normal flats at the three successive points P, P', P''. Its limiting position is a line through S, the point where these three normal flats meet the osculating plane at P; and it is the line in the direction SG, which is normal to the osculating flat at P. The point S is the centre of spherical curvature at P; the line CS, in the plane through C orthogonal to the osculating plane TPC, is perpendicular to PC.

Similarly for S', the centre of spherical curvature at P', for S'', the centre of spherical curvature at P''; and for successive points on the curve. But the lines CS, $C'S'$, $C''S''$, ... do not meet.

The line PF is drawn through P parallel to SG, it is the trinormal at P.

IV. The normal flats at the four successive points P, P', P'', P''', meet in a point G; and this is the point where the normal flat at P''' is met by the line SG, which is the line of intersection of the normal flats at P, P', P''.

The point G is the centre of globular curvature. The line SG is perpendicular at S, to the osculating flat at P; and thus SG is perpendicular to all directions in the flat, and therefore perpendicular to PT, PC, PB. The trinormal PF is perpendicular to the same three lines, for it is parallel to SG. We thus have the quadruply orthogonal frame at P.

Similarly for the frame at P', and for the frames at successive points.

As SG is the intersection of the three normal flats at P, P', P'', and as $S'G'$ is the intersection of the three normal flats at P', P'', P''', both SG and $S'G'$ lie in the plane which is the intersection of the normal flats at P' and P''.

Consequently the two lines SG and $S'G'$, lying in one plane, must meet; the limiting position of their point of meeting is G, the centre of globular curvature at P.

As regards these centres of the three curvatures C, S, G, we note that G is the intersection of four consecutive normal flats at P, P', P'', P''': that S is the intersection of three consecutive normal flats at P, P', P'', and the osculating flat at P: that C is the intersection of two consecutive normal flats at P, P', and two consecutive osculating flats at P, P'. The point P itself can be regarded as the intersection of the normal flat at P and three consecutive osculating flats at P, P', P'': it can also be regarded as the intersection of four consecutive osculating flats at P, P', P'', P'''.

Finally, the osculating flat at P is $PTCB$, with TP, CP, BP, as guiding lines; and the normal flat at P is $PCBF$, with CP, BP, FP, as guiding lines. The osculating flat at P' is $P'T'C'B'$, and the normal flat at P' is $P'C'B'F'$; and so for successive points on the curve.

Direction-cosines of a consecutive tangent, relative to the frame.

155. Later, we shall require the direction-cosines of $P'T'$ with respect to the principal lines of the orthogonal frame at P. Because $P'T'$ lies in the osculating plane at P, it is perpendicular to PB and to PF. Also, $P'T'$ makes an angle $d\epsilon$ with PT measured positively from PT to PC; it therefore makes an angle $\frac{1}{2}\pi - d\epsilon$ with PC. Hence, if λ_1, μ_1, ν_1, κ_1, denote the direction-cosines of $P'T'$ with respect to the principal lines at P, we have

$$\begin{aligned}\lambda_1 &= 1,\\ \mu_1 &= \cos\left(\tfrac{1}{2}\pi - d\epsilon\right) = d\epsilon,\\ \nu_1 &= 0,\\ \kappa_1 &= 0.\end{aligned}$$

These results can at once be verified analytically. For the direction-cosines of $P'T'$ with respect to the general axes OX, OY, OZ, OV, are

$$x' + x''ds, \quad y' + y''ds, \quad z' + z''ds, \quad v' + v''ds;$$

consequently

$$\begin{aligned}\lambda_1 &= \Sigma x'(x' + x''ds) = 1,\\ \mu_1 &= \Sigma\rho x''(x' + x''ds) = \frac{1}{\rho}ds = d\epsilon,\\ \nu_1 &= \Sigma\left\{\frac{\sigma}{\rho}(x' + \rho\rho' x'' + \rho^2 x''')\right\}(x' + x''ds) = 0,\\ \kappa_1 &= \Sigma\sigma\rho^2 J_x(x' + x''ds) = 0.\end{aligned}$$

Locus of the centre of circular curvature.

156. The locus of C, the centre of plane curvature of the given curve, is another curve; but it is not an evolute of the given curve, because the successive radii of plane curvature PC, $P'C'$, $P''C''$, ... do not meet one another in succession.

The coordinates of C are

$$\xi_1 = x + \rho^2 x'', \quad \eta_1 = y + \rho^2 y'', \quad \zeta_1 = z + \rho^2 z'', \quad v_1 = v + \rho^2 v''.$$

Hence

$$\frac{d\xi_1}{ds} = x' + 2\rho\rho' x'' + \rho^2 x''',$$

$$\frac{d\eta_1}{ds} = y' + 2\rho\rho' y'' + \rho^2 y''',$$

$$\frac{d\zeta_1}{ds} = z' + 2\rho\rho' z'' + \rho^2 z''',$$

$$\frac{dv_1}{ds} = v' + 2\rho\rho' v'' + \rho^2 v''';$$

and therefore, if ds_1 denote the arc of the locus of C corresponding to the arc $ds\ (= PP')$ of the given curve,

$$\left(\frac{ds_1}{ds}\right)^2 = \Sigma\,(x' + 2\rho\rho' x'' + \rho^2 x''')^2 = \frac{R^2}{\sigma^2}.$$

Consequently the *direction-cosines of* CC', being $\frac{d\xi_1}{ds_1}$, $\frac{d\eta_1}{ds_1}$, $\frac{d\zeta_1}{ds_1}$, $\frac{dv_1}{ds_1}$, are

$$\frac{\sigma}{R}(x' + 2\rho\rho' x'' + \rho^2 x'''), \quad \frac{\sigma}{R}(y' + 2\rho\rho' y'' + \rho^2 y'''),$$

$$\frac{\sigma}{R}(z' + 2\rho\rho' z'' + \rho^2 z'''), \quad \frac{\sigma}{R}(v' + 2\rho\rho' v'' + \rho^2 v''').$$

Now

$$\Sigma x' \frac{d\xi_1}{ds_1} = 0,$$

so that CC' is perpendicular to PT; and

$$\Sigma\,\sigma\rho^2 J_x \frac{d\xi_1}{ds_1} = 0,$$

so that CC' is perpendicular to PF. Consequently CC' is perpendicular to the plane TPF; and therefore it lies in the plane which is orthogonal to the plane TPF, that is, it lies in the normal plane CPB through the principal normal and the binormal.

Further, we have

$$ds_1{}^2 = \left(\rho'^2 + \frac{\rho^2}{\sigma^2}\right) ds^2 = d\rho^2 + \rho^2 d\eta^2,$$

for $\sigma d\eta = ds$ The line CP' is perpendicular to PP' in the osculating plane TPC, and the line $C'P'$ is perpendicular to the same line in the osculating plane $T'P'C'$; hence the angle $CP'C'$ is equal to $d\eta$. When we project $C'P'$ on CP, the point P' projects on P; if N be the projection of C', we have

$$C'N = \rho d\eta, \quad CN = d\rho,$$

these two portions giving the value $\rho^2 d\eta^2 + d\rho^2$ for CC'^2. Also, if θ denote the inclination of CC' to PC, and therefore $\frac{1}{2}\pi - \theta$ its inclination to PB, we have

$$\cos\theta = \frac{CN}{CC'} = \frac{d\rho}{ds_1} = \frac{\sigma\rho'}{R},$$

$$\sin\theta = \frac{NC'}{CC'} = \rho\frac{d\eta}{ds_1} = \frac{\rho}{R}.$$

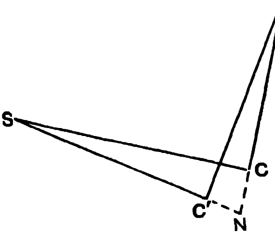

Fig 14.

But in the diagram, SCP is a right angle; $SP = R$, $PC = \rho$, $CS = \sigma\rho'$; and therefore when, in the normal plane CPB, a circle is described on PS as diameter, the tangent at C to the locus of C is a tangent to this circle.

Also from the diagram, we have $SC'P' = \frac{1}{2}\pi = SCP'$; thus a circle can be drawn through $SC'CP'$: the angle $CSC' =$ the angle $CP'C' = d\eta$; and thus we verify that

$$CSd\eta = CN = d\rho = \rho' ds,$$

so that $CS = \sigma\rho'$.

Direction-cosines of a consecutive principal normal, relative to the frame.

157. We shall require the direction-cosines of $C'P'$ with reference to the lines of the orthogonal frame at P. The arc CC' lies in the normal plane BPC: and the whole configuration in the figure on p. 211 lies in the osculating flat at P. Also $C'P'$ is perpendicular to $T'P'$, while $d\epsilon$ is the angle $TP'T'$; and in the plane $SC'CP'$, the angle between $P'C'$ and CS is $\frac{1}{2}\pi - d\eta$. Hence *if* $\lambda_2, \mu_2, \nu_2, \kappa_2$, *be the direction-cosines of* $P'C'$ with respect to the principal lines of the orthogonal frame at P, we have (up to the first order of small quantities)

$$\lambda_2 = \cos(\tfrac{1}{2}\pi + d\epsilon) = -d\epsilon,$$

$$\mu_2 = 1,$$

$$\nu_2 = \cos(\tfrac{1}{2}\pi - d\eta) = d\eta,$$

$$\kappa_2 = 0.$$

These results can easily be verified analytically. The direction-cosines of $P'C'$ with reference to the external axes OX, OY, OZ, OV, are

$$\rho x'' + (\rho' x'' + \rho x''')\,ds, \quad \rho y'' + (\rho' y'' + \rho y''')\,ds,$$
$$\rho z'' + (\rho' z'' + \rho z''')\,ds, \quad \rho v'' + (\rho' v'' + \rho v''')\,ds;$$

hence

$$\lambda_2 = \Sigma x' \left\{\rho x'' + (\rho' x'' + \rho x''')\,ds\right\} = -\frac{ds}{\rho} = -\,d\epsilon,$$
$$\mu_2 = \Sigma \rho x'' \left\{\rho x'' + (\rho' x'' + \rho x''')\,ds\right\} = 1,$$
$$\nu_2 = \Sigma \frac{\sigma}{\rho}(x' + \rho\rho' x'' + \rho^2 x''') \left\{\rho x'' + (\rho' x'' + \rho x''')\,ds\right\} = \frac{ds}{\sigma} = d\eta,$$
$$\kappa_2 = \Sigma \sigma\rho^2 J_x \left\{\rho x'' + (\rho' x'' + \rho x''')\,ds\right\} = 0.$$

Locus of the centre of spherical curvature.

158. The centre S of spherical curvature is given by the coordinates

$$\xi_2 = x + \sigma^2 \frac{\rho'}{\rho} x' + R^2 x'' + \sigma^2 \rho\rho' x''',$$
$$\eta_2 = y + \sigma^2 \frac{\rho'}{\rho} y' + R^2 y'' + \sigma^2 \rho\rho' y''',$$
$$\zeta_2 = z + \sigma^2 \frac{\rho'}{\rho} z' + R^2 z'' + \sigma^2 \rho\rho' z''',$$
$$v_2 = v + \sigma^2 \frac{\rho'}{\rho} v' + R^2 v'' + \sigma^2 \rho\rho' v'''.$$

Hence

$$\frac{d\xi_2}{ds} = \left\{1 + \frac{d}{ds}\left(\sigma^2 \frac{\rho'}{\rho}\right)\right\} x' + \left(\sigma^2 \frac{\rho'}{\rho} + 2RR'\right) x''$$
$$+ \left\{R^2 + \frac{d}{ds}(\sigma^2 \rho\rho')\right\} x''' + \sigma^2 \rho\rho' x^{\text{iv}}.$$

On substituting the value of x^{iv} as obtained (§ 151) in terms of x', x'', x''' and J_x, viz.

$$\frac{\rho}{\tau} J_x - \frac{1}{\rho^2}\left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho}\right) x' - \left(\frac{1}{\rho^2} + \frac{R}{\sigma^2 \rho}\frac{dR}{d\rho}\right) x'' - \left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right) x''',$$

the coefficient of x' in the modified expression for $\frac{d\xi_2}{ds}$ is

$$1 + \frac{d}{ds}\left(\sigma^2 \frac{\rho'}{\rho}\right) - \sigma^2 \rho\rho' \frac{1}{\rho^2}\left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho}\right), \quad = \frac{RR'}{\rho\rho'};$$

the coefficient of x'' is

$$\sigma^2 \frac{\rho'}{\rho} + 2RR' - \sigma^2 \rho\rho' \left(\frac{1}{\rho^2} + \frac{R}{\sigma^2 \rho}\frac{dR}{d\rho}\right), \quad = RR';$$

and the coefficient of x''' is

$$R^2 + \frac{d}{ds}(\sigma^2\rho\rho') - \sigma^2\rho\rho'\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right), \quad = \frac{\rho}{\rho'}RR'.$$

Thus

$$\frac{d\xi_2}{ds} = \frac{RR'}{\rho\rho'}(x' + \rho\rho'x'' + \rho^2x''') + \frac{\sigma\rho'}{\tau}\sigma\rho^2J_x;$$

and similarly

$$\frac{d\eta_2}{ds} = \frac{RR'}{\rho\rho'}(y' + \rho\rho'y'' + \rho^2y''') + \frac{\sigma\rho'}{\tau}\sigma\rho^2J_y,$$

$$\frac{d\zeta_2}{ds} = \frac{RR'}{\rho\rho'}(z' + \rho\rho'z'' + \rho^2z''') + \frac{\sigma\rho'}{\tau}\sigma\rho^2J_z,$$

$$\frac{dv_2}{ds} = \frac{RR'}{\rho\rho'}(v' + \rho\rho'v'' + \rho^2v''') + \frac{\sigma\rho'}{\tau}\sigma\rho^2J_v.$$

The form of these expressions for $\frac{d\xi_2}{ds}, \frac{d\eta_2}{ds}, \frac{d\zeta_2}{ds}, \frac{dv_2}{ds}$, which are proportional to the direction-cosines of the line SS', shews that they lie in a plane with guiding lines whose direction-cosines are proportional to

$$x' + \rho\rho'x'' + \rho^2x''', \quad y' + \rho\rho'y'' + \rho^2y''', \quad z' + \rho\rho'z'' + \rho^2z''', \quad v' + \rho\rho'v'' + \rho^2v''',$$

and

$$J_x, \quad J_y, \quad J_z, \quad J_v,$$

respectively. that is, whose guiding lines are parallel to PB and PF. Thus *SS' lies in a plane parallel to the orthogonal plane BPF.*

This result also follows from the properties

$$\Sigma x'\frac{d\xi_2}{ds} = 0, \quad \Sigma x''\frac{d\xi_2}{ds} = 0,$$

shewing that SS' is perpendicular to PT and to PC, that is, is perpendicular to the osculating plane at P, and therefore lies in a plane parallel to the orthogonal plane at P.

Again, let ds_2 denote the arc SS'; then

$$\left(\frac{ds_2}{ds}\right)^2 = \left(\frac{d\xi_2}{ds}\right)^2 + \left(\frac{d\eta_2}{ds}\right)^2 + \left(\frac{d\zeta_2}{ds}\right)^2 + \left(\frac{dv_2}{ds}\right)^2$$

$$= \left(\frac{RR'}{\rho\rho'}\right)^2 \Sigma(x' + \rho\rho'x'' + \rho^2x''')^2 + \left(\frac{\sigma\rho'}{\tau}\right)^2(\sigma\rho^2J_x)^2$$

$$= \left(\frac{RR'}{\sigma\rho'}\right)^2 + \left(\frac{\sigma\rho'}{\tau}\right)^2.$$

In the configuration of the normal flat at *P*, viz. the flat in the diagram on p. 211 with *PC, PB, PF*, as guiding lines, the significance of this arc *SS'* appears. The arc *CC'* lies in the normal plane *BPC*; the arc *SS'* lies in a plane parallel to the orthogonal plane *BPF*. When *C'*, *S'*, *G'*, are the centres of the three curvatures at the consecutive point *P'*, we know that *S'G'* and *SG* intersect; and as *SG* and *S'G'* are normal to the consecutive osculating flats, the angle *SGS'* is equal to $d\omega$. Thus, if *S'M* is the perpendicular from *S'* on *SG*, we have

$$S'M = GS' \, . \, d\omega$$

$$= \frac{\tau}{\sigma} R \frac{dR}{d\rho} \, d\omega,$$

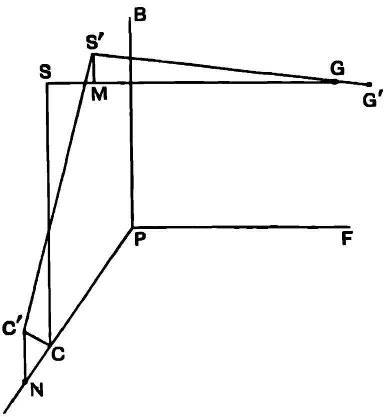

Fig. 15.

to small quantities of the first order: that is,

$$S'M = \frac{RR'}{\sigma\rho'} \, ds.$$

This value can also be obtained otherwise from the figure; for

$$\begin{aligned} S'M &= S'C' - SC' \\ &= S'C' - (SC - C'N) \\ &= (S'C' - SC) + C'N \\ &= \frac{d}{ds}(\sigma\rho')\, ds + \rho\, d\eta \\ &= \left\{\frac{d}{ds}(\sigma\rho') + \frac{\rho}{\sigma}\right\} ds = \frac{RR'}{\sigma\rho'}\, ds, \end{aligned}$$

as before.

Again *SG*, perpendicular to the plane *BPC*, is perpendicular to *C'S*; and *S'G'* is perpendicular to *C'S'*; hence the points *C'SS'G* lie on a circle and therefore the angle *SC'S'* is equal to *SGS'*, that is, is equal to $d\omega$. Consequently

$$SM = SC d\omega$$

$$= \sigma\rho' d\omega = \frac{\sigma\rho'}{\tau} \, ds.$$

We at once have

$$ds_2^2 = SS'^2 = S'M^2 + SM^2 = \left\{\left(\frac{RR'}{\sigma\rho'}\right)^2 + \left(\frac{\sigma\rho'}{\tau}\right)^2\right\} ds^2,$$

in agreement with the former result.

Direction-cosines of a consecutive binormal, relative to the frame.

159. We shall require the direction-cosines of the binormal $P'B'$ at the consecutive point P', this consecutive binormal being parallel to $C'S'$. In the normal flat, take a spherical triangle in which F, C, B, C', B', represent the directions $PF, PC, PB, P'C', P'B'$, respectively: let FC' and BC meet in n, and FB' and CB meet in m.

Then

$$Bm = nC = d\eta,$$

$$mB' = FF' = d\omega,$$

and the whole of this configuration lies in the normal flat, that is, every direction in the configuration is perpendicular to PT.

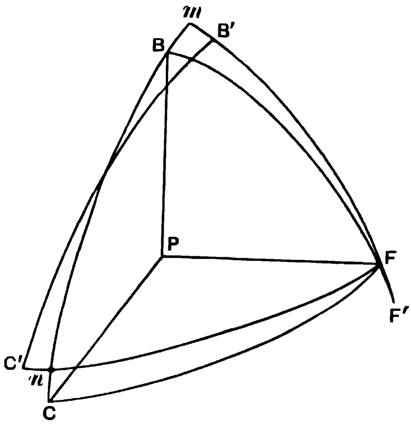

Fig. 16.

Let λ_3, μ_3, ν_3, κ_3, be the direction-cosines of $P'B'$ with respect to the principal lines of the orthogonal frame at P. Then

$$\lambda_3 = 0,$$

$$\mu_3 = \cos CB' = \cos(\tfrac{1}{2}\pi + d\eta) = -d\eta,$$

$$\nu_3 = \cos BB' = 1,$$

$$\kappa_3 = \cos FB' = \cos(\tfrac{1}{2}\pi - d\omega) = d\omega.$$

These results can be verified analytically. The direction-cosine of PB with reference to OX is

$$\frac{\sigma}{\rho}x' + \sigma\rho' x'' + \sigma\rho x''';$$

and therefore the direction-cosine of $P'B'$ with reference to OX is

$$\frac{\sigma}{\rho}x' + \sigma\rho' x'' + \sigma\rho x'''$$
$$+ \left[x'\frac{d}{ds}\left(\frac{\sigma}{\rho}\right) + x''\left\{\frac{\sigma}{\rho} + \frac{d}{ds}(\sigma\rho')\right\} + x'''\left\{\sigma\rho' + \frac{d}{ds}(\sigma\rho)\right\} + \sigma\rho x^{\text{iv}}\right] ds.$$

In this expression, substitute for x^{iv} the value obtained in § 151 and quoted above (p. 258). When the resulting expression is reduced so as to be expressible linearly in terms of x', x'', x''', and J_x, we find that the expression in square brackets becomes

$$-\frac{\rho}{\sigma}x'' + \frac{1}{\tau}\sigma\rho^2 J_x.$$

Similarly for the inclinations to OY, OZ, OV. Hence the direction-cosines $P'B'$ with respect to the external frame OX, OY, OZ, OV, are

$$\frac{\sigma}{\rho}(x'+\rho\rho' x''+\rho^2 x''')+\left(-\frac{\rho}{\sigma}x''+\frac{1}{\tau}\sigma\rho^2 J_x\right)ds,$$

$$\frac{\sigma}{\rho}(y'+\rho\rho' y''+\rho^2 y''')+\left(-\frac{\rho}{\sigma}y''+\frac{1}{\tau}\sigma\rho^2 J_y\right)ds,$$

$$\frac{\sigma}{\rho}(z'+\rho\rho' z''+\rho^2 z''')+\left(-\frac{\rho}{\sigma}z''+\frac{1}{\tau}\sigma\rho^2 J_z\right)ds,$$

$$\frac{\sigma}{\rho}(v'+\rho\rho' v''+\rho^2 v''')+\left(-\frac{\rho}{\sigma}v''+\frac{1}{\tau}\sigma\rho^2 J_v\right)ds.$$

Consequently

$$\lambda_3=\Sigma x'\left\{\frac{\sigma}{\rho}(x'+\rho\rho' x''+\rho^2 x''')+\left(-\frac{\rho}{\sigma}x''+\frac{1}{\tau}\sigma\rho^2 J_x\right)ds\right\}=0,$$

$$\mu_3=\Sigma\rho x''\left\{\frac{\sigma}{\rho}(x'+\rho\rho' x''+\rho^2 x''')+\left(-\frac{\rho}{\sigma}x''+\frac{1}{\tau}\sigma\rho^2 J_x\right)ds\right\}=-\frac{1}{\sigma}ds=-d\eta,$$

$$\nu_3=\Sigma\frac{\sigma}{\rho}(x'+\rho\rho' x''+\rho^2 x''')\left\{\frac{\sigma}{\rho}(x'+\rho\rho' x''+\rho^2 x''')+\left(-\frac{\rho}{\sigma}x''+\frac{1}{\tau}\sigma\rho^2 J_x\right)ds\right\}=$$

$$\kappa_3=\Sigma\sigma\rho^2 J_x\left\{\frac{\sigma}{\rho}(x'+\rho\rho' x''+\rho^2 x''')+\left(-\frac{\rho}{\sigma}x''+\frac{1}{\tau}\sigma\rho^2 J_x\right)ds\right\}=\frac{1}{\tau}ds=d\omega,$$

thus completing the verification as stated.

Locus of the centre of globular curvature.

160. The coordinates of G, the centre of globular curvature, are

$$\xi_3=x+\rho^2 x''+\sigma\rho'\frac{\sigma}{\rho}(x'+\rho\rho' x''+\rho^2 x''')+\frac{\tau}{\sigma}R\frac{dR}{d\rho}\sigma\rho^2 J_x,$$

$$\eta_3=y+\rho^2 y''+\sigma\rho'\frac{\sigma}{\rho}(y'+\rho\rho' y''+\rho^2 y''')+\frac{\tau}{\sigma}R\frac{dR}{d\rho}\sigma\rho^2 J_y,$$

$$\zeta_3=z+\rho^2 z''+\sigma\rho'\frac{\sigma}{\rho}(z'+\rho\rho' z''+\rho^2 z''')+\frac{\tau}{\sigma}R\frac{dR}{d\rho}\sigma\rho^2 J_z,$$

$$v_3=v+\rho^2 v''+\sigma\rho'\frac{\sigma}{\rho}(v'+\rho\rho' v''+\rho^2 v''')+\frac{\tau}{\sigma}R\frac{dR}{d\rho}\sigma\rho^2 J_v.$$

Hence

$$\begin{aligned}\frac{d\xi_3}{ds}&=x'+2\rho\rho' x''+\rho^2 x'''+(x'+\rho\rho' x''+\rho^2 x''')\frac{d}{ds}\left(\frac{\sigma^2\rho'}{\rho}\right)\\&\quad+\frac{\sigma^2\rho'}{\rho}\{(1+\rho\rho''+\rho'^2)x''+3\rho\rho' x'''+\rho^2 x^{\text{iv}}\}\\&\quad+\sigma\rho^2 J_x\frac{d}{ds}\left(\frac{\tau}{\sigma}R\frac{dR}{d\rho}\right)+\frac{\tau}{\sigma}R\frac{dR}{d\rho}\frac{d}{ds}(\sigma\rho^2 J_x),\end{aligned}$$

with similar expressions for $\dfrac{d\eta_3}{ds}$, $\dfrac{d\zeta_3}{ds}$, $\dfrac{dv_3}{ds}$.

We proceed to modify the expression for $\frac{d\xi_3}{ds}$ by substituting, for x^{iv}, its value (§ 151)

$$\frac{\rho}{\tau}J_x - \frac{1}{\rho^2}\left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho}\right)x' - \left(\frac{1}{\rho^2} + \frac{R}{\sigma^2\rho}\frac{dR}{d\rho}\right)x'' - \left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)x''',$$

and by evaluating the quantity $\frac{d}{ds}(\sigma\rho^2 J_x)$. For the latter, we first take

$$\frac{dJ_x}{ds} = \frac{d}{ds}\begin{vmatrix} y', & z', & v' \\ y'', & z'', & v'' \\ y''', & z''', & v''' \end{vmatrix}$$

$$= \begin{vmatrix} y', & z', & v' \\ y'', & z'', & v'' \\ y^{\text{iv}}, & z^{\text{iv}}, & v^{\text{iv}} \end{vmatrix}$$

$$= -\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)\begin{vmatrix} y', & z', & v' \\ y'', & z'', & v'' \\ y''', & z''', & v''' \end{vmatrix} + \frac{\rho}{\tau}\begin{vmatrix} y', & z', & v' \\ y'', & z'', & v'' \\ J_y, & J_z, & J_v \end{vmatrix}.$$

Now the determinant of the direction-cosines of the four lines PT, PC PB, PF, which constitute an orthogonal frame, is equal to 1. It is easy to verify, immediately, that

$$\begin{matrix} \sigma\rho^2 J_x & , & \sigma\rho^2 J_y & , & \sigma\rho^2 J_z & , & \sigma\rho^2 J_v \\ x' & , & y' & , & z' & , & v' \\ \rho x'' & , & \rho y'' & , & \rho z'' & , & \rho v'' \\ (x' + \rho\rho' x'' + \rho^2 x'''), & & \frac{\sigma}{\rho}(y' + \rho\rho' y'' + \rho^2 y'''), & & \frac{\sigma}{\rho}(z' + \rho\rho' z'' + \rho^2 z'''), & & \frac{\sigma}{\rho}(v' + \rho\rho' v'' + \rho^2 v''') \end{matrix}$$

and therefore, by the properties of the rows of this determinant which represent orthogonal directions taken in pairs,

$$-\frac{\sigma}{\rho}(x' + \rho\rho' x'' + \rho^2 x''') = \begin{vmatrix} \sigma\rho^2 J_y, & \sigma\rho^2 J_z, & \sigma\rho^2 J_v \\ y', & z', & v' \\ \rho y'', & \rho z'', & \rho v'' \end{vmatrix}.$$

Consequently

$$\frac{dJ_x}{ds} = -\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)J_x - \frac{1}{\rho^2\sigma\tau}\frac{\sigma}{\rho}(x' + \rho\rho' x'' + \rho^2 x''');$$

with corresponding expressions for $\frac{dJ_y}{ds}$, $\frac{dJ_z}{ds}$, $\frac{dJ_v}{ds}$.

When this value of $\frac{dJ_x}{ds}$ and the value of x^{iv} are inserted in the expression for $\frac{d\xi_3}{ds}$, and the resulting quantity is arranged as a linear combination of x', x'', x''', and J_x, we find that the coefficient of x' is

$$1 + \frac{d}{ds}\left(\sigma^2 \frac{\rho'}{\rho}\right) + \sigma^2 \rho\rho' \left\{-\frac{1}{\rho^2}\left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho}\right)\right\} - \frac{\tau}{\sigma} R \frac{dR}{d\rho} \cdot \sigma\rho^2 \cdot \frac{1}{\rho^2 \sigma\tau} \frac{\sigma}{\rho},$$

which vanishes: that the coefficient of x'' is

$$2\rho\rho' + \rho\rho' \frac{d}{ds}\left(\sigma^2 \frac{\rho'}{\rho}\right) + \sigma^2 \frac{\rho'}{\rho}(1 + \rho\rho'' + \rho'^2) - \sigma^2 \rho\rho' \left(\frac{1}{\rho^2} + \frac{R}{\sigma^2 \rho} \frac{dR}{d\rho}\right)$$
$$- \frac{\tau}{\sigma} R \frac{dR}{d\rho} \cdot \sigma\rho^2 \cdot \frac{1}{\rho^2 \sigma\tau} \frac{\sigma}{\rho} \rho\rho',$$

which vanishes: that the coefficient of x''' is

$$\rho^2 + \rho^2 \frac{d}{ds}\left(\sigma^2 \frac{\rho'}{\rho}\right) + 3\sigma^2 \rho'^2 - \left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right) \sigma^2 \rho\rho' - \frac{\tau}{\sigma} R \frac{dR}{d\rho} \cdot \sigma\rho^2 \cdot \frac{1}{\rho^2 \sigma\tau} \frac{\sigma}{\rho} \rho^2,$$

which vanishes: and that the coefficient of J_x is

$$\sigma\rho^2 \frac{d}{ds}\left(\frac{\tau}{\sigma} R \frac{dR}{d\rho}\right) + \sigma^2 \rho' \rho \frac{\rho}{\tau} + \frac{\tau}{\sigma} R \frac{dR}{d\rho} \left\{\frac{d}{ds}(\sigma\rho^2) - \sigma\rho^2 \left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)\right\},$$

which is equal to

$$\left\{\frac{d}{ds}\left(\frac{\tau}{\sigma} R \frac{dR}{d\rho}\right) + \frac{\sigma\rho'}{\tau}\right\} \sigma\rho^2.$$

Consequently we have

$$\frac{d\xi_3}{ds} = \left\{\frac{d}{ds}\left(\frac{\tau}{\sigma} R \frac{dR}{d\rho}\right) + \frac{\sigma\rho'}{\tau}\right\} \sigma\rho^2 J_x;$$

and, similarly,

$$\frac{d\eta_3}{ds} = \left\{\frac{d}{ds}\left(\frac{\tau}{\sigma} R \frac{dR}{d\rho}\right) + \frac{\sigma\rho'}{\tau}\right\} \sigma\rho^2 J_y,$$

$$\frac{d\zeta_3}{ds} = \left\{\frac{d}{ds}\left(\frac{\tau}{\sigma} R \frac{dR}{d\rho}\right) + \frac{\sigma\rho'}{\tau}\right\} \sigma\rho^2 J_z,$$

$$\frac{d\upsilon_3}{ds} = \left\{\frac{d}{ds}\left(\frac{\tau}{\sigma} R \frac{dR}{d\rho}\right) + \frac{\sigma\rho'}{\tau}\right\} \sigma\rho^2 J_v$$

We therefore infer, without further calculation, that the arc ds_3, being the distance GG' between two consecutive centres of globular curvature, is given by

$$\frac{ds_3}{ds} = \frac{d}{ds}\left(\frac{\tau}{\sigma} R \frac{dR}{d\rho}\right) + \frac{\sigma\rho'}{\tau}.$$

We also infer that the normal to the osculating flat, drawn through the centre of spherical curvature, is the tangent at G to the locus of the centre of globular curvature: that is, the tangent to the locus of G is parallel to the trinormal.

These results can also be inferred from the geometry of the configuration, as depicted in the figure on p. 211. When regard is paid to the normal $S'G'$

at S' to the consecutive osculating flat, it is clear that GG' is in the line SG which thus, in the limit, is the tangent at G to the locus of G. Also

$$GG' = S'G' - S'G = S'G' - (SG - SM)$$
$$= (S'G' - SG) + SM;$$

but

$$S'G' - SG = \frac{d}{ds}(SG)\,ds = \frac{d}{ds}\left(\frac{\tau}{\sigma} R \frac{dR}{d\rho}\right), \quad SM = \frac{\sigma\rho'}{\tau}\,ds,$$

and therefore

$$ds_3 = GG' = \left\{\frac{d}{ds}\left(\frac{\tau}{\sigma} R \frac{dR}{d\rho}\right) + \frac{\sigma\rho'}{\tau}\right\} ds,$$

in accordance with the foregoing result.

NOTE. A geometrical construction, connected with the magnitude SG, may be noted in passing We have

$$SG = \frac{\tau}{\sigma} R \frac{dR}{d\rho}$$
$$= R \frac{\tau R'}{\sigma\rho'}.$$

Now in the normal plane PCS,

$$CS = \sigma\rho', \quad SP = R;$$

and therefore

$$\tau R' . SP = CS . SG.$$

Along SC from S, measure a distance $S\bar{G}$ equal to SG, and from $\bar{G}$ draw the perpendicular $\bar{G}\bar{P}$ upon SP Then P, C, $\bar{G}$, $\bar{P}$, lie on a circle in the normal plane; and therefore

$$S\bar{P} . SP = SC . S\bar{G} = SC . SG,$$

that is,

$$\tau R' = S\bar{P}.$$

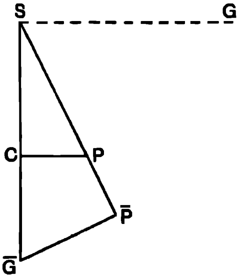

Fig 17.

Direction-cosines of a consecutive trinormal, relative to the frame.

161. We shall require the direction-cosines of $P'F'$, the trinormal at the consecutive point P', this line being parallel to $S'G'$.

In the diagram (p. 261) connected with the direction-cosines of the consecutive binormal, the point F' represents the direction $P'F'$. Now $P'F'$ is perpendicular to the tangent PT, while PT itself is perpendicular to the normal flat at P containing the whole configuration in that diagram; also, $P'F'$ is perpendicular to the principal normal PC, for the line $S'G'$ (to which $P'F'$ is parallel) is perpendicular to that principal normal. The inclination of $P'F'$ to PB is measured by the arc BF' which is $\frac{1}{2}\pi + FF'$, that is, $\frac{1}{2}\pi + d\omega$; and the inclination of $P'F'$ to PF is measured by the arc FF', that is, $d\omega$.

Let λ_4, μ_4, ν_4, κ_4, denote the direction-cosines of $P'F'$ with respect to the principal lines of the orthogonal frame at P. Then

$$\begin{aligned}
\lambda_4 &= 0,\\
\mu_4 &= 0,\\
\nu_4 &= \cos\left(\tfrac{1}{2}\pi + d\omega\right) = -\,d\omega,\\
\kappa_4 &= \cos d\omega = 1.
\end{aligned}$$

These results also can be verified analytically. The direction-cosine of PF with reference to OX is

$$\sigma\rho^2 J_x\,;$$

and therefore the direction-cosine of $P'F'$ with reference to OX is

$$\sigma\rho^2 J_x + \frac{d}{ds}\left(\sigma\rho^2 J_x\right) ds.$$

Now it has been proved that

$$\frac{dJ_x}{ds} = -\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right) J_x - \frac{1}{\rho^2\sigma\tau}\,\frac{\sigma}{\rho}\left(x' + \rho\rho' x'' + \rho^2 x'''\right),$$

and therefore

$$\begin{aligned}
\frac{d}{ds}\left(\rho^2\sigma J_x\right) &= \rho^2\sigma\,\frac{dJ_x}{ds} + \rho^2\sigma\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right) J_x\\
&= -\frac{1}{\tau}\,\frac{\sigma}{\rho}\left(x' + \rho\rho' x'' + \rho^2 x'''\right).
\end{aligned}$$

Consequently, the direction-cosine of $P'F'$ with respect to OX is

$$\sigma\rho^2 J_x - \frac{1}{\tau}\,\frac{\sigma}{\rho}\left(x' + \rho\rho' x'' + \rho^2 x'''\right) ds\,;$$

and similarly its direction-cosines with respect to OY, OZ, OV, are

$$\begin{aligned}
&\sigma\rho^2 J_y - \frac{1}{\tau}\,\frac{\sigma}{\rho}\left(y' + \rho\rho' y'' + \rho^2 y'''\right) ds,\\
&\sigma\rho^2 J_z - \frac{1}{\tau}\,\frac{\sigma}{\rho}\left(z' + \rho\rho' z'' + \rho^2 z'''\right) ds,\\
&\sigma\rho^2 J_v - \frac{1}{\tau}\,\frac{\sigma}{\rho}\left(v' + \rho\rho' v'' + \rho^2 v'''\right) ds.
\end{aligned}$$

Hence

$$\lambda_4 = \Sigma x' \left\{\sigma\rho^2 J_x - \frac{1}{\tau}\,\frac{\sigma}{\rho}\left(x' + \rho\rho' x'' + \rho^2 x'''\right) ds\right\} = 0,$$

$$\nu_4 = \Sigma \frac{\sigma}{\rho}(x' + \rho\rho' x'' + \rho^2 x''') \left\{ \sigma\rho^2 J_x - \frac{1}{\tau}\frac{\sigma}{\rho}(x' + \rho\rho' x'' + \rho^2 x''')\, ds \right\}$$

$$= -\frac{ds}{\tau} = -d\omega,$$

$$\kappa_4 = \Sigma\sigma\rho^2 J_x \left\{ \sigma\rho^2 J_x - \frac{1}{\tau}\frac{\sigma}{\rho}(x' + \rho\rho' x'' + \rho^2 x''')\, ds \right\} = 1,$$

in accordance with the preceding results.

NOTE. The four sets of direction-cosines λ_r, μ_r, ν_r, κ_r, for $r = 1, 2, 3, 4$, have been obtained directly from the configuration, account being taken of small quantities up to the first order only. It is easy to verify that they satisfy the necessary conditions of orthogonality for the new frame which their directions constitute.

Curvature of twist.

162. In § 131, the angle between consecutive principal normals was investigated; it led to a nominal curvature, *the curvature of screw*, which has neither centre, nor any line for radius, in the diagram. Similarly, the angle between consecutive binormals can be investigated; it likewise leads to a nominal curvature, which may be called *the curvature of twist*, and which also has neither centre, nor any line for radius, in the diagram.

Let $d\phi$ be the angle between consecutive binormals, that is, between PB and $P'B'$; it is represented by the arc BB' in the diagram on p. 261. Thus

$$d\phi^2 = BB'^2 = Bm^2 + B'm^2 = d\eta^2 + d\omega^2,$$

and therefore

$$\left(\frac{d\phi}{ds}\right)^2 = \frac{1}{\sigma^2} + \frac{1}{\tau^2},$$

the measure of the curvature of twist, as defined.

This result can be verified analytically. As l_3, m_3, n_3, k_3, are the direction-cosines of PB, the direction-cosines of $P'B'$ are (§ 159)

$$l_3 + \left(\frac{1}{\tau}\sigma\rho^2 J_x - \frac{\rho}{\sigma}x''\right)ds = l_3 + l_3' ds,$$

$$m_3 + \left(\frac{1}{\tau}\sigma\rho^2 J_y - \frac{\rho}{\sigma}y''\right)ds = m_3 + m_3' ds,$$

$$n_3 + \left(\frac{1}{\tau}\sigma\rho^2 J_z - \frac{\rho}{\sigma}z''\right)ds = n_3 + n_3' ds,$$

$$k_3 + \left(\frac{1}{\tau}\sigma\rho^2 J_v - \frac{\rho}{\sigma}v''\right)ds = k_3 + k_3' ds;$$

and then

$$\left(\frac{d\phi}{ds}\right)^2 = \Sigma l_3^2 \Sigma l_3'^2 - (\Sigma l_3 l_3')^2 = \Sigma l_3'^2 = \frac{1}{\tau^2} + \frac{1}{\sigma^2},$$

in accordance with the above measure.

Equation of the globe of curvature.

163. It remains to find the equation of the globe of curvature, whether in a form more explicit than

$$(\bar{x}-\xi_3)^2+(\bar{y}-\eta_3)^2+(\bar{z}-\zeta_3)^2+(\bar{v}-\upsilon_3)^2=\Gamma^2,$$

or in a form connected more closely with the principal lines of the orthogonal frame.

By taking $\bar{x}-\xi_3=\bar{x}-x-(\xi_3-x)$, and by noting the relation

$$\Sigma\,(\xi_3-x)^2=\Gamma^2,$$

the former equation becomes

$$\Sigma\,(\bar{x}-x)^2=2\,\Sigma\,(\bar{x}-x)\,(\xi_3-x).$$

We substitute on the right-hand side the values of ξ_3-x, η_3-y, $\zeta_3=z$, υ_3-v, as given in § 149, and then the equation of the globe of curvature becomes

$$\Sigma\,(\bar{x}-x)^2=2\rho\,\Sigma\,(\bar{x}-x)\,\rho x''+2\sigma\rho'\,\Sigma\,(\bar{x}-x)\left\{\frac{\sigma}{\rho}\,(x'+\rho\rho' x''+\rho^2 x''')\right\}$$
$$+\,2\,\frac{\tau}{\sigma}\,R\,\frac{dR}{d\rho}\,\Sigma\,(\bar{x}-x)\,\sigma\rho^2 J_x,$$

which is another explicit form.

To refer the globe to the principal lines in the orthogonal frame, we can represent any point $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{v}$, in the quadruple space referred to those lines as axes, by taking

(i) a distance t along the tangent,

(ii) a distance n along the principal normal,

(iii) a distance b along the binormal,

and (iv) a distance $\bar{t}$ along the trinormal.

In these coordinates, any point in space is given by

$$\left.\begin{aligned}
\bar{x}-x&=tx'+n\rho x''+b\,\frac{\sigma}{\rho}\,(x'+\rho\rho' x''+\rho^2 x''')+\bar{t}\sigma\rho^2 J_x\\
\bar{y}-y&=ty'+n\rho y''+b\,\frac{\sigma}{\rho}\,(y'+\rho\rho' y''+\rho^2 y''')+\bar{t}\sigma\rho^2 J_y\\
\bar{z}-z&=tz'+n\rho z''+b\,\frac{\sigma}{\rho}\,(z'+\rho\rho' z''+\rho^2 z''')+\bar{t}\sigma\rho^2 J_z\\
\bar{v}-v&=tv'+n\rho v''+b\,\frac{\sigma}{\rho}\,(v'+\rho\rho' v''+\rho^2 v''')+\bar{t}\sigma\rho^2 J_v
\end{aligned}\right\};$$

and the coordinates of the centre of the globe, referred to PT, PC, PB, PF, as axes, are 0, ρ, $\sigma\rho'$, $R\,\dfrac{\tau R'}{\sigma\rho'}$.

Because the coefficients of t, n, b, $\bar{t}$, in the expressions for $\bar{x}-x$, $\bar{y}-y$, $\bar{z}-z$, $\bar{v}-v$, are the direction-cosines of the four orthogonal principal lines referred to OX, OY, OZ, OV, as axes, we have

$$(\bar{x}-x)^2+(\bar{y}-y)^2+(\bar{z}-z)^2+(\bar{v}-v)^2=t^2+n^2+b^2+\bar{t}^2.$$

Also

$$\Sigma\,(\bar{x}-x)\,\rho x''=n,$$

$$\Sigma\,(\bar{x}-x)\,\frac{\sigma}{\rho}\,(x'+\rho\rho' x''+\rho^2 x''')=b,$$

$$\Sigma\,(\bar{x}-x)\,\sigma\rho^2 J_x=\bar{t};$$

and therefore the equation becomes

$$t^2+n^2+b^2+\bar{t}^2=2n\rho+2b\sigma\rho'+2\bar{t}R\frac{\tau R'}{\sigma\rho'},$$

that is,

$$\begin{aligned} t^2+(n-\rho)^2+(b-\sigma\rho')^2+\left(\bar{t}-R\frac{\tau R'}{\sigma\rho'}\right)^2 &= \rho^2+\sigma^2\rho'^2+\left(R\frac{\tau R'}{\sigma\rho'}\right)^2\\ &= R^2+\left(R\frac{\tau R'}{\sigma\rho'}\right)^2\\ &= \Gamma^2, \end{aligned}$$

the equation of the globe of curvature at P, the form to be expected when the globe is referred to the four principal lines of the orthogonal frame at P as axes.

From this form of the equation of the globe of curvature, two inferences can be drawn.

In the first place, the sphere of curvature is the intersection of the globe of curvature by the osculating flat; for that flat is given by the equation

$$\bar{t}=0,$$

and the intersection of the globe in that flat is

$$t^2+(n-\rho)^2+(b-\sigma\rho')^2=\rho^2+\sigma^2\rho'^2=R^2,$$

which is the sphere of curvature in the osculating flat.

In the second place, the circle of plane curvature is the section of the globe of curvature by the osculating plane: or, otherwise stated, the circle of plane curvature is the section of the sphere of curvature by the osculating plane. For the osculating plane is given by the equations

$$b=0,\quad \bar{t}=0;$$

and the intersection of the globe by that plane, being the intersection of the sphere of curvature in the osculating flat made by the osculating plane in that flat, is

$$t^2+(n-\rho)^2=\rho^2,$$

which is the circle of plane curvature in the osculating plane.

Frenet formulæ in quadruple space.

164. In the preceding discussion of the curve, most of the investigations have been concerned with the various amplitudes associated with the curve at, or in the immediate vicinity of, a point P. Special attention has been devoted to the orthogonal frame, particularly to its four chief lines (the tangent, the principal normal, the binormal, and the trinormal), to three of its chief planes (the osculating plane, the normal plane, and the orthogonal plane), and to two of its chief flats (the osculating flat and the normal flat). Throughout, the direction-cosines of the chief lines have been of significant importance. We now proceed to exhibit the relation of these direction-cosines of successive sets of chief lines by means of equations, which are a manifest extension of the Serret-Frenet formulæ for twisted curves in triple space.

For convenience of notation, we denote the direction-cosines of the tangent by l_1, m_1, n_1, k_1, these being respectively equal to x', y', z', v'; the direction-cosines of the principal normal by l_2, m_2, n_2, k_2, these being respectively equal to $\rho x'', \rho y'', \rho z'', \rho v''$; the direction-cosines of the binormal by l_3, m_3, n_3, k_3, these being respectively equal to

$$\frac{\sigma}{\rho}(x'+\rho\rho' x''+\rho^2 x'''),\quad \frac{\sigma}{\rho}(y'+\rho\rho' y''+\rho^2 y'''),$$

$$\frac{\sigma}{\rho}(z'+\rho\rho' z''+\rho^2 z'''),\quad \frac{\sigma}{\rho}(v'+\rho\rho' v''+\rho^2 v''');$$

and the direction-cosines of the trinormal by l_4, m_4, n_4, k_4, these being respectively equal to $\sigma\rho^2 J_x, \sigma\rho^2 J_y, \sigma\rho^2 J_z, \sigma\rho^2 J_v$. Also, it will be found convenient to denote any one of a set of direction-cosines l, m, n, k, by c; thus c_1, c_2, c_3, c_4, can stand indifferently for l_1, l_2, l_3, l_4; or for m_1, m_2, m_3, m_4; or for n_1, n_2, n_3, n_4; or for k_1, k_2, k_3, k_4.

The direction-cosines of the principal lines at a point P of a curve are shewn in the tableau

$$\left(\begin{array}{cccc} l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \\ l_4, & m_4, & n_4, & k_4 \end{array}\right).$$

When the principal lines at a point P', consecutive to P on the curve, are referred to the principal lines at P, they have direction-cosines

$\lambda_1, \mu_1, \nu_1, \kappa_1, = \ \ 1\ ,\ d\epsilon\ ,\ 0\ ,\ 0\ ,$ for the tangent (§ 155),

$\lambda_2, \mu_2, \nu_2, \kappa_2, = -d\epsilon,\ 1\ ,\ d\eta\ ,\ 0\ ,$ for the principal normal (§ 157),

$\lambda_3, \mu_3, \nu_3, \kappa_3, = \ \ 0\ ,\ -d\eta,\ 1\ ,\ d\omega,$ for the binormal (§ 159),

$\lambda_4, \mu_4, \nu_4, \kappa_4, = \ \ 0\ ,\ 0\ ,\ -d\omega,\ 1\ ,$ for the trinormal (§ 161).

Then for the four direction-cosines $c_1 + dc_1$, $c_2 + dc_2$, $c_3 + dc_3$, $c_4 + dc_4$, for the four lines at Q, corresponding to the four direction-cosines c_1, c_2, c_3, c_4, for the four lines at P, the transformation-law of coordinates gives

$$\begin{array}{lllll}
c_1 + dc_1 = (c_1, c_2, c_3, c_4 \between\ 1, & d\epsilon, & 0, & 0) = & c_1 + c_2\, d\epsilon, \\
c_2 + dc_2 = (c_1, c_2, c_3, c_4 \between - d\epsilon, & 1, & d\eta, & 0) = & - c_1\, d\epsilon + c_2 + c_3\, d\eta, \\
c_3 + dc_3 = (c_1, c_2, c_3, c_4 \between\ 0, & - d\eta, & 1, & d\omega) = & - c_2\, d\eta + c_3 + c_4\, d\omega, \\
c_4 + dc_4 = (c_1, c_2, c_3, c_4 \between\ 0, & 0, & - d\omega, & 1) = & - c_3\, d\omega + c_4.
\end{array}$$

(The four points x, y, z, v, in the diagram on p. 197 may be taken to represent the principal lines at P; and the four points x', y', z', v', in that diagram represent the principal lines at P' after the most general displacement in space. The equations, giving the relations for the actual displacement of the frame of the curve under consideration, are the sums of the expressions of the projections of the coordinates along the respective principal lines at P' upon the axes OX, OY, OZ, OV.) Now

$$\frac{d\epsilon}{ds} = \frac{1}{\rho}, \quad \frac{d\eta}{ds} = \frac{1}{\sigma}, \quad \frac{d\omega}{ds} = \frac{1}{\tau};$$

hence the foregoing relations become

$$\left.\begin{array}{llll}
\dfrac{dc_1}{ds} = & & \dfrac{c_2}{\rho} & \\[2ex]
\dfrac{dc_2}{ds} = - \dfrac{c_1}{\rho} & & & + \dfrac{c_3}{\sigma} \\[2ex]
\dfrac{dc_3}{ds} = & & - \dfrac{c_2}{\sigma} & + \dfrac{c_4}{\tau} \\[2ex]
\dfrac{dc_4}{ds} = & & & - \dfrac{c_3}{\tau}
\end{array}\right\},$$

which are the extension, to quadruple space, of the Serret-Frenet formulæ for triple space*.

We shall call these relations the Frenet formulæ.

Centres, and radii, of circular curvature, spherical curvature, globular curvature.

165. The Frenet formulæ can be used, in connection with the explanations already given, to obtain the coordinates of the various centres, and the various radii, of the circular, the spherical, and the globular curvatures.

The various centres arise in connection with the intersection of successive normal flats at consecutive points on the curve.

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* Serret, *Liouville's Journal*, t. xvi (1851), p. 193; Frenet, *ib.*, t. xvii (1852), p. 437.

The normal flat at the point O is given by the equation

$$\Sigma(\bar{x}-x)\,x'=0,$$

that is, by

$$\Sigma(\bar{x}-x)\,l_1=0.$$

The intersection of this flat by the consecutive normal flat is the plane, the two equations of which can be taken

$$\Sigma(\bar{x}-x)\,l_1=0,\quad \Sigma(\bar{x}-x)\,l_1'-\Sigma x'l_1=0.$$

The latter equation, by the use of Frenet's equations, is

$$\Sigma(\bar{x}-x)\,l_2=\rho\Sigma x'l_1=\rho\Sigma l_1^2=\rho\,;$$

or the two equations of the plane are, together,

$$\Sigma(\bar{x}-x)\,l_1=0,\quad \Sigma(\bar{x}-x)\,l_2=\rho.$$

The intersection of this plane by another consecutive normal flat—that is, the intersection of three consecutive normal flats—is the line, the three equations of which are obtained by associating, with the two preceding equations, the further equation

$$\Sigma(\bar{x}-x)\,l_2'-\Sigma x'l_2=\rho'.$$

Now

$$\Sigma x'l_2=\Sigma l_1 l_2=0,$$

and

$$\Sigma(\bar{x}-x)\,l_2'=-\frac{1}{\rho}\Sigma(\bar{x}-x)\,l_1+\frac{1}{\sigma}\Sigma(\bar{x}-x)\,l_3,$$

and, in the equations of the line, we have $\Sigma(\bar{x}-x)\,l_1=0$. Hence the third equation is

$$\Sigma(\bar{x}-x)\,l_3=\sigma\rho'\,;$$

and therefore the three equations of the line, that is, the intersection of three consecutive normal flats, are

$$\Sigma(\bar{x}-x)\,l_1=0,\quad \Sigma(\bar{x}-x)\,l_2=\rho,\quad \Sigma(\bar{x}-x)\,l_3=\sigma\rho'.$$

The intersection of this line by a fourth consecutive normal flat, being the centre of globular curvature arising as the point of intersection of four consecutive normal flats, is given by associating, with the three equations of the line, the further equation

$$\Sigma(\bar{x}-x)\,l_3'-\Sigma x'l_3=\frac{d}{ds}(\sigma\rho').$$

Now

$$\Sigma x'l_3=\Sigma l_1 l_3=0\,;$$

and

$$\Sigma(\bar{x}-x)\,l_3'=-\frac{1}{\sigma}\Sigma(\bar{x}-x)\,l_2+\frac{1}{\tau}\Sigma(\bar{x}-x)\,l_4,$$

and, for these intersections of consecutive flats, $\Sigma(\bar{x}-x)l_2=\rho$; hence the fourth equation becomes

$$\Sigma(\bar{x}-x)l_4=\tau\left\{\frac{\rho}{\sigma}+\frac{d}{ds}(\sigma\rho')\right\}$$

$$=\frac{\tau}{\sigma\rho'}\left\{\rho\rho'+\sigma\rho'\frac{d}{ds}(\sigma\rho')\right\}.$$

But, because $R^2=\rho^2+\sigma^2\rho'^2$, we have

$$RR'=\rho\rho'+\sigma\rho'\frac{d}{ds}(\sigma\rho');$$

and therefore the point of intersection of four consecutive normal flats is given by the four equations

$$\Sigma(\bar{x}-x)l_1=0,\quad \Sigma(\bar{x}-x)l_2=\rho,\quad \Sigma(\bar{x}-x)l_3=\sigma\rho',\quad \Sigma(\bar{x}-x)l_4=R\frac{\tau R'}{\sigma\rho'}.$$

The various centres of curvature, as regards their position, can be described as follows.

The centre of circular curvature is the intersection of two consecutive normal flats with the osculating plane of the curve, which is given by the equations

$$\left\|\begin{matrix}\bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v\\ x', & y', & z', & v'\\ x'', & y'', & z'', & v''\end{matrix}\right\|,$$

or by the equivalent equations

$$\left\|\begin{matrix}\bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v\\ l_1, & m_1, & n_1, & k_1\\ l_2, & m_2, & n_2, & k_2\end{matrix}\right\|=0.$$

Any point in this plane is given by equations

$$\bar{x}-x=l_1\Lambda_1+l_2\Lambda_2,$$

with three like equations, where Λ_1 and Λ_2 are parameters of the plane; and the particular point, which is the centre of curvature, is obtained by choosing Λ_1 and Λ_2 to satisfy the equations of the two consecutive normal flats

$$\Sigma(\bar{x}-x)l_1=0,\quad \Sigma(\bar{x}-x)l_2=\rho.$$

Thus

$$\Sigma l_1(l_1\Lambda_1+l_2\Lambda_2)=0,\quad \Sigma l_2(l_1\Lambda_1+l_2\Lambda_2)=\rho,$$

that is,

$$\Lambda_1=0,\quad \Lambda_2=\rho;$$

and therefore the coordinates of the centre of circular curvature are given by

$$\bar{x}-x=\rho l_2,\quad \bar{y}-y=\rho m_2,\quad \bar{z}-z=\rho n_2,\quad \bar{v}-v=\rho k_2.$$

The radius of circular curvature is

$$=\{\Sigma(\bar{x}-x)^2\}^{\frac{1}{2}}=\rho,$$

as is to be expected.

The centre of spherical curvature is the point where the line, which is the intersection of three consecutive normal flats, meets the osculating flat. The equation of this flat is

$$\begin{vmatrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ x' \;, & y' \;, & z' \;, & v' \\ x'' \;, & y'' \;, & z'' \;, & v'' \\ x''' \;, & y''' \;, & z''' \;, & v''' \end{vmatrix}=0;$$

or, as

$$l_3=\frac{\sigma}{\rho}(x'+\rho\rho'x''+\rho^2x'''),$$

with like expressions for m_3, n_3, k_3, the equation of the osculating flat can be written

$$\begin{vmatrix} \bar{x}-x, & y-y, & \bar{z}-z, & \bar{v}-v \\ l_1 \;, & m_1 \;, & n_1 \;. & k_1 \\ l_2 \;, & m_2 \;, & n_2 \;, & k_2 \\ l_3 \;, & m_3 \;, & n_3 \;, & k_3 \end{vmatrix}=0.$$

The coordinates of any point in this flat can be taken in the form

$$\bar{x}-x=l_1\Lambda_1+l_2\Lambda_2+l_3\Lambda_3,$$

with like expressions for $\bar{y}-y$, $\bar{z}-z$, $\bar{v}-v$; and the centre of spherical curvature is the point among this aggregate the coordinates of which satisfy the three equations

$$\Sigma(\bar{x}-x)\,l_1=0,\quad \Sigma(\bar{x}-x)\,l_2=\rho,\quad \Sigma(\bar{x}-x)\,l_3=\sigma\rho'.$$

Hence we have

$$\Sigma l_1(l_1\Lambda_1+l_2\Lambda_2+l_3\Lambda_3)=0,$$

$$\Sigma l_2(l_1\Lambda_1+l_2\Lambda_2+l_3\Lambda_3)=\rho,$$

$$\Sigma l_3(l_1\Lambda_1+l_2\Lambda_2+l_3\Lambda_3)=\sigma\rho',$$

that is,

$$\Lambda_1=0,\quad \Lambda_2=\rho,\quad \Lambda_3=\sigma\rho',$$

and therefore the coordinates of the centre of spherical curvature are given by

$$\bar{x}-x=\rho l_2+\sigma\rho' l_3,\quad \bar{y}-y=\rho m_2+\sigma\rho' m_3,$$

$$\bar{z}-z=\rho n_2+\sigma\rho' n_3,\quad \bar{v}-v=\rho k_2+\sigma\rho' k_3.$$

The radius of spherical curvature, R, is such that

$$\begin{aligned} R^2 &= \Sigma(\bar{x}-x)^2 \\ &= \Sigma(\rho l_2 + \sigma\rho' l_3)^2 \\ &= \rho^2 + \sigma^2\rho'^2. \end{aligned}$$

The centre of globular curvature has been proved to be the point of intersection of four consecutive normal flats

$$\Sigma(\bar{x}-x)\,l_1 = 0, \quad \Sigma(\bar{x}-x)\,l_2 = \rho, \quad \Sigma(\bar{x}-x)\,l_3 = \sigma\rho', \quad \Sigma(\bar{x}-x)\,l_4 = R\frac{\tau R'}{\sigma\rho'}.$$

Now any point in space, referred to the orthogonal frame of the curve, can be represented by expressions

$$\bar{x} - x = l_1\Lambda_1 + l_2\Lambda_2 + l_3\Lambda_3 + l_4\Lambda_4,$$

with like expressions for $\bar{y}-y$, $\bar{z}-z$, $\bar{v}-v$. Hence, for the centre of globular curvature, we have

$$\begin{aligned} \Sigma l_1(l_1\Lambda_1 + l_2\Lambda_2 + l_3\Lambda_3 + l_4\Lambda_4) &= 0, \\ \Sigma l_2(l_1\Lambda_1 + l_2\Lambda_2 + l_3\Lambda_3 + l_4\Lambda_4) &= \rho, \\ \Sigma l_3(l_1\Lambda_1 + l_2\Lambda_2 + l_3\Lambda_3 + l_4\Lambda_4) &= \sigma\rho', \\ \Sigma l_4(l_1\Lambda_1 + l_2\Lambda_2 + l_3\Lambda_3 + l_4\Lambda_4) &= R\frac{\tau R'}{\sigma\rho'}: \end{aligned}$$

that is,

$$\Lambda_1 = 0, \quad \Lambda_2 = \rho, \quad \Lambda_3 = \sigma\rho', \quad \Lambda_4 = R\frac{\tau R'}{\sigma\rho'}.$$

Thus the coordinates of the centre of globular curvature are given by

$$\bar{x} - x = \rho l_2 + \sigma\rho' l_3 + R\frac{\tau R'}{\sigma\rho'}\,l_4,$$

with like expressions for $\bar{y}-y$, $\bar{z}-z$, $\bar{v}-v$.

The radius of globular curvature, Γ, is such that

$$\begin{aligned} \Gamma^2 &= \Sigma(\bar{x}-x)^2 \\ &= \Sigma\left(\rho l_2 + \sigma\rho' l_3 + R\frac{\tau R'}{\sigma\rho'}\,l_4\right)^2 \\ &= \rho^2 + \sigma^2\rho'^2 + \left(R\frac{\tau R'}{\sigma\rho'}\right)^2 \\ &= R^2\left\{1 + \left(\frac{\tau R'}{\sigma\rho'}\right)^2\right\}, \end{aligned}$$

in accordance with the result already (§ 147) obtained.

Infinitesimal displacement of the orthogonal frame.

166. The preceding equations for the variations of sets of direction-cosines, each set being constituted by the inclination-cosines for the four principal lines of the curve referred to one and the same external axis of reference (to OX for $c=l$, to OY for $c=m$, to OZ for $c=n$, and to OV for $c=k$), can be taken in the form

$$\left.\begin{aligned} x' &= x + y\,d\epsilon \\ y' &= -x\,d\epsilon + y + z\,d\eta \\ z' &= -y\,d\eta + z + v\,d\omega \\ v' &= -z\,d\omega + v \end{aligned}\right\}.$$

Thus an infinitesimal transformation (§ 123) is constituted. We know (§ 125) that this transformation can be effected by a small rotation α' round the plane

$$\left.\begin{aligned} y\cos\beta' + v\sin\beta' &= 0 \\ -x\sin\beta + z\cos\beta &= 0 \end{aligned}\right\},$$

and an independent small rotation α round the plane

$$\left.\begin{aligned} x\cos\beta + z\sin\beta &= 0 \\ -y\sin\beta' + v\cos\beta' &= 0 \end{aligned}\right\},$$

where

$$\tan 2\beta = \frac{2\,d\epsilon\,d\eta}{d\omega^2 + d\eta^2 - d\epsilon^2}, \quad \tan 2\beta' = \frac{2\,d\omega\,d\eta}{d\omega^2 - d\eta^2 - d\epsilon^2},$$

$$\left.\begin{aligned} 2\alpha' &= (d\epsilon^2 + d\eta^2 + d\omega^2 + 2d\epsilon\,d\omega)^{\frac{1}{2}} + (d\epsilon^2 + d\eta^2 + d\omega^2 - 2d\epsilon\,d\omega)^{\frac{1}{2}} \\ 2\alpha &= (d\epsilon^2 + d\eta^2 + d\omega^2 + 2d\epsilon\,d\omega)^{\frac{1}{2}} - (d\epsilon^2 + d\eta^2 + d\omega^2 - 2d\epsilon\,d\omega)^{\frac{1}{2}} \end{aligned}\right\}.$$

It should be noted that, when $d\omega = 0$, so that there is no tilt and the curve therefore is a three-dimensional curve (§ 146), we have

$$\tan\beta = \frac{d\epsilon}{d\eta} = \frac{\sigma}{\rho}, \quad \beta' = 0, \quad \alpha = 0.$$

We can take $v=0$ as the flat in which the curve lies: and then the infinitesimal transformation is effected in that flat by a rotation round the line $z = x\dfrac{\sigma}{\rho}$ in the plane $y=0$, that is, is effected round the rectifying line of the skew three-dimensional curve.

Fourth-order derivatives of point-coordinates.

167. It is to be noted that, as the values of λ_r, μ_r, ν_r, κ_r, (for $r = 1, 2, 3, 4$) were obtained directly from the geometry of the moving orthogonal frame, they can be regarded as having been established independently of the analysis; and thus they can be used to obtain the analytical expressions for l_r, m_r, n_r, k_r, (for $r = 1, 2, 3, 4$). For example, take c to denote l; then

$$c_1 = l_1 = x';$$

hence

$$l_2 = c_2 = \rho \frac{dc_1}{ds} = \rho x'',$$

$$l_3 = c_3 = \sigma \left(\frac{dc_2}{ds} + \frac{c_1}{\rho} \right) = \frac{\sigma}{\rho} (x' + \rho\rho' x'' + \rho^2 x''').$$

Also, we have

$$c_4 = l_4 = \sigma\rho^2 J_x;$$

and thus the fourth equation gives

$$\frac{d}{ds} (\sigma\rho^2 J_x) = -\frac{1}{\tau} \frac{\sigma}{\rho} (x' + \rho\rho' x'' + \rho^2 x''').$$

The third equation gives

$$\begin{aligned} \frac{l_4}{\tau} &= \frac{l_2}{\sigma} + \frac{dl_3}{ds} \\ &= \frac{\rho}{\sigma} x'' + \frac{d}{ds} \left\{ \frac{\sigma}{\rho} (x' + \rho\rho' x'' + \rho^2 x''') \right\} \\ &= \frac{\sigma}{\rho'} \left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho} \right) x' + \left\{ \frac{\rho}{\sigma} + \frac{\sigma}{\rho} + \frac{d}{ds} (\sigma\rho') \right\} x'' + (2\rho'\sigma + \rho\sigma') x''' + \rho\sigma x^{\text{iv}}, \end{aligned}$$

leading to the equation

$$x^{\text{iv}} = \frac{\rho}{\tau} J_x - \frac{1}{\rho^2} \left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho} \right) x' - \left\{ \frac{1}{\rho^2} + \frac{1}{\sigma^2} + \frac{1}{\rho\sigma} \frac{d}{ds} (\sigma\rho') \right\} x'' - \left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma} \right) x''',$$

together with the corresponding expressions for y^{iv}, z^{iv}, v^{iv}.

All these relations are in accordance with results already (§ 151) established.

Fifth-order derivatives of point-coordinates.

168. The last expression, giving the fourth derivative of a coordinate in terms of the first three derivatives and of the intrinsic magnitudes of the curve which are invariantive for all axes of reference, will now be modified so as to make that fourth derivative linearly expressible in terms of the direction-cosines of the chief lines at the point. The Frenet formulæ can then be used to express all higher derivatives of the variables, which are the point-coordinates referred to the initial general system of axes in quadruple space

In the preceding expression for x^{iv}, we use the relation

$$x''' = \frac{1}{\rho\sigma} \left[\frac{\sigma}{\rho} (x' + \rho\rho' x'' + \rho^2 x''') - \sigma\rho' x'' - \frac{\sigma}{\rho} x' \right]$$

to remove the term in x'''; and we easily find

$$\begin{aligned}x^{\text{iv}} = \frac{1}{\rho\sigma\tau}\sigma\rho^2 J_x + 3\frac{\rho'}{\rho^3}x' + \left(-\frac{\rho''}{\rho} + 2\frac{\rho'^2}{\rho^2} - \frac{1}{\rho^2} - \frac{1}{\sigma^2}\right)x'' \\ - \frac{1}{\rho\sigma}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)\frac{\sigma}{\rho}(x' + \rho\rho' x'' + \rho^2 x''')\end{aligned}$$

$$= \frac{1}{\rho\sigma\tau} l_4 - \frac{1}{\rho\sigma}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right) l_3 + M\rho l_2 + 3\frac{\rho'}{\rho^3} l_1$$

and, similarly,

$$\left.\begin{aligned} y^{\text{iv}} &= \frac{1}{\rho\sigma\tau} m_4 - \frac{1}{\rho\sigma}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right) m_3 + M\rho m_2 + 3\frac{\rho'}{\rho^3} m_1 \\ z^{\text{iv}} &= \frac{1}{\rho\sigma\tau} n_4 - \frac{1}{\rho\sigma}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right) n_3 + M\rho n_2 + 3\frac{\rho'}{\rho^3} n_1 \\ v^{\text{iv}} &= \frac{1}{\rho\sigma\tau} k_4 - \frac{1}{\rho\sigma}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right) k_3 + M\rho k_2 + 3\frac{\rho'}{\rho^3} k_1 \end{aligned}\right\}.$$

In these expressions, the value of M is given by

$$M = -\frac{\rho''}{\rho^3} + 2\frac{\rho'^2}{\rho^4} - \frac{1}{\rho^4} - \frac{1}{\rho^2\sigma^2},$$

it can also be expressed in the subsequently useful form

$$M = 2\frac{\rho'^2}{\rho^4} + \frac{\rho'\sigma'}{\rho^3\sigma} - \frac{1}{\rho^4} - \frac{R}{\sigma^2\rho^3}\frac{dR}{d\rho}.$$

When obtaining a measure of the closeness of contact of the curve with its globe of curvature at any point, we shall require values of x^{v}, y^{v}, z^{v}, v^{v}. These can be derived at once from the preceding result; and, as the convenient form is that which expresses them linearly in terms of the direction-cosines of the chief lines, the Frenet formulæ are used for the construction of this form. We have

$$\begin{aligned} x^{\text{v}} = {} & l_4\frac{d}{ds}\left(\frac{1}{\rho\sigma\tau}\right) + \frac{1}{\rho\sigma\tau}\frac{dl_4}{ds} \\ & - l_3\frac{d}{ds}\left\{\frac{1}{\rho\sigma}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)\right\} - \frac{1}{\rho\sigma}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)\frac{dl_3}{ds} \\ & + l_2\frac{d}{ds}(M\rho) + M\rho\frac{dl_2}{ds} \\ & + l_1\frac{d}{ds}\left(3\frac{\rho'}{\rho^3}\right) - 3\frac{\rho'}{\rho^3}\frac{dl_1}{ds}. \end{aligned}$$

When the values of $\frac{dl_4}{ds}, \frac{dl_3}{ds}, \frac{dl_2}{ds}, \frac{dl_1}{ds}$, as given by the Frenet equations (on taking $c = l$) are substituted, and the coefficients of l_1, l_2, l_3, l_4, are collected, we have, in all,

$$\left.\begin{aligned} x^{\text{v}} &= \Delta l_4 + W l_3 + V l_2 + U l_1 \\ y^{\text{v}} &= \Delta m_4 + W m_3 + V m_2 + U m_1 \\ z^{\text{v}} &= \Delta n_4 + W n_3 + V n_2 + U n_1 \\ v^{\text{v}} &= \Delta k_4 + W k_3 + V k_2 + U k_1 \end{aligned}\right\},$$

where

$$\left.\begin{aligned} \Delta &= -\frac{1}{\rho\sigma\tau}\left(3\frac{\rho'}{\rho} + 2\frac{\sigma'}{\sigma} + \frac{\tau'}{\tau}\right) \\ W &= M\frac{\rho}{\sigma} - \frac{1}{\rho\sigma\tau^2} - \frac{d}{ds}\left\{\frac{1}{\rho\sigma}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)\right\} \\ V &= 3\frac{\rho'}{\rho^4} + \frac{1}{\rho\sigma^2}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right) + \frac{d}{ds}(M\rho) \\ U &= -M + \frac{d}{ds}\left(3\frac{\rho'}{\rho^3}\right) \end{aligned}\right\}.$$

These are the required values of x^{v}, y^{v}, z^{v}, v^{v}.

The derivatives of successive orders can be obtained in the same way, each expressed linearly in terms of the direction-cosines of the principal lines.

A curve is determinate, save as to orientation and position, by assigned curvatures.

169. By means of the Frenet formulæ, we establish the theorem that *a curve in quadruple space is determinate, save as to orientation and position, by the assignment of its circular curvature, its curvature of torsion* (*or its spherical curvature*), *and its curvature of tilt* (*or its globular curvature*).

As data, the spherical curvature is equivalent to the torsion and the globular curvature to the tilt, through the relations

$$R^2 = \rho^2 + \sigma^2\rho'^2, \quad \Gamma^2 = R^2 + R^2\left(\frac{\tau R'}{\sigma\rho'}\right)^2.$$

From the Frenet equations, we have

$$l_2 = \rho\frac{dl_1}{ds}, \quad l_3 = \rho\sigma\frac{d^2 l_1}{ds^2} + \sigma\rho'\frac{dl_1}{ds} + \frac{\sigma}{\rho}l_1,$$

$$l_4 = \rho\sigma\tau\left\{\frac{d^3 l_1}{ds^3} + \left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)\frac{d^2 l_1}{ds^2}\right\} + \tau\left\{\frac{\rho}{\sigma} + \frac{\sigma}{\rho} + \frac{d}{ds}(\sigma\rho')\right\}\frac{dl_1}{ds} + \tau\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)l_1.$$

Then l_1 satisfies an ordinary linear differential equation of the fourth order with variable coefficients, obtained by substituting these values of l_4 and l_3 in the fourth Frenet equation

$$\frac{dl_4}{ds} = -\frac{l_3}{\tau}.$$

The primitive of this linear differential equation, of order four, is of the form

$$l_1 = a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4,$$

where a_1, a_2, a_3, a_4, are arbitrary constants. The values of these constants are uniquely determinable, and so the primitive is made particular to l_1, by the assignment of values of l_1, $\frac{dl_1}{ds}$, $\frac{d^2l_1}{ds^2}$, $\frac{d^3l_1}{ds^3}$, for any initial value of s: that is, by the assignment of values of l_1, l_2, l_3, l_4, for that initial value of s and therefore at an initial point O. The current values of l_2, l_3, l_4, are then derived from this current value of l_1 by means of the Frenet equations.

The same linear equation of the fourth order is satisfied by m_1, n_1, k_1. By corresponding assignments of initial values in each instance, the current values of m_1, n_1, k_1, are determined, and the values of the quantities m_r, n_r, k_r, for $r = 2$, 3, 4, are similarly derived. It is, of course, necessary that the whole set of assigned values at O should there conform to the conditions of orthogonality.

We must prove that these current values of l_r, m_r, n_r, k_r, representing initially an orthogonal system, and satisfying the Frenet equations, continue to represent an orthogonal system. Let α and β represent any two of the quantities l, m, n, k. Then

$$\frac{d}{ds}\left(\sum_r \alpha_r^2\right) = 2\left(\alpha_1\frac{d\alpha_1}{ds} + \alpha_2\frac{d\alpha_2}{ds} + \alpha_3\frac{d\alpha_3}{ds} + \alpha_4\frac{d\alpha_4}{ds}\right) = 0,$$

$$\begin{aligned}\frac{d}{ds}\left(\sum_r \alpha_r\beta_r\right) = \; & \alpha_1\frac{d\beta_1}{ds} + \alpha_2\frac{d\beta_2}{ds} + \alpha_3\frac{d\beta_3}{ds} + \alpha_4\frac{d\beta_4}{ds} \\ & + \beta_1\frac{d\alpha_1}{ds} + \beta_2\frac{d\alpha_2}{ds} + \beta_3\frac{d\alpha_3}{ds} + \beta_4\frac{d\alpha_4}{ds} = 0,\end{aligned}$$

in both instances, by substitution from the Frenet equations. Hence

$$\Sigma\alpha_r^2 = \text{constant} = 1, \quad \Sigma\alpha_r\beta_r = \text{constant} = 0,$$

the respective constant values being the initial values at O. Hence the current system of values constitutes an orthogonal system.

If now we define coordinates x, y, z, v, by the relations

$$dx = l_1 ds, \quad dy = m_1 ds, \quad dz = n_1 ds, \quad dv = k_1 ds,$$

so that $dx^2 + dy^2 + dz^2 + dv^2 = ds^2$, the variable s of the Frenet equations becomes an arc in the quadruple space. We thus have

$$x - x_0 = \int_{s_0}^{s} l_1 ds = \Sigma\, a_r \int_{s_0}^{s} \lambda_r ds,$$

where x_0 is the value of x at the initial point. We have similar values for $y - y_0$, $z - z_0$, $v - v_0$, all these quantities being expressed as functions of an arc s. Consequently, one curve certainly exists, determined by the assigned curvatures.

Next consider two curves, thus determined, but having different orientations and positions in the quadruple space. On the second curve, let an initial point $\overline{O}$ be selected corresponding to the initial point O on the first curve; and let the two curves be brought together so that (i) the point $\overline{O}$ coincides with the point O and (ii) the orthogonal frame of the second curve at $\overline{O}$ coincides with the orthogonal frame of the first curve at O Let $\overline{l}_1, \overline{l}_2, \overline{l}_3, \overline{l}_4$, be the four quantities for the second curve, corresponding to l_1, l_2, l_3, l_4, for the first curve. Then as the magnitudes ρ, σ, τ, are the same for the two curves, the equation of the fourth order satisfied by $\overline{l}_1$ is formally the same as that satisfied by l_1; and therefore its primitive is

$$\overline{l}_1 = \overline{a}_1\lambda_1 + \overline{a}_2\lambda_2 + \overline{a}_3\lambda_3 + \overline{a}_4\lambda_4,$$

where $\overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}_4$, are arbitrary constants so far as that linear differential equation is concerned. The values of these constants are uniquely determinable, and so this primitive is made particular to $\overline{l}_1$, by the assignment of values of $\overline{l}_1, \overline{l}_2, \overline{l}_3, \overline{l}_4$, at the initial point $\overline{O}$ which coincides with O. But at that initial point, the orthogonal frames of the two curves coincide, so that $\overline{l}_1 = l_1$, $\overline{l}_2 = l_2$, $\overline{l}_3 = l_3$, $\overline{l}_4 = l_4$, at O; consequently $\overline{a}_1 = a_1$, $\overline{a}_2 = a_2$, $\overline{a}_3 = a_3$, $\overline{a}_4 = a_4$, and therefore

$$\overline{l}_1 = l_1,$$

throughout the range. Similarly, we have

$$\overline{m}_1 = m_1, \quad \overline{n}_1 = n_1, \quad \overline{k}_1 = k_1,$$

throughout the range.

If then $\overline{x}, \overline{y}, \overline{z}, \overline{v}$, be the coordinates of a point $\overline{P}$ on the second curve at the same arc-distance as a point P, x, y, z, v, on the first curve, we have

$$\frac{d\overline{x}}{ds} = \frac{dx}{ds}, \quad \frac{d\overline{y}}{ds} = \frac{dy}{ds}, \quad \frac{d\overline{z}}{ds} = \frac{dz}{ds}, \quad \frac{d\overline{v}}{ds} = \frac{dv}{ds}.$$

Hence $\overline{x} - x$, $\overline{y} - y$, $\overline{z} - z$, $\overline{v} - v$, are constants along the two curves. But $\overline{O}$ and O have been made to coincide, owing to the displacement of the second curve; and, in that coincident position, each of these constants is zero Hence

$$\overline{x} = x, \quad \overline{y} = y, \quad \overline{z} = z, \quad \overline{v} = v,$$

everywhere along the two curves. Thus the second curve coincides with the first when it has been suitably displaced, in position and in orientation; and therefore, in its undisplaced position, it is the same as the first curve.

Hence the two curves are the same, except as to position and orientation; and the theorem is established.

Curves having their curvature, torsion, and tilt, constant.

170. It is known that every plane curve of constant curvature is a circle: and that every curve in homaloidal three-dimensional space, having its circular curvature and its torsion each constant, is a cylindrical helix. We proceed to obtain the curve in homaloidal quadruple space which has its circular curvature, its torsion, and its tilt, each constant. Such a curve is uniquely determinate except as to position and orientation.

For the present purpose, the quantities ρ, σ, τ, in the Frenet equations for the direction-cosines of a curve

$$\frac{dc_1}{ds} = \frac{1}{\rho} c_2, \quad \frac{dc_2}{ds} = -\frac{1}{\rho} c_1 + \frac{1}{\sigma} c_3, \quad \frac{dc_3}{ds} = -\frac{1}{\sigma} c_2 + \frac{1}{\tau} c_4, \quad \frac{dc_4}{ds} = -\frac{1}{\tau} c_3,$$

($c = l, m, n, k$, in turn) are constant; and therefore each of the quantities c satisfies the equation

$$\frac{d^4 c}{ds^2} + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2} + \frac{1}{\tau^2}\right) \frac{d^2 c}{ds^2} + \frac{1}{\rho^2 \tau^2} = 0.$$

Then

$$c_1 = A \cos \mu_1 s + B \sin \mu_1 s + C \cos \mu_2 s + D \sin \mu_2 s,$$

where μ_1, $-\mu_1$, μ_2, $-\mu_2$, are the (necessarily real) roots of the equation

$$\mu^4 - \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2} + \frac{1}{\tau^2}\right) \mu^2 + \frac{1}{\rho^2 \tau^2} = 0;$$

and at the moment A, B, C, D, are arbitrary constants in the primitive of the equation, which have to be determined by assigned conditions.

We shall take $c_1 = l_1$, m_1, n_1, k_1; that is, $= x'$, y', z', v'; in turn. The constants will be determined by the initial values of c_1, $\frac{dc_1}{ds}$, $\frac{d^2 c_1}{ds^2}$, $\frac{d^3 c_1}{ds^3}$, at any initial place on the curve: the arc s will be supposed measured from the initial place. Also, the axes of reference will be taken to be the four principal lines of the curve at that place; so we must have, when $s = 0$,

$$\begin{aligned} l_1, m_1, n_1, k_1, &= 1, 0, 0, 0, \\ l_2, m_2, n_2, k_2, &= 0, 1, 0, 0; \\ l_3, m_3, n_3, k_3, &= 0, 0, 1, 0; \\ l_4, m_4, n_4, k_4, &= 0, 0, 0, 1. \end{aligned}$$

Now

$$\frac{dc_1}{ds} = \frac{1}{\rho} c_2, \quad \frac{d^2 c_1}{ds^2} = -\frac{1}{\rho^2} c_1 + \frac{1}{\rho\sigma} c_3, \quad \frac{d^3 c_1}{ds^3} = -\frac{1}{\rho}\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) c_2 + \frac{1}{\rho\sigma\tau} c_4;$$

and therefore we have, when $s = 0$,

$$\left.\begin{array}{llll} l_1 = 1, & \dfrac{dl_1}{ds} = 0, & \dfrac{d^2 l_1}{ds^2} = -\dfrac{1}{\rho^2}, & \dfrac{d^3 l_1}{ds^3} = 0 \\[2ex] m_1 = 0, & \dfrac{dm_1}{ds} = \dfrac{1}{\rho}, & \dfrac{d^2 m_1}{ds^2} = 0, & \dfrac{d^3 m_1}{ds^3} = -\dfrac{1}{\rho}\left(\dfrac{1}{\rho^2} + \dfrac{1}{\sigma^2}\right) \\[2ex] n_1 = 0, & \dfrac{dn_1}{ds} = 0, & \dfrac{d^2 n_1}{ds^2} = \dfrac{1}{\rho\sigma}, & \dfrac{d^3 n_1}{ds^3} = 0 \\[2ex] k_1 = 0, & \dfrac{dk_1}{ds} = 0, & \dfrac{d^2 k_1}{ds^2} = 0, & \dfrac{d^3 k_1}{ds^3} = \dfrac{1}{\rho\sigma\tau} \end{array}\right\}.$$

With these values, we find

$$\left.\begin{aligned}
(\mu_2^2-\mu_1^2)\,l_1 &= \left(\mu_2^2-\frac{1}{\rho^2}\right)\cos\mu_1 s-\left(\mu_1^2-\frac{1}{\rho^2}\right)\cos\mu_2 s\\
(\mu_2^2-\mu_1^2)\,m_1 &= \left(\mu_2^2-\frac{1}{\rho^2}-\frac{1}{\sigma^2}\right)\frac{\sin\mu_1 s}{\rho\mu_1}-\left(\mu_1^2-\frac{1}{\rho^2}-\frac{1}{\sigma^2}\right)\frac{\sin\mu_2 s}{\rho\mu_2}\\
(\mu_2^2-\mu_1^2)\,n_1 &= \frac{1}{\rho\sigma}(\cos\mu_1 s-\cos\mu_2 s)\\
(\mu_2^2-\mu_1^2)\,k_1 &= \frac{1}{\rho\sigma\tau}\left(\frac{\sin\mu_1 s}{\mu_1}-\frac{\sin\mu_2 s}{\mu_2}\right)
\end{aligned}\right\};$$

and therefore, noting that the origin and the axes of reference are taken where $s=0$,

$$\left.\begin{aligned}
(\mu_2^2-\mu_1^2)\,x &= \left(\mu_2^2-\frac{1}{\rho^2}\right)\frac{\sin\mu_1 s}{\mu_1}-\left(\mu_1^2-\frac{1}{\rho^2}\right)\frac{\sin\mu_2 s}{\mu_2}\\
(\mu_2^2-\mu_1^2)\,y &= \left(\mu_2^2-\frac{1}{\rho^2}-\frac{1}{\sigma^2}\right)\frac{1-\cos\mu_1 s}{\rho\mu_1^2}-\left(\mu_1^2-\frac{1}{\rho^2}-\frac{1}{\sigma^2}\right)\frac{1-\cos\mu_2 s}{\rho\mu_2^2}\\
(\mu_2^2-\mu_1^2)\,z &= \frac{1}{\rho\sigma}\left(\frac{\sin\mu_1 s}{\mu_1}-\frac{\sin\mu_2 s}{\mu_2}\right)\\
(\mu_2^2-\mu_1^2)\,v &= \frac{1}{\rho\sigma\tau}\left(\frac{1-\cos\mu_1 s}{\mu_1^2}-\frac{1-\cos\mu_2 s}{\mu_2^2}\right)
\end{aligned}\right\}.$$

Now by combining x and z, and by combining y and v, these equations can be expressed in the simpler form

$$\left.\begin{aligned}
x-\rho\sigma\left(\mu_1^2-\frac{1}{\rho^2}\right)z &= \frac{\sin\mu_1 s}{\mu_1}\\
x-\rho\sigma\left(\mu_2^2-\frac{1}{\rho^2}\right)z &= \frac{\sin\mu_2 s}{\mu_2}\\
y-\sigma\tau\left(\mu_1^2-\frac{1}{\rho^2}-\frac{1}{\sigma^2}\right)v &= \frac{1-\cos\mu_1 s}{\rho\mu_1^2}\\
y-\sigma\tau\left(\mu_2^2-\frac{1}{\rho^2}-\frac{1}{\sigma^2}\right)v &= \frac{1-\cos\mu_2 s}{\rho\mu_2^2}
\end{aligned}\right\}.$$

As the radii ρ, σ, τ, are constant, we take three equivalent constants a, α, β, such that

$$\frac{2}{\rho}=\frac{\cot\alpha}{a}\sin(2\alpha-2\beta),\quad \frac{2}{\tau}=\frac{\cot\alpha}{a}\sin 2\beta,\quad \frac{2}{\sigma}=\frac{1}{a}\{\sin(2\alpha-2\beta)-\sin 2\beta\}.$$

(It will be noticed that when $\beta=0$, we have

$$\frac{1}{\rho}=\frac{\cos^2\alpha}{a},\quad \frac{1}{\sigma}=\frac{\sin\alpha\cos\alpha}{a},\quad \frac{1}{\tau}=0;$$

which are characteristic of the cylindrical helix in homaloidal three-dimensional space.) With these values, we find

$$\mu_1 = \frac{\cot\alpha}{a}\sin(\alpha-\beta)\cos\beta, \qquad \mu_2 = \frac{\cot\alpha}{a}\cos(\alpha-\beta)\sin\beta,$$

$$\left(\mu_1^2 - \frac{1}{\rho^2}\right)\rho\sigma = \tan(\alpha-\beta), \qquad \left(\mu_2^2 - \frac{1}{\rho^2}\right)\rho\sigma = -\cot(\alpha-\beta),$$

$$\left(\mu_1^2 - \frac{1}{\rho^2} - \frac{1}{\sigma^2}\right)\sigma\tau = \tan\beta, \qquad \left(\mu_2^2 - \frac{1}{\rho^2} - \frac{1}{\sigma^2}\right)\sigma\tau = -\cot\beta,$$

$$\rho\mu_1 = \frac{\cos\beta}{\cos(\alpha-\beta)}, \qquad \rho\mu_2 = \frac{\sin\beta}{\sin(\alpha-\beta)}.$$

We now change the axes from the principal lines of the curve at the place $s=0$ to another set of orthogonal axes, and we change the origin, according to the relations

$$x\cos(\alpha-\beta) - z\sin(\alpha-\beta) = X,$$

$$x\sin(\alpha-\beta) + z\cos(\alpha-\beta) = Z,$$

$$y\cos\beta - v\sin\beta = \frac{\cos(\alpha-\beta)}{\mu_1} - Y,$$

$$y\sin\beta + v\cos\beta = V,$$

transformations which, merely changing the position and the orientation of the curve, do not affect its character: and then the equations of the curve become

$$\left.\begin{aligned} X &= a\frac{\tan\alpha}{\tan(\alpha-\beta)\cos\beta}\sin\mu_1 s \\ Y &= a\frac{\tan\alpha}{\tan(\alpha-\beta)\cos\beta}\cos\mu_1 s \\ Z &= a\frac{\tan\alpha\tan(\alpha-\beta)}{\sin\beta}\sin\mu_2 s \\ V &= a\frac{\tan\alpha\tan(\alpha-\beta)}{\sin\beta}(1-\cos\mu_2 s) \end{aligned}\right\},$$

the values of μ_1 and μ_2 being given above, in terms of a, α, β.

Accordingly these are the equations of *the curve which has constant circular* (*or plane*) *curvature, constant torsion, and constant tilt.* Hereafter (§ 219) it will be seen that such a curve is a geodesic on a cylindro-cylindric surface represented by the two equations

$$x^2 + y^2 = b^2, \quad z^2 + v^2 = c^2.$$

The values of ρ, σ, τ, in terms of a, α, β, have already been stated;

conversely, the values of a, α, β, in terms of ρ, σ, τ, are expressible by the relations

$$a\left\{\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}-\frac{1}{\tau^2}\right)^2+\frac{4}{\sigma^2\tau^2}\right\}^{\frac{1}{2}}=\frac{1}{\rho}-\frac{1}{\tau},$$

$$\frac{1}{\sigma\tau}\cot\alpha=\frac{1}{\rho}-\frac{1}{\tau},\quad \frac{2}{\sigma\tau}\cot 2\beta=\frac{1}{\rho^2}+\frac{1}{\sigma^2}-\frac{1}{\tau^2}.$$

NOTE 1. When there is no tilt, the equations of the curve become

$$X=a\sin\left(\frac{s}{a}\cos\alpha\right),\quad Y=a\cos\left(\frac{s}{a}\cos\alpha\right),\quad Z=s\sin\alpha,\quad V=0:$$

they represent a cylindrical helix in the flat $V=0$.

NOTE 2. The four-dimensional curve of constant curvatures ρ, σ, τ, is a closed curve, when $\tan\beta\cot(\alpha-\beta)$ is a commensurable quantity different from zero. When the constant tilt is finite but not zero, the curve remains within a finite range.

COROLLARY. The preceding analysis can be used to investigate *the curves the ratios of whose circular curvature, torsion, and tilt, are constants.*

Let these constants be a and b, where

$$\frac{1}{\sigma}=a\frac{1}{\rho},\quad \frac{1}{\tau}=b\frac{1}{\rho}.$$

Instead of using s as the independent variable in Frenet's equations, let the angle of contingence ϵ be used, where

$$\frac{1}{\rho}=\frac{d\epsilon}{ds}.$$

The Frenet equations now are

$$\frac{dl_1}{d\epsilon}=l_2,\quad \frac{dl_2}{d\epsilon}=-l_1+al_3,\quad \frac{dl_3}{d\epsilon}=-al_2+bl_4,\quad \frac{dl_4}{d\epsilon}=-bl_3.$$

The preceding analysis for the determination of l_1, m_1, n_1, k_1, can be adapted to the present case by changing

$$s \text{ into } \epsilon,\quad \rho \text{ into } 1,\quad \frac{1}{\sigma} \text{ into } a,\quad \frac{1}{\tau} \text{ into } b;$$

and therefore

$$\left.\begin{aligned}
\frac{dx}{ds}\qquad\quad -\frac{1}{a}(\mu_1{}^2-1)\frac{dz}{ds}&=\cos\mu_1\epsilon\\
\frac{dx}{ds}\qquad\quad -\frac{1}{a}(\mu_2{}^2-1)\frac{dz}{ds}&=\cos\mu_2\epsilon\\
\frac{dy}{ds}-\frac{1}{ab}(\mu_1{}^2-1-a^2)\frac{dv}{ds}&=\frac{\sin\mu_1\epsilon}{\mu_1}\\
\frac{dy}{ds}-\frac{1}{ab}(\mu_2{}^2-1-a^2)\frac{dv}{ds}&=\frac{\sin\mu_2\epsilon}{\mu_2}
\end{aligned}\right\},$$

where $\pm \mu_1$ and $\pm \mu_2$ are the roots of the equation

$$\mu^4 - \mu^2 (1 + a^2 + b^2) + b^2 = 0.$$

To specify a curve completely, the value of ρ must be assigned: if it be $\rho = f(s)$, the relation between s and ϵ is

$$\frac{ds}{d\epsilon} = f(s),$$

which can be used to determine s as a function of ϵ. The integration of the preceding equations is then merely a matter of quadrature.

Hereafter (§ 302, Ex. 2), it will be seen that such curves can be taken as geodesics in a sphero-cylindrical region, such as is represented by the equation

$$x^2 + y^2 + z^2 = c^2.$$

Deviations of a curve from its tangent, its osculating plane, and its osculating flat.

171. Various configurations have been introduced, usually defined in association with some degree of contact with the curve. Among these, are the tangent line, the osculating plane, the osculating flat, all homaloidal; and the circle of curvature, the sphere of curvature, and the globe of curvature. It is desirable to have an estimate of the closeness of contact of each such amplitude with the curve: and such an estimate is provided by the smallness of the order in magnitude of the deviation of a consecutive point of the curve from each of the amplitudes in question.

Let u denote the arc-distance of a point Q on the curve from the point P, and let $\bar{x}, \bar{y}, \bar{z}, \bar{v}$, be the coordinates of that point Q. Then

$$\bar{x} = x + ux' + \frac{u^2}{2!}x'' + \frac{u^3}{3!}x''' + \frac{u^4}{4!}x^{\text{iv}} + \frac{u^5}{5!}x^{\text{v}} + \dots,$$

$$\bar{y} = y + uy' + \frac{u^2}{2!}y'' + \frac{u^3}{3!}y''' + \frac{u^4}{4!}y^{\text{iv}} + \frac{u^5}{5!}y^{\text{v}} + \dots,$$

$$\bar{z} = z + uz' + \frac{u^2}{2!}z'' + \frac{u^3}{3!}z''' + \frac{u^4}{4!}z^{\text{iv}} + \frac{u^5}{5!}z^{\text{v}} + \dots,$$

$$\bar{v} = v + uv' + \frac{u^2}{2!}v'' + \frac{u^3}{3!}v''' + \frac{u^4}{4!}v^{\text{iv}} + \frac{u^5}{5!}v^{\text{v}} + \dots.$$

We proceed to find the perpendicular from Q upon the three homaloidal tangential amplitudes at P: also to find the distance of Q from the boundary circumference of the circle of plane curvature at P, the boundary surface of the sphere of curvature at P, and the boundary region of the globe of curvature at P.

We shall require the projection w of the arc PQ upon the tangent at P. As the direction-cosines of the tangent are x', y', z', v', this projection is equal to $\Sigma (\bar{x} - x) x'$; thus

$$w = \Sigma (\bar{x} - x) x' = u - \frac{1}{6\rho^2} u^3 + \frac{1}{8}\frac{\rho'}{\rho^3} u^4 + \dots.$$

I. The *projection of PQ upon the principal normal at P* is equal to the *distance of Q from the tangent at P*, and thus measures the deviation of the curve from that tangent. As the direction-cosines of the principal normal at P are $\rho x''$, $\rho y''$, $\rho z''$, $\rho v''$, this projection, ν, is equal to

$$\begin{aligned}\nu &= \Sigma\,(\bar{x} - x)\,\rho x'' \\ &= \frac{1}{2\rho}u^2 - \frac{1}{6}\frac{\rho'}{\rho^2}u^3 + \frac{1}{24}\left(2\frac{\rho'^2}{\rho^3} - \frac{\rho''}{\rho^2} - \frac{1}{\rho^3} - \frac{1}{\rho\sigma^2}\right)u^4 + \dots.\end{aligned}$$

By direct substitution, we find the length c of the straight chord PQ; the result is

$$c^2 = \Sigma\,(\bar{x} - x)^2 = u^2 - \frac{1}{12}\frac{u^4}{\rho^2} + \frac{1}{12}\frac{\rho'}{\rho^3}u^5 + \dots.$$

It is easy to verify that the relation $c^2 = w^2 + \nu^2$ is satisfied.

Manifestly, in the expression for ν, the deviation from the tangent, the important term in the immediate vicinity of P is the customary quantity $\dfrac{u^2}{2\rho}$.

II. The *projection of PQ upon the binormal at P* is equal to the *distance of Q from the osculating plane at P*, and thus measures the deviation of the curve from that osculating plane. As the direction-cosines of the binormal are $\dfrac{\sigma}{\rho}(x' + \rho\rho' x'' + \rho^2 x''')$, with the three similar quantities, the projection of PQ on the binormal is

$$\begin{aligned}&= \Sigma\,(\bar{x} - x)\frac{\sigma}{\rho}(x' + \rho\rho' x'' + \rho^2 x''') \\ &= \frac{1}{6\rho\sigma}u^3 - \frac{1}{24\rho\sigma}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)u^4 + \dots.\end{aligned}$$

III. The *projection of PQ upon the trinormal at P* is equal to the *distance of Q from the osculating flat at P*, and thus measures the deviation of the curve from that osculating flat. As the direction-cosines of the trinormal are $\sigma\rho^2 J_x$, $\sigma\rho^2 J_y$, $\sigma\rho^2 J_z$, $\sigma\rho^2 J_v$, the projection of PQ on the trinormal is

$$\begin{aligned}&= \Sigma\,(\bar{x} - x)\,\sigma\rho^2 J_x \\ &= \sigma\rho^2\Sigma\,(\bar{x} - x)\begin{vmatrix} y', & z', & v' \\ y'', & z'', & v'' \\ y''', & z''', & v''' \end{vmatrix} \\ &= \frac{1}{24}\sigma\rho^2 u^4 \Sigma x^{\text{iv}}\begin{vmatrix} y', & z', & v' \\ y'', & z'', & v'' \\ y''', & z''', & v''' \end{vmatrix} + \text{higher powers of } u \\ &= -\frac{1}{24}\sigma\rho^2 u^4 \Omega + \text{higher powers of } u \\ &= \frac{1}{24}\frac{u^4}{\rho\sigma\tau} + \text{higher powers of } u.\end{aligned}$$

It is an immediate corollary that, if the curve be referred to its principal lines at P as axes, so that $t, n, b, \bar{t}$, are the coordinates parallel to the tangent, the principal normal, the binormal, and the trinormal, respectively, the coordinates of a point Q at a distance u along the curve from P, are

$$\begin{aligned} t &= u - \frac{1}{6\rho^2}u^3 + \frac{1}{8}\frac{\rho'}{\rho^3}u^4 + \ldots, \\ n &= \frac{1}{2\rho}u^2 - \frac{1}{6}\frac{\rho'}{\rho^2}u^3 + \frac{1}{24}\left(2\frac{\rho'^2}{\rho^3} - \frac{\rho''}{\rho^2} - \frac{1}{\rho^3} - \frac{1}{\rho\sigma^2}\right)u^4 + \ldots, \\ b &= \frac{1}{6\rho\sigma}u^3 - \frac{1}{24\rho\sigma}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)u^4 + \ldots, \\ \bar{t} &= \frac{1}{24}\frac{u^4}{\rho\sigma\tau} + \ldots. \end{aligned}$$

Deviations of a curve from its circle of curvature, its sphere of curvature, and its globe of curvature.

172. We proceed to estimate the distance of Q from the respective curved amplitudes of contact at P, each with its own measure of curvature and with its own degree of contact, and thus to estimate the deviations of the curve from those amplitudes in the immediate vicinity of P.

IV. We first find the *distance of Q from the circle of plane curvature at P*. Let Q' be the projection of Q on the osculating plane at P, so that (§ 171)

$$QQ' = \frac{u^3}{6\rho\sigma} + \ldots.$$

Let $Q'C$ intersect the circle of curvature in U, where C is the centre of that circle. Then $CUQ'Q$ is a plane, which lies in the osculating flat and is perpendicular to the osculating plane at P; and the tangent to the circle at U is perpendicular to this plane $CUQ'Q$, being perpendicular to CU and to QQ'. Hence QU is perpendicular to that tangent and therefore, being normal to the circle, is the shortest distance of Q from the circle. Obviously

$$QU^2 = QQ'^2 + Q'U^2.$$

Further, the coordinates of C are $\xi_1, \eta_1, \zeta_1, v_1, = x + \rho^2 x'', y + \rho^2 y'', z + \rho^2 z'', v + \rho^2 v''$; hence

$$QC^2 = \Sigma(\bar{x} - \xi_1)^2 = \Sigma(\xi_1 - x)^2 - 2\Sigma(\bar{x} - x)(\xi_1 - x) + \Sigma(\bar{x} - x)^2.$$

Now

$$\Sigma(\xi_1 - x)^2 = CP^2 = \rho^2,$$

$$\begin{aligned} \Sigma(\bar{x} - x)(\xi_1 - x) &= \rho^2 \Sigma(\bar{x} - x)x'' \\ &= \rho^2\left\{\frac{1}{2\rho^2}u^2 - \frac{1}{6}\frac{\rho'}{\rho^3}u^3 + \frac{1}{24}\left(-\frac{\rho''}{\rho^3} + 2\frac{\rho'^2}{\rho^4} - \frac{1}{\rho^2} - \frac{1}{\sigma^2\rho^2}\right)u^4 + \ldots\right\}; \end{aligned}$$

and therefore, with the foregoing value of $\Sigma(\bar{x}-x)^2$ on p. 287, we have

$$QC^2 = \rho^2 - \left(u^2 - \frac{1}{3}\frac{\rho'}{\rho}u^3 + \dots\right) + u^2 - \frac{1}{3}\frac{u^4}{\rho^2} + \dots = \rho^2 + \frac{1}{3}\frac{\rho'}{\rho}u^3 + \dots,$$

so that

$$QC = \rho + \frac{1}{6}\frac{\rho'}{\rho^2}u^3 + \dots.$$

But $Q'C^2 = QC^2 - QQ'^2$, and QQ' is of the third order in u; hence, certainly up to the fifth order inclusive, $Q'C$ is equal to QC, and we have

$$Q'C = \rho + \frac{1}{6}\frac{\rho'}{\rho^2}u^3 + \dots,$$

so that

$$Q'U = Q'C - CU = Q'C - \rho = \frac{1}{6}\frac{\rho'}{\rho^2}u^3 + \dots.$$

Consequently, up to the third order,

$$QU = (QQ'^2 + Q'U^2)^{\frac{1}{2}} = \frac{u^3}{6\rho\sigma}\left(1 + \frac{\sigma^2\rho'^2}{\rho^2}\right)^{\frac{1}{2}} = \frac{R}{6\rho^2\sigma}u^3,$$

which accordingly measures the deviation of the curve from its circle of curvature at P.

V. In a similar way, we obtain the *distance of Q from the sphere of curvature at P*. Let Q'' be the projection of Q on the osculating flat at P, so that (§ 171)

$$QQ'' = \frac{u^4}{24\rho\sigma\tau} + \dots.$$

Let $Q''S$ intersect the sphere of curvature in V, where S is the centre of the sphere. Then $Q''S$ is perpendicular to the tangent plane at V to the sphere; $Q''Q$, normal to the flat, also is perpendicular to that tangent plane; hence the plane $QQ''VS$ is orthogonal to the tangent plane. Thus QV is perpendicular to the tangent plane at V and therefore is the shortest distance of Q from the sphere. Obviously

$$QV^2 = QQ''^2 + Q''V^2.$$

Further, the coordinates of S are ξ_2, η_2, ζ_2, υ_2, as given in § 142; and therefore

$$QS^2 = \Sigma(\bar{x}-\xi_2)^2 = \Sigma(\xi_2 - x)^2 - 2\Sigma(\bar{x}-x)(\xi_2 - x) + \Sigma(\bar{x}-x)^2.$$

Now $\Sigma(\xi_2 - x)^2 = SP^2 = R^2$; and

$$\Sigma(\bar{x}-x)(\xi_2 - x) = \Sigma(\bar{x}-x)\left\{\sigma^2\frac{\rho'}{\rho}x' + R^2x'' + \sigma^2\rho\rho'x'''\right\}$$

$$= \frac{1}{2}u^2 - \frac{1}{24}\left(\frac{1}{\sigma^2\rho}R\frac{dR}{d\rho} + \frac{1}{\rho^2}\right)u^4 + \dots;$$

and therefore, with the foregoing value of $\Sigma(\bar{x}-x)^2$, we have

$$QS^2 = R^2 + \frac{1}{12}\frac{1}{\sigma^2\rho}R\frac{dR}{d\rho}u^4 + \text{higher powers of } u.$$

But $Q''S^2 = QS^2 - QQ''^2$, and QQ'' is of the fourth order in u; hence, certainly up to the seventh order inclusive, $Q''S$ is equal to QS, and we have, certainly up to the fourth order,

$$Q'V = Q''S - SV = QS - R = \frac{R'}{24\sigma^2\rho\rho'}u^4 + \dots.$$

Consequently

$$\begin{aligned} QV &= (QQ''^2 + Q'V^2)^{\frac{1}{2}} \\ &= \frac{u^4}{24\sigma^2\rho\rho'\tau}(\sigma^2\rho'^2 + \tau^2R'^2)^{\frac{1}{2}} = \frac{\Gamma}{24R\rho\sigma\tau}u^4, \end{aligned}$$

which accordingly measures, to the fourth order, the deviation of the curve from its sphere of curvature at P.

VI. Finally, the *distance of Q from the globe of curvature at P* is manifestly equal to $QG - PG$, where G is the centre of globular curvature at P; for QG is normal to the globe. The coordinates of G are ξ_3, η_3, ζ_3, v_3 as given in § 149; and therefore

$$\begin{aligned} QG^2 &= \Sigma(\bar{x} - \xi_3)^2 = \Sigma(\xi_3 - x)^2 - 2\Sigma\{(\bar{x} - x)(\xi_3 - x)\} + \Sigma(\bar{x} - x)^2 \\ &= \Gamma^2 + u^2 - \frac{1}{12}\frac{1}{\rho^2}u^4 + \frac{1}{12}\frac{\rho'}{\rho^3}u^5 - 2\Sigma\{(\bar{x} - x)(\xi_3 - x)\}, \end{aligned}$$

up to the fifth power of u inclusive.

It is necessary to evaluate the quantity $\Sigma\{(\bar{x} - x)(\xi_3 - x)\}$, up to the fifth power of u inclusive. Now

$$\bar{x} - x = x'u + \frac{1}{2\,!}x''u^2 + \frac{1}{3\,!}x'''u^3 + \frac{1}{4\,!}x^{\text{iv}}u^4 + \frac{1}{5\,!}x^{\text{v}}u^5 + \dots,$$

and

$$\begin{aligned} \xi_3 - x &= \rho\,.\,\rho x'' + \sigma\rho'\frac{\sigma}{\rho}(x' + \rho\rho'x'' + \rho^2x''') + \frac{\tau}{\sigma}R\frac{dR}{d\rho}\sigma\rho^2J_x \\ &= \rho l_2 + \sigma\rho' l_3 + \frac{\tau}{\sigma}R\frac{dR}{d\rho}l_4; \end{aligned}$$

with corresponding expressions for $\bar{y} - y$, $\eta_3 - y$; $\bar{z} - z$, $\zeta_3 - z$; and $\bar{v} - v$, $v_3 - v$ Hence, in the quantity $\Sigma\{(\bar{x} - x)(\xi_3 - x)\}$, the coefficient of u is

$$= \Sigma l_1(\xi_3 - x) = \Sigma l_1\left(\rho l_2 + \sigma\rho' l_3 + \frac{\tau}{\sigma}R\frac{dR}{d\rho}l_4\right) = 0,$$

the coefficient of $\frac{1}{2}u^2$ is

$$\begin{aligned} &= \Sigma x''\left(\rho l_2 + \sigma\rho' l_3 + \frac{\tau}{\sigma}R\frac{dR}{d\rho}l_4\right) \\ &= \frac{1}{\rho}\Sigma l_2\left(\rho l_2 + \sigma\rho' l_3 + \frac{\tau}{\sigma}R\frac{dR}{d\rho}l_4\right) = 1; \end{aligned}$$

and the coefficient of $\frac{1}{6}u^3$ is

$$\begin{aligned} &= \Sigma x'''\left(\rho l_2 + \sigma\rho' l_3 + \frac{\tau}{\sigma}R\frac{dR}{d\rho}l_4\right) \\ &= \frac{1}{\rho\sigma}\Sigma\left(l_3 - \frac{\sigma\rho'}{\rho}l_2 - \frac{\sigma}{\rho}l_1\right)\left(\rho l_2 + \sigma\rho' l_3 + \frac{\tau}{\sigma}R\frac{dR}{d\rho}l_4\right) = 0: \end{aligned}$$

by using the relations $\Sigma l_\mu^2 = 1$, $\Sigma\Sigma l_\mu l_\nu = 0$, for $\mu = 1, 2, 3, 4$, and $\nu = 1, 2, 3, 4$, with ν not equal to μ. For the coefficient of u^4, we use the formula of p. 278, viz.

$$x^{\text{iv}} = \frac{1}{\rho\sigma\tau} l_4 - \frac{1}{\rho\sigma}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right) l_3 + M\rho l_2 + 3\frac{\rho'}{\rho^3} l_1;$$

and therefore the coefficient of $\frac{1}{24}u^4$ in $\Sigma\,\{(\bar{x}-x)(\xi_3-x)\}$ is

$$= \Sigma x^{\text{iv}}\left(\rho l_2 + \sigma\rho' l_3 + \frac{\tau}{\sigma} R\frac{dR}{d\rho} l_4\right)$$

$$= M\rho^2 - \frac{\rho'}{\rho}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right) + \frac{1}{\rho\sigma^2} R\frac{dR}{d\rho} = -\frac{1}{\rho^2}.$$

For the coefficient of u^5, we use the formula of p. 279, viz.

$$x^{\text{v}} = \Delta l_4 + W l_3 + V l_2 + U l_1;$$

and therefore the coefficient of $\frac{1}{120}u^5$ in $\Sigma\,\{(\bar{x}-x)(\xi_3-x)\}$ is

$$= \Sigma x^{\text{v}}\left(\rho l_2 + \sigma\rho' l_3 + \frac{\tau}{\sigma} R\frac{dR}{d\rho} l_4\right)$$

$$= \rho V + \sigma\rho' W + \frac{\tau}{\sigma} R\frac{dR}{d\rho}\Delta = \Theta.$$

Hence

$$\begin{aligned} QG^2 &= \Sigma\,(\bar{x}-\xi_3)^2 \\ &= \Gamma^2 - \frac{1}{60}u^5\Theta + \frac{1}{12}u^5\frac{\rho'}{\rho^3} + \text{higher powers} \\ &= \Gamma^2 + \frac{u^5}{60}\left(5\frac{u'}{\rho^3} - \Theta\right) + \text{higher powers.} \end{aligned}$$

The deviation of Q from the globe of curvature

$$\begin{aligned} &= QG - PG \\ &= \Gamma\left\{1 + \frac{u^5}{120\Gamma^2}\left(5\frac{\rho'}{\rho^3} - \Theta\right) + \text{higher powers of } u\right\} - \Gamma \\ &= \frac{1}{120\Gamma}\left(5\frac{\rho'}{\rho^3} - \Theta\right)u^5; \end{aligned}$$

and it is necessary to obtain an equivalent expression for the quantity $5\frac{\rho'}{\rho^3} - \Theta$. On substituting for V, W, Δ, in Θ, we have

$$\begin{aligned} \Theta - 5\frac{\rho'}{\rho^3} &= -5\frac{\rho'}{\rho^3} + \rho V + \sigma\rho' W + \frac{\tau}{\sigma} R\frac{dR}{d\rho}\Delta \\ &= \rho\left\{-2\frac{\rho'}{\rho^4} + \frac{1}{\rho\sigma^2}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right) + \frac{d}{ds}(M\rho)\right\} \\ &\quad + \sigma\rho'\left[-\frac{1}{\rho\sigma\tau^2} + M\frac{\rho}{\sigma} - \frac{d}{ds}\left\{\frac{1}{\rho\sigma}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)\right\}\right] \\ &\quad - \frac{\tau}{\sigma} R\frac{dR}{d\rho}\frac{1}{\rho\sigma\tau}\left(3\frac{\rho'}{\rho} + 2\frac{\sigma'}{\sigma} + \frac{\tau'}{\tau}\right). \end{aligned}$$

Now
$$-2\frac{\rho'}{\rho^3}+\rho\frac{d}{ds}(M\rho)+\sigma\rho' M\frac{\rho}{\sigma}$$
$$=-2\frac{\rho'}{\rho^3}+\frac{d}{ds}(M\rho^2)$$
$$=-2\frac{\rho'}{\rho^3}+\frac{d}{ds}\left\{\frac{\rho'}{\rho}\left(2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right)-\frac{1}{\sigma^2\rho}R\frac{dR}{d\rho}-\frac{1}{\rho^2}\right\}$$
$$=\frac{d}{ds}\left\{\frac{\rho'}{\rho}\left(2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right)-\frac{1}{\sigma^2\rho}R\frac{dR}{d\rho}\right\},$$
on using the value for $M\rho^2$ given on p. 278; also
$$\frac{1}{\sigma^2}\left(2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right)-\sigma\rho'\frac{d}{ds}\left\{\frac{1}{\rho\sigma}\left(2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right)\right\}$$
$$=\left(2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right)\left(\frac{1}{\sigma^2}+\frac{\rho'^2}{\rho^2}+\frac{\rho'\sigma'}{\rho\sigma}\right)-\frac{\rho'}{\rho}\frac{d}{ds}\left(2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right).$$
Hence
$$\Theta-5\frac{\rho'}{\rho^3}=\frac{d}{ds}\left(-\frac{1}{\sigma^2\rho}R\frac{dR}{d\rho}\right)+\left(2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right)\left\{\frac{d}{ds}\left(\frac{\rho'}{\rho}\right)+\frac{1}{\sigma^2}+\frac{\rho'^2}{\rho^2}+\frac{\rho'\sigma'}{\rho\sigma}\right\}$$
$$-\frac{\rho'}{\rho\tau^2}-\frac{1}{\sigma^2\rho}R\frac{dR}{d\rho}\left(3\frac{\rho'}{\rho}+2\frac{\sigma'}{\sigma}+\frac{\tau'}{\tau}\right).$$
But
$$\frac{d}{ds}\left(\frac{\rho'}{\rho}\right)+\frac{1}{\sigma^2}+\frac{\rho'^2}{\rho^2}+\frac{\rho'\sigma'}{\rho\sigma}=\frac{1}{\sigma^2\rho}R\frac{dR}{d\rho};$$
and therefore
$$\Theta-5\frac{\rho'}{\rho^3}=\frac{d}{ds}\left(-\frac{1}{\sigma^2\rho}R\frac{dR}{d\rho}\right)-\frac{1}{\sigma^2\rho}R\frac{dR}{d\rho}\left(\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}+\frac{\tau'}{\tau}\right)-\frac{\rho'}{\rho\tau^2}$$
$$=-\frac{1}{\sigma^2\rho}\frac{d}{ds}\left(R\frac{dR}{d\rho}\right)+\frac{1}{\sigma^2\rho}R\frac{dR}{d\rho}\left(\frac{\sigma'}{\sigma}-\frac{\tau'}{\tau}\right)-\frac{\rho'}{\rho\tau^2}$$
Now the radius Γ of globular curvature is given by
$$\Gamma^2=R^2+\left(\frac{\tau}{\sigma}R\frac{dR}{d\rho}\right)^2,$$
and therefore
$$\Gamma\Gamma'=RR'+\frac{\tau}{\sigma\rho'}RR'\frac{d}{ds}\left(\frac{\tau}{\sigma}R\frac{dR}{d\rho}\right),$$
so that
$$\frac{\Gamma\Gamma'}{RR'}=1+\frac{\tau}{\sigma\rho'}\frac{d}{ds}\left(\frac{\tau}{\sigma}R\frac{dR}{d\rho}\right)$$
$$=1+\frac{\tau^2}{\sigma^2\rho'}\frac{d}{ds}\left(R\frac{dR}{d\rho}\right)+R\frac{dR}{d\rho}\frac{\tau}{\sigma\rho'}\left(\frac{\tau'}{\sigma}-\frac{\sigma'\tau}{\sigma^2}\right).$$
Consequently
$$\Theta-5\frac{\rho'}{\rho^3}=-\frac{\rho'}{\rho\tau^2}\frac{\Gamma\Gamma'}{RR'}.$$

Hence at Q, distant u along the curve from P, the *deviation of the curve from the globe of curvature* at P
$$=\frac{1}{120}\frac{\rho'\Gamma'}{\tau^2\rho RR'}u^5.$$

Descriptive relation of a curve to its amplitudes of contact.

173. Thus the three deviations at Q from the curved amplitudes of different dimensions which have closest contact with the curve at P are:

from the circle of plane curvature, $\frac{R}{\rho}\frac{1}{6\rho\sigma}u^3$;

from the sphere of curvature, $\frac{\Gamma}{R}\frac{1}{24\rho\sigma\tau}u^4$;

from the globe of curvature, $\Gamma'\frac{\sigma\rho'}{\tau R'}\frac{1}{120R\rho\sigma\tau}u^5$.

The three deviations at Q from the homaloidal amplitudes of different dimensions which have closest contact with the curve at P are:

from the tangent, $\frac{1}{2\rho}u^2$;

from the osculating plane, $\frac{1}{6\rho\sigma}u^3$;

from the osculating flat, $\frac{1}{24\rho\sigma\tau}u^4$.

These results can be summarised, qualitatively, in a more descriptive form: the description being an interpretation of the fact that an even power of u is unchanged in sign and an odd power of u is changed in sign when the sign of u is changed. In each instance of the relation between the curve and the tangential amplitude, we are dealing solely with the near vicinity of a point P. Also, it is assumed throughout that P is not a singularity of any kind upon the curve: thus ρ^{-1} is taken to be different from zero, so that P is not a position of instantaneous linearity on the curve: σ^{-1} is taken to be different from zero, so that P is not a position of instantaneous planarity on the curve: τ^{-1} is taken to be different from zero, so that P is not a position of instantaneous flatness on the curve. Then the preceding results shew that *the curve, in passing through P, continues on only one side of its tangent and on only one side of its osculating flat and crosses its osculating plane*; and that *the curve, in passing through P, continues on one side of its sphere of curvature (that is, either remains without the spherical surface or remains within the spherical surface) and that it crosses from the inside to the outside both of its circle of plane curvature and of its globe of curvature.*

CHAPTER X.

CURVES: ASSOCIATED REGIONS: DEVELOPABLE REGIONS.

Envelopes of associated flats.

174. Among the homaloidal amplitudes associated with a curve, and passing through a point on the curve, there are three types. The first of these types consists of the chief lines of the curve at the point, which are the tangent, the principal normal, the binormal, and the trinormal. The third of the types consists of the four chief flats of the curve at the point, each of the flats having three of the chief lines for guiding directions; and, of the four flats, the two of more immediate importance are the osculating flat and the normal flat. The second of the types consists of the six chief planes of the curve at the point, each of the planes having two of the chief lines for guiding directions; and of the six planes, the three of more immediate importance are the osculating plane, the normal plane, and the orthogonal plane. But the osculating plane can be regarded as generated in two ways: as the plane through the point containing two consecutive tangent directions, and as the intersection of two consecutive osculating flats: and it has been discussed almost entirely through the former generation The normal plane is the intersection of two distinct (and non-consecutive) flats, the osculating flat and the normal flat at the point. The orthogonal plane is the plane through the point of the curve parallel to the intersection of two consecutive normal flats.

Thus two of the chief planes of the curve are determined by the intersection of two consecutive flats: in the one instance, of the osculating flats: in the other instance, of the normal flats. In each instance, the equation of the flat involves only a single parameter. and the association of any such equation with the equation of the consecutive flat is, in effect, an essential stage in determining the region which is the envelope of the flat in question.

The two instances will be discussed in succession.

Envelope of osculating flat.

175. When we deal with the osculating flat, and its intersection with consecutive osculating flats, we shall, first of all, take its equation in the form

$$(\bar{x}-x)\,l_4+(\bar{y}-y)\,m_4+(\bar{z}-z)\,n_4+(\bar{v}-v)\,k_4=\Sigma\,(\bar{x}-x)\,l_4=0$$

in order to exhibit its association with the curve, partly in connection with the Frenet formulæ (§ 164).

Its intersection by a consecutive osculating flat is the osculating plane; and the equations of the osculating plane arise by combining the equation

$$\Sigma \left\{ (\bar{x} - x) \frac{dl_4}{ds} - x' l_4 \right\} = 0$$

with the foregoing equation. Now $x' = l_1$, and so $\Sigma x' l_4 = \Sigma l_1 l_4 = 0$; and $\dfrac{dl_4}{ds} = -\dfrac{1}{\tau} l_3$; hence, unless the tilt is zero (and we shall assume it not to be zero), the equations of the osculating plane arise in the form

$$\Sigma (\bar{x} - x) l_4 = 0, \quad \Sigma (\bar{x} - x) l_3 = 0.$$

To express the equations in a form more suggestive of the earlier property that the osculating plane contains the tangent and the principal normal to the curve, we note that

$$\left.\begin{array}{l} \Sigma l_1 l_4 = 0 \\ \Sigma l_1 l_3 = 0 \end{array}\right\}, \quad \left.\begin{array}{l} \Sigma l_2 l_4 = 0 \\ \Sigma l_2 l_3 = 0 \end{array}\right\},$$

so that the two directions l_1, m_1, n_1, k_1; and l_2, m_2, n_2, k_2, lie in the osculating plane. Its equations can therefore be taken in the (more customary) form

$$\left\| \begin{array}{cccc} \bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{array} \right\| = 0.$$

The intersection of three consecutive osculating flats is a line: it is the tangent to the curve. Its equations are obtained by associating with the two preceding equations—which, combined, are the equivalent of the original flat and the first consecutive flat—the equation of the next consecutive flat. These equations are

$$\Sigma (\bar{x} - x) l_4 = 0,$$

$$\Sigma (\bar{x} - x) l_4 + \Sigma \left\{ (\bar{x} - x) \frac{dl_4}{ds} - x' l_4 \right\} ds = 0,$$

which, combined with the first, leads to

$$\Sigma (\bar{x} - x) l_3 = 0;$$

and

$$\Sigma (\bar{x} - x) l_4 + \Sigma \left\{ (\bar{x} - x) \frac{dl_4}{ds} - x' l_4 \right\} ds$$
$$+ \Sigma \left\{ (\bar{x} - x) \frac{d^2 l_4}{ds^2} - 2x' \frac{dl_4}{ds} - x'' l_4 \right\} \frac{ds^2}{2} = 0.$$

Now, because

$$\Sigma x'' l_4 = \frac{1}{\rho} \Sigma l_2 l_4 = 0,$$

$$\Sigma x' \frac{dl_4}{ds} = -\frac{1}{\tau} \Sigma l_1 l_3 = 0,$$

and

$$\frac{d^2 l_4}{ds^2} = -\frac{d}{ds}\left(\frac{l_3}{\tau}\right) = \frac{l_2}{\sigma\tau} + \frac{1}{\tau^2}\frac{d\tau}{ds} l_3 - \frac{1}{\tau^2} l_4;$$

hence the third equation can, by the use of the first two equations with which it coexists for a common intersection, be transformed to

$$\Sigma (\bar{x} - x) l_2 = 0.$$

Thus the intersection of the three flats is given by the equations

$$\Sigma (\bar{x} - x) l_4 = 0, \quad \Sigma (\bar{x} - x) l_3 = 0, \quad \Sigma (\bar{x} - x) l_2 = 0.$$

Now l_1, m_1, n_1, k_1; l_2, m_2, n_2, k_2; l_3, m_3, n_3, k_3; and l_4, m_4, n_4, k_4; are a quadruply orthogonal system such that

$$\begin{vmatrix} l_1, & m_1, & n_1, & k_1 \\ l_2, & m_2, & n_2, & k_2 \\ l_3, & m_3, & n_3, & k_3 \\ l_4, & m_4, & n_4, & k_4 \end{vmatrix} = 1;$$

and therefore

$$l_1 = \begin{vmatrix} m_2, & n_2, & k_2 \\ m_3, & n_3, & k_3 \\ m_4, & n_4, & k_4 \end{vmatrix},$$

with corresponding values for m_1, n_1, k_1; hence the foregoing three equations are equivalent to

$$\frac{\bar{x} - x}{l_1} = \frac{\bar{y} - y}{m_1} = \frac{\bar{z} - z}{n_1} = \frac{\bar{v} - v}{k_1},$$

which are the equations of the tangent.

The intersection of four consecutive osculating flats is the initial point P of the curve. This result can be obtained by associating, with the equations of the first three flats which are equivalent to

$$\Sigma (\bar{x} - x) l_4 = 0, \quad \Sigma (\bar{x} - x) l_3 = 0, \quad \Sigma (\bar{x} - x) l_2 = 0,$$

the equation of the next consecutive flat, which is easily seen to be reducible, in combination with these three equations, to the additional equation

$$\Sigma (\bar{x} - x) l_1 = 0.$$

These are four equations linear and homogeneous in $\bar{x} - x$, $\bar{y} - y$, $\bar{z} - z$, $\bar{v} - v$; the determinant of their coefficients does not vanish—actually, it is equal to 1; hence the only solution of the equations is

$$\bar{x} = x, \quad \bar{y} = y, \quad \bar{z} = z, \quad \bar{v} = v,$$

being the point P of the curve.

The preceding analysis, with a different interpretation, gives the tangent as the intersection of two consecutive osculating planes. The osculating plane can be taken as represented by the equations

$$\Sigma (\bar{x} - x) l_4 = 0, \quad \Sigma (\bar{x} - x) l_3 = 0;$$

the consecutive osculating plane is then given by the equations

$$\Sigma(\bar{x}-x)l_4+\Sigma\left\{(\bar{x}-x)\frac{dl_4}{ds}-x'l_4\right\}ds=0,$$

$$\Sigma(\bar{x}-x)l_3+\Sigma\left\{(\bar{x}-x)\frac{dl_3}{ds}-x'l_3\right\}ds=0.$$

For the purpose of intersection, the last two equations can be combined with the first two; hence, for them, we can substitute the equations

$$\Sigma(\bar{x}-x)l_3=0,\quad \Sigma(\bar{x}-x)l_2=0:$$

that is, there are three equations, and they provide the tangent.

Similarly, the intersection of three consecutive osculating planes gives the point on the curve.

Finally, the point on the curve can be obtained as the intersection of two consecutive tangents. For, when the first tangent is given by the equations

$$\Sigma(\bar{x}-x)l_4=0,\quad \Sigma(\bar{x}-x)l_3=0,\quad \Sigma(\bar{x}-x)l_2=0,$$

the consecutive tangent is given by the equations

$$\Sigma(\bar{x}-x)l_4+\Sigma\left\{(\bar{x}-x)\frac{dl_4}{ds}-x'l_4\right\}ds=0,$$

$$\Sigma(\bar{x}-x)l_3+\Sigma\left\{(\bar{x}-x)\frac{dl_3}{ds}-x'l_3\right\}ds=0,$$

$$\Sigma(\bar{x}-x)l_2+\Sigma\left\{(\bar{x}-x)\frac{dl_2}{ds}-x'l_2\right\}ds=0;$$

and now the intersection of the tangents is given by associating, with the three equations of the former tangent, the additional equation

$$\Sigma(\bar{x}-x)l_1=0.$$

As before, the four equations determine the point on the curve.

Developable region.

176. An entirely different interpretation can be given to these equations The equation of the osculating flat

$$\Sigma(\bar{x}-x)l_4=0$$

contains a single parameter s. In order to obtain the envelope of the flat, we associate, with this equation, the derived equation

$$\frac{d}{ds}\{\Sigma(\bar{x}-x)l_4\}=0,$$

that is, the equation

$$\Sigma(\bar{x}-x)l_3=0.$$

Between the two equations, we could eliminate the parameter and obtain an equation

$$F(\bar{x},\bar{y},\bar{z},\bar{v})=0,$$

which accordingly is the (non-homaloidal) region enveloped by the flat. But the equation $F=0$ is satisfied by virtue of the two equations $\Sigma(\bar{x}-x)\,l_4=0$, $\Sigma(\bar{x}-x)\,l_3=0$, which are those of the osculating plane for a particular value of s; and these equations, for the succession of parametric values, represent the family of osculating planes. Thus the region $F=0$ contains all the osculating planes of the curve: it can be regarded as a region generated by the moving osculating plane: we can describe it as a *planar region*, in the sense that it contains a continuous family of planes. The osculating flat of the curve is a flat which is tangential to the region: and a plane, through any point of the region and lying entirely within the region, is the intersection of the flat tangential to the region at that point, by the flat through a consecutive point and also tangential to the region.

Now consider the succession of these planes, as they are contained in the region. Take any such plane, which is the intersection of two consecutive flats tangential to the region; and round this plane effect the infinitesimal rotation $d\omega$ (of § 145) which, while leaving the plane unaltered, brings the two flats into coincidence. The result of this rotation is to move the elementary portion common to the region and the first flat, so that it lies in the second flat: this second flat now contains two elementary portions of the region. Effect the same process by an infinitesimal rotation round the next osculating plane, which is the intersection of the second flat with a third flat, choosing the rotation so that these two flats are brought into coincidence: the result is that the two former elementary portions of the region are brought into the third flat which itself contains an elementary portion of the region: that is, the third flat now contains three elementary portions of the region. Proceeding in this way by successive infinitesimal rotations round successive osculating planes, we bring at each stage all the earlier portions of the region into the latest flat considered, which itself contains a further elementary portion of the region. The result of the whole completed process is to bring the whole of the region into one flat: that is, without extension and without rupture, the region has been deformed into a flat. It thus follows that the region is of a special type: we have seen that it contains a family of planes: and it can be developed into a flat by rotation round these planes. On the analogy of the corresponding phenomenon in three dimensions, it will be called a *developable region*.

Osculating developable surface.

177. Again, the equations of the osculating plane are

$$\Sigma(\bar{x}-x)\,l_4=0,\quad \Sigma(\bar{x}-x)\,l_3=0\,;$$

and they contain a single parameter. In order to obtain the envelope of the plane, we associate the equations

$$\frac{d}{ds}\{\Sigma(\bar{x}-x)\,l_4\}=0,\quad \frac{d}{ds}\{\Sigma(\bar{x}-x)\,l_3\}=0,$$

with these equations: that is, there are, in all, the three equations

$$\Sigma(\bar{x}-x)\,l_4=0,\quad \Sigma(\bar{x}-x)\,l_3=0,\quad \Sigma(\bar{x}-x)\,l_2=0,$$

or their equivalent

$$\frac{\bar{x}-x}{l_1}=\frac{\bar{y}-y}{m_1}=\frac{\bar{z}-z}{n_1}=\frac{\bar{v}-v}{k_1},$$

representing the tangent as they stand. When we eliminate the parameter s between these three equations, the eliminant is constituted by two equations which accordingly constitute a surface; and the two equations can be expressed in a variety of equivalent forms. Whatever be the form adopted, the two equations of the surface are satisfied in virtue of these three equations: in other words, a parametric form of equations for the surface is

$$\bar{x}=x+l_1r,\quad \bar{y}=y+m_1r,\quad \bar{z}=z+n_1r,\quad \bar{v}=v+v_1r,$$

where s and r are the two parameters. The tangent to the curve thus lies wholly in the surface, which therefore is a ruled surface. On this surface, each generating line (being a tangent) meets the consecutive generating line (being the consecutive tangent): the surface is a developable surface.

On this developable surface, the intersection of consecutive generators (that is, the intersection of consecutive tangents to the curve) is the edge of regression: that is, the original curve is a knife-edge—the edge of regression—of this surface.

Deformation of the developable region and surface to a flat and a plane.

178. In the deformation of the developable region which is the envelope of the osculating flat, the successive osculating flats (which are tangential to that region) are brought into coincidence, each with the next, by rotation about the osculating plane common to the two flats as an instantaneous plane. Thus the successive osculating planes are brought into coincidence, each with the next, by rotation round the line which is common, alike to the first flat that was moved into coincidence with the second flat, to the second flat that is to be moved into coincidence with the third flat, and to this third flat. But this minor operation, as regards the planes, which is seen to be involved in the major operation as regards the flats, is the deformation of the developable surface which envelopes the osculating planes.

Thus we have, connected with the curve, the osculating flat and its envelope, a developable region: and the osculating plane and its envelope, a developable surface. The deformation of the developable region into a flat causes, simultaneously, the deformation of the developable surface into a plane; and as the developable ruled surface is contained in the developable region, so the plane which is the ultimate deformed shape of the developable surface is contained in the flat which is the ultimate deformed shape of the developable region.

The curvatures of the osculating developable region.

179. The osculating developable region, being the envelope of the osculating flat, has for its equation the eliminant of the two equations

$$\left\| \begin{matrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ l_1\ , & m_1\ , & n_1\ , & k_1 \\ l_2\ , & m_2\ , & n_2\ , & k_2 \end{matrix} \right\| = 0,$$

which are the equations of the osculating plane: for that plane at once is the intersection of two consecutive osculating flats and is the amplitude determined by two consecutive tangents as guiding lines. The region thus contains the osculating plane at all points of the curve; and therefore the coordinates of any point in the region can be expressed by reference, first to its position in the proper osculating plane, and next to the point upon the curve at which it is osculated by the plane. Accordingly, if we take the point P on the curve determined by the variable s, and in the osculating plane to the curve at P measure a distance t along the tangent and a distance u along the principal normal to the curve, where t and u are arbitrary independent quantities, the coordinates of a general point Q in the osculating developable region are given by the equations

$$\left. \begin{aligned} \bar{x} &= x + tl_1 + ul_2 \\ \bar{y} &= y + tm_1 + um_2 \\ \bar{z} &= z + tn_1 + un_2 \\ \bar{v} &= v + tk_1 + uk_2 \end{aligned} \right\},$$

where x, y, z, v; l_1, m_1, n_1, k_1; l_2, m_2, n_2, k_2, are functions of s. These equations may be taken as equivalent to the single equation of the region; in them, s, t, u, are the current parameters of the region. The tangent flat of the region is the osculating flat of the curve

$$(\bar{x}-x)\,l_4 + (\bar{y}-y)\,m_4 + (\bar{z}-z)\,n_4 + (\bar{v}-v)\,k_4 = 0.$$

In order to have an independent verification that the region thus defined is planar, we shall anticipate the later discussion (Chapter XVI) of the properties of regions so far as to use the results there established concerning the principal curvatures at a point of a region.

For this purpose, the primary magnitudes and the secondary magnitudes connected with a region are required. We have

$$\frac{\partial \bar{x}}{\partial s} = \left(1 - \frac{u}{\rho}\right) l_1 + \frac{t}{\rho} l_2 + \frac{u}{\sigma} l_3, \quad \frac{\partial \bar{x}}{\partial t} = l_1, \quad \frac{\partial \bar{x}}{\partial u} = l_2,$$

with corresponding values for the first derivatives of $\bar{y}$, $\bar{z}$, $\bar{v}$; and therefore the primary magnitudes for the region are

$$A = \Sigma \left(\frac{\partial \bar{x}}{\partial s}\right)^2 = \left(1 - \frac{u}{\rho}\right)^2 + \frac{t^2}{\rho^2} + \frac{u^2}{\sigma^2}, \qquad F = \Sigma \frac{\partial \bar{x}}{\partial t} \frac{\partial \bar{x}}{\partial u} = 0,$$

$$B = \Sigma \left(\frac{\partial \bar{x}}{\partial t}\right)^2 = 1, \qquad G = \Sigma \frac{\partial \bar{x}}{\partial s} \frac{\partial \bar{x}}{\partial u} = \frac{t}{\rho},$$

$$C = \Sigma \left(\frac{\partial \bar{x}}{\partial u}\right)^2 = 1, \qquad H = \Sigma \frac{\partial \bar{x}}{\partial s} \frac{\partial \bar{x}}{\partial t} = 1 - \frac{u}{\rho}.$$

Again, using the Frenet equations,

$$\frac{\partial^2 \bar{x}}{\partial s^2} = \left(-\frac{t}{\rho^2} + \frac{\rho'}{\rho^2} u\right) l_1 + \left\{\frac{1}{\rho} - \frac{\rho'}{\rho^2} t - \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) u\right\} l_2 + \left(\frac{t}{\rho\sigma} - u \frac{\sigma'}{\sigma^2}\right) l_3 + \frac{u}{\sigma\tau} l_4,$$

$$\frac{\partial^2 \bar{x}}{\partial s \partial t} = \frac{l_2}{\rho}, \quad \frac{\partial^2 \bar{x}}{\partial s \partial u} = -\frac{l_1}{\rho} + \frac{l_3}{\sigma}, \quad \frac{\partial^2 \bar{x}}{\partial t^2} = 0 = \frac{\partial^2 \bar{x}}{\partial t \partial u} = \frac{\partial^2 \bar{x}}{\partial u^2},$$

with corresponding values for the second derivatives of $\bar{y}, \bar{z}, \bar{v}$. Also l_4, m_4, n_4, k_4, are the direction-cosines of the normal to the region, because they are the direction-cosines of the normal to the flat which, osculating the curve, is tangential to the region. Hence the secondary magnitudes for the region are

$$L = \Sigma l_4 \frac{\partial^2 \bar{x}}{\partial s^2} = \frac{u}{\sigma\tau}, \quad M = \Sigma l_4 \frac{\partial^2 \bar{x}}{\partial s \partial t} = 0, \quad K = \Sigma l_4 \frac{\partial^2 \bar{x}}{\partial s \partial u} = 0,$$

$$N = \Sigma l_4 \frac{\partial^2 \bar{x}}{\partial t^2} = 0, \quad I = \Sigma l_4 \frac{\partial^2 \bar{x}}{\partial t \partial u} = 0, \quad J = \Sigma l_4 \frac{\partial^2 \bar{x}}{\partial u^2} = 0.$$

Now the principal radii of curvature of any region at a point are (§ 281) the roots of the cubic equation

$$\begin{vmatrix} \frac{A}{R} - L, & \frac{H}{R} - M, & \frac{G}{R} - K \\ \frac{H}{R} - M, & \frac{B}{R} - N, & \frac{F}{R} - I \\ \frac{G}{R} - K, & \frac{F}{R} - I, & \frac{C}{R} - J \end{vmatrix} = 0;$$

and therefore, in the present instance, these principal radii are the roots of the equation

$$\frac{1}{R^3} \frac{u^2}{\sigma^2} - \frac{L}{R^2} = 0.$$

Hence two of the values of $\frac{1}{R}$, being the principal curvatures of the region at a point, are zero everywhere: that is, the region contains planes, the property to be verified. For the non-vanishing curvature, we have

$$R = \frac{\tau}{\sigma} u.$$

Again, the direction at a point in the region at which one of the principal radii is a root of the foregoing general equation is determined (§ 283) by any two of the three equations

$$\left(\frac{A}{R} - L\right) ds + \left(\frac{H}{R} - M\right) dt + \left(\frac{G}{R} - K\right) du = 0,$$

$$\left(\frac{H}{R} - M\right) ds + \left(\frac{B}{R} - N\right) dt + \left(\frac{F}{R} - I\right) du = 0,$$

$$\left(\frac{G}{R} - K\right) ds + \left(\frac{F}{R} - I\right) dt + \left(\frac{C}{R} - J\right) du = 0.$$

Hence for the directions of the principal zero curvatures, the equations require

$$ds = 0,$$

and then are satisfied for any values of dt and du: again verifying the planar character of the surface. For the non-zero curvature of the surface, given as above by $R = \frac{\tau}{\sigma} du$, the second and the third equations become

$$H ds + dt = 0, \quad G ds + du = 0:$$

or the direction is given by the equations

$$\frac{dt}{1 - \frac{u}{\rho}} = \frac{du}{\frac{t}{\rho}} = -ds.$$

A flat, the equation of which involves only one parameter, as the fundamental element of a curve.

180. The investigation in § 175, whereby the reverse passage was made from the osculating flat of the given curve back to the curve itself, manifestly suggests that a flat may initially be taken as the fundamental element for the creation of the whole configuration. There must, of course, always be a proviso that it belongs to a singly-infinite family of flats: and the proviso can be satisfied by the analytical property that the equation of the typical flat of the family contains only a single parameter. We accordingly take a flat of this type, represented by an equation

$$l\bar{x} + m\bar{y} + n\bar{z} + k\bar{v} = P,$$

in which l, m, n, k, P, are functions of a single parameter t; and we proceed to take three flats next consecutive to this flat and to find their point of intersection. This point, so determined, is a point on the edge of regression of the region which envelopes the flat: that is, it is a point on a curve which can be regarded as thus generated.

At each stage, when a consecutive flat is introduced and is retained, the quantities $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{v}$, are current coordinates: initially, in the first flat:

secondly, also in the second flat (and thus in the osculating plane): thirdly, also in the third flat (and thus on the tangent): and fourthly, also in the fourth flat (thus becoming the representative point on the curve). In all these stages, these current coordinate variables $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{v}$, are not affected by the variations of t, which enable us to secure each successive flat.

To simplify the form of the analysis, we take a quantity θ, where

$$\theta = \frac{ds}{dt},$$

ds being now an element of arc of the edge of regression: obviously ds is as yet unknown, and θ is a magnitude to be determined. The introduction of this quantity θ enables us to compare each stage with the corresponding stage in the previous discussion (§ 175) of the osculating flat.

In the first place*, we take μ such that

$$l^2 + m^2 + n^2 + k^2 = \mu^2,$$

so that μ is known, we write

$$l = \mu l_4, \quad m = \mu m_4, \quad n = \mu n_4, \quad k = \mu k_4, \quad P = \mu p.$$

The equation of the flat now is

$$\bar{x} l_4 + \bar{y} m_4 + \bar{z} n_4 + \bar{v} k_4 = p,$$

where now $l_4^2 + m_4^2 + n_4^2 + k_4^2 = 1$; thus l_4, m_4, n_4, k_4, are direction-cosines. They can be taken as the direction-cosines of the trinormal; and they, as well as p, are known functions of t.

The association of the given flat with the consecutive flat can be effected by combining, with the given equation, the derived equation

$$\bar{x}\frac{dl_4}{dt} + \bar{y}\frac{dm_4}{dt} + \bar{z}\frac{dn_4}{dt} + \bar{v}\frac{dk_4}{dt} = \frac{dp}{dt}.$$

Let

$$\left(\frac{dl_4}{dt}\right)^2 + \left(\frac{dm_4}{dt}\right)^2 + \left(\frac{dn_4}{dt}\right)^2 + \left(\frac{dk_4}{dt}\right)^2 = \alpha^2,$$

so that α is a known function of t; and introduce another magnitude τ, expressible in terms of θ by the relation

$$\frac{\theta}{\tau} = \alpha.$$

We now write

$$\frac{dl_4}{dt} = -\frac{l_3}{\tau}\theta, \quad \frac{dm_4}{dt} = -\frac{m_3}{\tau}\theta, \quad \frac{dn_4}{dt} = -\frac{n_3}{\tau}\theta, \quad \frac{dk_4}{dt} = -\frac{k_3}{\tau}\theta,$$

so that

$$l_3^2 + m_3^2 + n_3^2 + k_3^2 = 1;$$

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* It might happen that we can dispense with this first stage; it has already been attained if, in the given equation of the flat, l, m, n, k, satisfy the condition $l^2+m^2+n^2+k^2=1$, so that they actually are direction-cosines.

and, since

$$l_4^2+m_4^2+n_4^2+k_4^2=1,$$

so that

$$l_4\frac{dl_4}{dt}+m_4\frac{dm_4}{dt}+n_4\frac{dn_4}{dt}+k_4\frac{dk_4}{dt}=0,$$

we have

$$l_3l_4+m_3m_4+n_3n_4+k_3k_4=0.$$

Because

$$\frac{dl_4}{dt}=-\alpha l_3,\quad \frac{dm_4}{dt}=-\alpha m_3,\quad \frac{dn_4}{dt}=-\alpha n_3,\quad \frac{dk_4}{dt}=-\alpha k_3,$$

and because $\frac{dl_4}{dt}$, $\frac{dm_4}{dt}$, $\frac{dn_4}{dt}$, $\frac{dk_4}{dt}$, are known functions of t, as well as α, we can regard l_3, m_3, n_3, k_3, as known functions of t. Thus the second equation can be taken in the form

$$\bar{x}l_3+\bar{y}m_3+\bar{z}n_3+\bar{v}k_3=-\frac{1}{\alpha}\frac{dp}{dt}=p_3.$$

The quantities l_3, m_3, n_3, k_3, are the direction-cosines of the binormal; and when θ comes to be known, the curvature of tilt is known, being $\frac{1}{\tau}, =\frac{\alpha}{\theta}$. Also, we have established the equations

$$\Sigma l_3^2=1,\quad \Sigma l_3l_4=0.$$

The association of these two flats with the next consecutive flat can be effected by combining, with the two preceding equations linear in $\bar{x}, \bar{y}, \bar{z}, \bar{v}$, the further derived equation

$$\bar{x}\frac{dl_3}{dt}+\bar{y}\frac{dm_3}{dt}+\bar{z}\frac{dn_3}{dt}+\bar{v}\frac{dk_3}{dt}=\frac{dp_3}{dt}.$$

Now, as $\Sigma l_3l_4=0$, we have

$$\Sigma l_4\frac{dl_3}{dt}=-\Sigma l_3\frac{dl_4}{dt}=\alpha;$$

and therefore we take quantities $l_2, m_2, n_2, k_2, \beta$, such that

$$\frac{dl_3}{dt}=-\beta l_2+\alpha l_4,\quad \frac{dm_3}{dt}=-\beta m_2+\alpha m_4,\quad \frac{dn_3}{dt}=-\beta n_2+\alpha n_4,\quad \frac{dk_3}{dt}=-\beta k_2+\alpha k_4.$$

In the first place, we have

$$\Sigma l_4\frac{dl_3}{dt}=-\beta\Sigma l_2l_4+\alpha,$$

and therefore

$$\Sigma l_2l_4=0.$$

Again, we take $l_2^2+m_2^2+n_2^2+k_2^2=1$, and then

$$\left(\frac{dl_3}{dt}\right)^2+\left(\frac{dm_3}{dt}\right)^2+\left(\frac{dn_3}{dt}\right)^2+\left(\frac{dk_3}{dt}\right)^2=\beta^2+\alpha^2;$$

or, since the left-hand side is a known function of t, as also is α, we can regard β as a known function of t. Further, because $\Sigma l_3^2 = 1$, we have

$$\Sigma l_3 \frac{dl_3}{dt} = 0,$$

and therefore

$$-\beta \Sigma l_2 l_3 + \alpha \Sigma l_3 l_4 = 0,$$

that is,

$$\Sigma l_2 l_3 = 0.$$

Thus the third equation can be taken in the form

$$\bar{x} l_2 + \bar{y} m_2 + \bar{z} n_2 + v k_2 = \frac{1}{\beta}\left(\alpha p - \frac{dp_3}{dt}\right) = p_2.$$

The quantities l_2, m_2, n_2, k_2, are the direction-cosines of the principal normal; and when θ comes to be known, the curvature of torsion is known, being $\frac{1}{\sigma}, = \frac{\beta}{\theta}$. Also, we have established the equations

$$\Sigma l_2^2 = 1, \quad \Sigma l_2 l_4 = 0, \quad \Sigma l_2 l_3 = 0.$$

The association of these three flats with the next (and only remaining) consecutive flat can be effected by combining, with the three preceding equations which together represent the first three flats, the further derived equation

$$\bar{x} \frac{dl_2}{dt} + \bar{y} \frac{dm_2}{dt} + \bar{z} \frac{dn_2}{dt} + \bar{v} \frac{dk_2}{dt} = \frac{dp_2}{dt}.$$

Now as $\Sigma l_2 l_3 = 0$, we have

$$\Sigma \frac{dl_2}{dt} l_3 = -\Sigma l_2 \frac{dl_3}{dt} = \beta;$$

and therefore we take quantities $l_1, m_1, n_1, k_1, \gamma$, such that

$$\frac{dl_2}{dt} = \beta l_3 - \gamma l_1, \quad \frac{dm_2}{dt} = \beta m_3 - \gamma m_1, \quad \frac{dn_2}{dt} = \beta n_3 - \gamma n_1, \quad \frac{dk_2}{dt} = \beta k_3 - \gamma k_1.$$

In the first place, we have

$$\Sigma l_3 \frac{dl_2}{dt} = \beta - \gamma \Sigma l_1 l_3,$$

and therefore

$$\Sigma l_1 l_3 = 0.$$

Again, we take $l_1^2 + m_1^2 + n_1^2 + k_1^2 = 1$, and then

$$\left(\frac{dl_2}{dt}\right)^2 + \left(\frac{dm_2}{dt}\right)^2 + \left(\frac{dn_2}{dt}\right)^2 + \left(\frac{dk_2}{dt}\right)^2 = \beta^2 + \gamma^2;$$

or, since the left-hand side is a known function of t, as also is β, we can regard γ as a known function of t. Further, because $\Sigma l_2^2 = 1$, we have

$$\Sigma l_2 \frac{dl_2}{dt} = 0.$$

so that

$$\beta \Sigma l_2 l_3 - \gamma \Sigma l_1 l_2 = 0,$$

and therefore

$$\Sigma l_1 l_2 = 0.$$

Further, because $\Sigma l_2 l_4 = 0$, we have

$$\Sigma l_2 \frac{dl_4}{dt} + \Sigma l_4 \frac{dl_2}{dt} = 0,$$

that is,

$$- \alpha \Sigma l_2 l_3 + \beta \Sigma l_3 l_4 - \gamma \Sigma l_1 l_4 = 0;$$

and therefore

$$\Sigma l_1 l_4 = 0.$$

The fourth equation can thus be taken in the form

$$\bar{x} l_1 + \bar{y} m_1 + \bar{z} n_1 + \bar{v} k_1 = \frac{1}{\gamma}\left(\beta p_3 - \frac{dp_2}{dt}\right) = p_1.$$

The quantities l_1, m_1, n_1, k_1, are the direction-cosines of the tangent to the edge of regression; and when θ comes to be known, the plane curvature is known, being $\frac{1}{\rho}$, $= \frac{\gamma}{\theta}$. Also, we have established the equations

$$\Sigma l_1^2 = 1, \quad \Sigma l_1 l_2 = 0, \quad \Sigma l_1 l_3 = 0, \quad \Sigma l_1 l_4 = 0.$$

We thus have four equations,

$$\bar{x} l_r + \bar{y} m_r + \bar{z} n_r + \bar{v} k_r = p_r,$$

(for $r = 1, 2, 3, 4$); the quantities $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{v}$, being the current coordinates throughout, now are the coordinates of the single point common to the four flats, that is, they now are the coordinates of the point on the edge of regression of the developable region which is the envelope of the original flat.

It remains to determine the (still unknown) quantity θ which has persisted throughout. The four lines l_r, m_r, n_r, k_r, for $r = 1, 2, 3, 4$, are an orthogonal system: hence l_1, l_2, l_3, l_4, are the direction-cosines of OX with respect to the tangent, the principal normal, the binormal, and the trinormal: hence

$$l_1^2 + l_2^2 + l_3^2 + l_4^2 = 1,$$

and therefore

$$l_1 \frac{dl_1}{dt} + l_2 \frac{dl_2}{dt} + l_3 \frac{dl_3}{dt} + l_4 \frac{dl_4}{dt} = 0,$$

that is,

$$l_1 \frac{dl_1}{dt} + l_2 (\beta l_3 - \gamma l_1) + l_3 (-\beta l_2 - \alpha l_4) + l_4 (-\alpha l_3) = 0,$$

and therefore

$$\frac{dl_1}{dt} = \gamma l_2.$$

Similarly,

$$\frac{dm_1}{dt} = \gamma m_2, \quad \frac{dn_1}{dt} = \gamma n_2, \quad \frac{dk_1}{dt} = \gamma k_2.$$

Next, the four equations determine the point on the edge of regression which may be denoted by x, y, z, v. Let ds be the element of arc of this edge, so that

$$x'^2 + y'^2 + z'^2 + v'^2 = 1,$$

while

$$\frac{dx}{dt} = x'\theta, \quad \frac{dy}{dt} = y'\theta, \quad \frac{dz}{dt} = z'\theta, \quad \frac{dv}{dt} = v'\theta.$$

Now x, y, z, v, as functions of t, satisfy the four equations

$$\Sigma l_4 x = p_4, \quad \Sigma l_3 x = p_3, \quad \Sigma l_2 x = p_2, \quad \Sigma l_1 x = p_1.$$

From the first of these equations, we have

$$\Sigma l_4 \frac{dx}{dt} + \Sigma x \frac{dl_4}{dt} = \frac{dp_4}{dt};$$

also

$$\frac{dl_4}{dt} = -\alpha l_3, \quad \frac{dp_4}{dt} = -\alpha p_3;$$

and therefore

$$\Sigma l_4 \frac{dx}{dt} = 0,$$

in virtue of the second of the four equations. that is, $\theta \Sigma l_4 x' = 0$. From the second equation, we have

$$\Sigma l_3 \frac{dx}{dt} + \Sigma x \frac{dl_3}{dt} = \frac{dp_3}{dt};$$

also

$$\frac{dl_3}{dt} = -\beta l_2 + \alpha l_4, \quad \frac{dp_3}{dt} = -\beta p_2 + \alpha p_4,$$

and therefore

$$\Sigma l_3 \frac{dx}{dt} = 0,$$

in virtue of the first and the third of the four equations: that is, $\theta \Sigma l_3 x' = 0$. From the third equation, we have

$$\Sigma l_2 \frac{dx}{dt} + \Sigma x \frac{dl_2}{dt} = \frac{dp_2}{dt};$$

also

$$\frac{dl_2}{dt} = \beta l_3 - \gamma l_1, \quad \frac{dp_2}{dt} = \beta p_3 - \gamma p_1;$$

and therefore

$$\Sigma l_2 \frac{dx}{dt} = 0,$$

in virtue of the second and the fourth of the four equations: that is, $\theta \Sigma l_2 x' = 0$. Hence, as

$$\Sigma l_4 x' = 0, \quad \Sigma l_3 x' = 0, \quad \Sigma l_2 x' = 0,$$

we have

$$\frac{x'}{l_1} = \frac{y'}{m_1} = \frac{z'}{n_1} = \frac{v'}{k_1}$$

and $\Sigma x'^2 = 1$, $\Sigma l_1^2 = 1$, so that

$$l_1 = x', \quad m_1 = y', \quad n_1 = z', \quad k_1 = v'.$$

Finally, we have, from the equation $\Sigma x l_1 = p_1$, the relation

$$\Sigma \frac{dx}{dt} l_1 + \Sigma x \frac{dl_1}{dt} = \frac{dp_1}{dt},$$

that is,

$$\theta \Sigma x' l_1 + \gamma \Sigma x l_2 = \frac{dp_1}{dt};$$

and therefore

$$\theta = \frac{dp_1}{dt} - \gamma p_2.$$

Thus θ is known, as a function of t.

Summary of the preceding investigation, determining a curve from a flat.

181. The net results of the investigation are as follows:

The initial equation of the flat is

$$\bar{x} l + \bar{y} m + \bar{z} n + \bar{v} k = P,$$

where l, m, n, k, P, are functions of a parameter t. We take

$$\Sigma l^2 = \mu^2, \quad P = p\mu,$$

$$\frac{l_4}{l} = \frac{m_4}{m} = \frac{n_4}{n} = \frac{k_4}{k} = \frac{1}{\mu},$$

so that l_4, m_4, n_4, k_4, are definite functions of t. We next take

$$\left(\frac{dl_4}{dt}\right)^2 + \left(\frac{dm_4}{dt}\right)^2 + \left(\frac{dn_4}{dt}\right)^2 + \left(\frac{dk_4}{dt}\right)^2 = \alpha^2,$$

defining α, and

$$\frac{1}{l_3}\frac{dl_4}{dt} = \frac{1}{m_3}\frac{dm_4}{dt} = \frac{1}{n_3}\frac{dn_4}{dt} = \frac{1}{k_3}\frac{dk_4}{dt} = -\alpha,$$

so that l_3, m_3, n_3, k_3, are definite functions of t. We next take

$$\left(\frac{dl_3}{dt}\right)^2 + \left(\frac{dm_3}{dt}\right)^2 + \left(\frac{dn_3}{dt}\right)^2 + \left(\frac{dk_3}{dt}\right)^2 = \alpha^2 + \beta^2,$$

defining β, and

$$\beta l_2 = -\frac{dl_3}{dt} + \alpha l_4, \quad \beta m_2 = -\frac{dm_3}{dt} + \alpha m_4, \quad \beta n_2 = -\frac{dn_3}{dt} + \alpha n_4, \quad \beta k_2 = -\frac{dk_3}{dt} + \alpha k_4,$$

so that l_2, m_2, n_2, k_2, are definite functions of t. We next take

$$\left(\frac{dl_2}{dt}\right)^2 + \left(\frac{dm_2}{dt}\right)^2 + \left(\frac{dn_2}{dt}\right)^2 + \left(\frac{dk_2}{dt}\right)^2 = \beta^2 + \gamma^2,$$

defining γ, and

$$\gamma l_1 = -\frac{dl_2}{dt} + \beta l_3, \quad \gamma m_1 = -\frac{dm_2}{dt} + \beta m_3, \quad \gamma n_1 = -\frac{dn_2}{dt} + \beta n_3, \quad \gamma k_1 = -\frac{dk_2}{dt} + \beta k_3,$$

so that l_1, m_1, n_1, k_1, are definite functions of t. We write

$$p_3 = -\frac{1}{\alpha}\frac{d}{dt}\left(\frac{P}{\mu}\right) = -\frac{1}{\alpha}\frac{dp}{dt},$$

$$p_2 = \frac{1}{\beta}\left(-\frac{dp_3}{dt} + \alpha p\right),$$

$$p_1 = \frac{1}{\gamma}\left(-\frac{dp_2}{dt} + \beta p_3\right),$$

and, finally, we take a quantity θ, given by

$$\theta = \frac{dp_1}{dt} - \gamma p_2.$$

Then *the four equations*

$$\Sigma l_4 x = p, \quad \Sigma l_3 x = p_3, \quad \Sigma l_2 x = p_2, \quad \Sigma l_1 x = p_1,$$

give coordinates x, y, z, v, of a point upon a curve, being the edge of regression of the developable region which is the envelope of the initial flat $\Sigma l_4 x = p$. The quantity θ is equal to $\frac{ds}{dt}$, where ds is an elementary arc of this curve. The four sets of lines l_r, m_r, n_r, k_r, for $r = 1, 2, 3, 4$, are the chief lines of the curve at the point, being the tangent for $r = 1$, the principal normal for $r = 2$, the binormal for $r = 3$, and the trinormal for $r = 4$. The radii of curvature of the curve, (viz., ρ, the radius of plane curvature; σ, the radius of torsion; τ, the radius of tilt), are

$$\rho = \frac{\theta}{\gamma}, \quad \sigma = \frac{\theta}{\beta}, \quad \tau = \frac{\theta}{\alpha}.$$

Finally, the relations by which the successive quantities l, m, n, k, are obtained, being

$$\frac{dc_4}{dt} = -\alpha c_3, \quad \frac{dc_3}{dt} = -\beta c_2 + \alpha c_4, \quad \frac{dc_2}{dt} = -\gamma c_1 + \beta c_3, \quad \frac{dc_1}{dt} = \gamma c_2,$$

are the Frenet equations when we substitute $\frac{ds}{dt}$ for θ.

The osculating flat of the curve is represented by the equation $\Sigma l_4 \bar{x} = p$; the osculating plane of the curve by the two equations $\Sigma l_4 \bar{x} = p$, $\Sigma l_3 \bar{x} = p_3$; and the tangent to the curve by the three equations $\Sigma l_4 \bar{x} = p$, $\Sigma l_3 \bar{x} = p_3$, $\Sigma l_2 \bar{x} = p_2$.

NOTE. Manifestly the developable region, which is the envelope of the original flat, is the t-eliminant of the two equations

$$\Sigma \bar{x} l_4 = p, \quad \Sigma \bar{x} l_3 = p_3.$$

As these two equations, for parametric values of t, represent the osculating planes of the edge of regression, it follows that the developable region is planar and that it contains, as its generating planes, all the osculating planes of that curve.

Again, the developable surface, which is the envelope of the osculating plane, is given by the two equations which, combined, constitute the t-eliminant of the three equations

$$\Sigma \bar{x} l_4 = p, \quad \Sigma \bar{x} l_3 = p_3, \quad \Sigma \bar{x} l_2 = p_2.$$

As these three equations, for parametric values of t, represent the tangents to the edge of regression, it follows that the developable surface has for its generators all the tangents to that curve.

Lastly, the edge of the regression, being the aggregate of the individual points given by the four equations

$$\Sigma \bar{x} l_4 = p, \quad \Sigma \bar{x} l_3 = p_3, \quad \Sigma \bar{x} l_2 = p_2, \quad \Sigma \bar{x} l_1 = p_1,$$

is represented by the three equations which, combined, constitute the t-eliminant of these three equations.

The edge of regression lies on the developable surface which, in turn, lies within the developable region.

There is, in fact, a reciprocally complete association between a skew curve and the developable region of which it is the edge of regression, the association being secured by means of the flat which, on the one hand, is tangential to the region and, on the other hand, is osculant to the curve

Ex. 1. Verify that the four equations

$$\Sigma l_4 x = p, \quad \Sigma l_3 x = p_1, \quad \Sigma l_2 x = p_2, \quad \Sigma l_1 x = p_1,$$

are equivalent to

$$\Sigma l x = P, \quad \Sigma \frac{dl}{dt} x = \frac{dP}{dt}, \quad \Sigma \frac{d^2 l}{dt^2} x = \frac{d^2 P}{dt^2}, \quad \Sigma \frac{d^3 l}{dt^3} x = \frac{d^3 P}{dt^3},$$

with the notation of the text; and deduce a value of θ.

Ex. 2. The equation of a flat is given in the form

$$v = px + qy + rz - t,$$

where p, q, r, are functions of the parameter t; shew that the direction-cosines l, m, n, λ, of the tangent to the edge of regression of the developable region, which is the envelope of the flat, are given by the relations

$$\frac{l}{\begin{vmatrix} q', & r' \\ q'', & r'' \end{vmatrix}} = \frac{m}{\begin{vmatrix} r', & p' \\ r'', & p'' \end{vmatrix}} = \frac{n}{\begin{vmatrix} p', & q' \\ p'', & q'' \end{vmatrix}} = \frac{\lambda}{\begin{vmatrix} p, & q, & r \\ p', & q', & r' \\ p'', & q'', & r'' \end{vmatrix}} = \frac{1}{\Delta},$$

where Δ^2 is the sum of the squares of the denominators, while p' denotes $\frac{dp}{dt}$, and so for p'', q', q'', r', r''; and that the element of arc of the edge of regression is given by

$$\begin{vmatrix} p', & q', & r' \\ p'', & q'', & r'' \\ p''', & q''', & r''' \end{vmatrix}^2 \frac{ds}{dt} = -\Delta \begin{vmatrix} p'', & q'', & r'' \\ p''', & q''', & r''' \\ p^{\text{iv}}, & q^{\text{iv}}, & r^{\text{iv}} \end{vmatrix}.$$

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* For the process of deformation of a developable region into a flat, see hereafter (§ 190).

Ex. 3 The equation of a flat is given in the form

$$\Theta = u_0 a^4 + 4u_1 a^3 + 6u_2 a^2 + 4u_3 a + u_4 = 0,$$

where u_0, u_1, u_2, u_3, u_4, are linear functions of x, y, z, v, while a is a parameter; the quadrinvariant and the cubinvariant of Θ are denoted by I and J, where

$$I = u_0 u_4 - 4u_1 u_3 + 3u_2^2, \quad J = u_0 u_2 u_4 + 2u_1 u_2 u_3 - u_1^2 u_4 - u_0 u_3^2 - u_2^3.$$

Shew that the developable region enveloped by the flat is represented by the equation $I^3 - 27J^2 = 0$: that the associated developable surface is given by the equations $I = 0$, $J = 0$; and that the edge of regression is given by the equations

$$\frac{u_0}{u_1} = \frac{u_1}{u_2} = \frac{u_2}{u_3} = \frac{u_3}{u_4},$$

(which make the Hessian of Θ vanish).

Ex. 4. A family of flats is given by the equation

$$x + 4ay + 6a^2 z + 4a^3 v + a^4 = 0;$$

and their developable envelope is constructed. Shew that, when this envelope is developed into a flat and the developable surface into a plane, the edge of regression is developed into a curve in this plane given intrinsically by the equations

$$\left.\begin{aligned} \left(\frac{ds}{da}\right)^2 &= 1 + 4a^2 + 9a^4 + 16a^6 \\ \frac{1}{4}\left(\frac{d\psi}{da}\right)^2 \left(\frac{ds}{da}\right)^4 &= 1 + 9a^2 + 45a^4 + 64a^6 + 36a^8 \end{aligned}\right\}.$$

Normal flat: developable region.

182 We now proceed to consider the normal flat at any point P of the curve, together with the region which envelopes the normal flats, and the developable surface accompanying this region. The edge of regression of this region and of this surface is the locus of G, the centre of globular curvature of the curve for we have seen (§ 148) that G is the intersection of four consecutive normal flats.

The equation of the normal flat at P is

$$\Sigma (\bar{x} - x) x' = 0.$$

The association, with this initial normal flat, of the normal flat at a consecutive point, that is, of the consecutive normal flat, is effected by combining, with this equation, the derived equation

$$\Sigma (\bar{x} - x) x'' = \Sigma x'^2 = 1.$$

The association of a (third and) consecutive normal flat is effected by associating, with the two preceding equations, the further derived equation

$$\Sigma (\bar{x} - x) x''' = \Sigma x' x'' = 0.$$

The association of the fourth (and last) consecutive normal flat is effected by associating, with the three preceding equations, the further derived equation

$$\Sigma (\bar{x} - x) x^{\text{iv}} = \Sigma x' x''' = -\frac{1}{\rho^2}.$$

We know (§ 148) that the common values of $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{v}$, provided by these equations are the coordinates of the centre G of globular curvature, denoted by ξ_3, η_3, ζ_3, v_3; and thus the four equations can be written

$$\begin{aligned}\Sigma(\bar{x}-\xi_3)\,x' &= 0,\\ \Sigma(\bar{x}-\xi_3)\,x'' &= 0,\\ \Sigma(\bar{x}-\xi_3)\,x''' &= 0,\\ \Sigma(\bar{x}-\xi_3)\,x^{\text{iv}} &= 0.\end{aligned}$$

The first equation is

$$\Sigma(\bar{x}-\xi_3)\,l_1 = 0.$$

The second equation is

$$\Sigma(\bar{x}-\xi_3)\,l_2 = 0;$$

and these two equations are, together, the plane of cleavage of the first normal flat and its prime consecutive.

Instead of the given third equation associated with the first two equations, we can take a new third equation

$$\Sigma(\bar{x}-\xi_3)\,l_3 = 0,$$

by using the values of l_3, m_3, n_3, k_3, which are represented by

$$l_3 = \frac{\sigma}{\rho}(x' + \rho\rho' x'' + \rho^2 x''');$$

and these three equations, together, are the line which is the intersection of the first normal flat with its two next consecutives.

Finally, having regard to the relations (§ 151)

$$x^{\text{iv}} = \frac{1}{\rho\sigma\tau}\,l_4 - \frac{1}{\rho\sigma}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)l_3 + M\rho l_2 + 3\frac{\rho'}{\rho^3}\,l_1,$$

with like values of y^{iv}, z^{iv}, v^{iv}, we can change the fourth equation, when taken in combination with the first three, into the equation

$$\Sigma(\bar{x}-\xi_3)\,l_4 = 0.$$

The four modified equations, taken together, give the point which is the intersection of the four consecutive normal flats, that is, they give the point G.

The globular centric of a curve.

183. We shall call the locus of G the *globular centric* of the original curve.

The first equation $\Sigma(\bar{x}-\xi_3)\,l_1 = 0$ is the osculating flat at G of the globular centric; after the investigation of § 180, we shall write this

$$\Sigma(\bar{x}-\xi_3)\,\lambda_4 = 0,$$

so that λ_4, μ_4, ν_4, κ_4, $= l_1$, m_1, n_1, k_1, are the direction-cosines of the tri-normal to the globular centric.

The two equations $\Sigma(\bar{x}-\xi_3)l_1=0$, and $\Sigma(\bar{x}-\xi_3)l_2=0$, are the intersection of the osculating flat at G and the osculating flat at the consecutive point: that is, together they represent the osculating plane of the globular centric at G. Accordingly, after the same investigation, we write the second equation in the form

$$\Sigma(\bar{x}-\xi_3)\lambda_3=0,$$

so that $\lambda_3, \mu_3, \nu_3, \kappa_3, = l_2, m_2, n_2, k_2$, are the direction-cosines of the binormal to the globular centric at G.

The three equations $\Sigma(\bar{x}-\xi_3)l_1=0$, $\Sigma(\bar{x}-\xi_3)l_2=0$, and $\Sigma(\bar{x}-\xi_3)l_3=0$, are the intersection of the osculating flat at G and the osculating flats at the two consecutive points: that is, together they represent the tangent at G to the globular centric. In accordance with the same investigation, we write the third equation in the form

$$\Sigma(\bar{x}-\xi_3)\lambda_2=0,$$

so that $\lambda_2, \mu_2, \nu_2, \kappa_2, = l_3, m_3, n_3, k_3$, are the direction-cosines of the principal normal to the globular centric at G.

The three directions $\lambda_r, \mu_r, \nu_r, \kappa_r$, for $r=4, 3, 2$, are orthogonal to one another: the fourth direction, completing the orthogonal direction at G, is that of the tangent, and may be denoted by $\lambda_1, \mu_1, \nu_1, \kappa_1$. But the fourth direction, perpendicular to each of the three specified directions, which are l_p, m_p, n_p, k_p, for $p=1, 2, 3$, respectively, is l_4, m_4, n_4, k_4; so we take the fourth equation in the form

$$\Sigma(\bar{x}-\xi_3)\lambda_1=0,$$

so that $\lambda_1, \mu_1, \nu_1, \kappa_1, = l_4, m_4, n_4, k_4$, are the direction-cosines of the tangent at G to the globular centric.

Curvatures of the globular centric.

184. In all these equations, the sets of direction-cosines and the coordinates of G are expressed in terms of s, the arc of the original curve, but now only a parameter for the globular centric. We require the arc-element at G of the globular centric, in order to obtain the curvatures; and this arc, ds_3, has already (§ 160) been expressed in terms of s. It was proved that

$$\theta=\frac{ds_3}{ds}=\frac{d}{ds}\left(R\frac{\tau R'}{\sigma\rho'}\right)+\frac{\sigma\rho'}{\tau},$$

which can be expressed in various forms, thus we can regard ds_3 as known.

We denote by ρ_3, σ_3, τ_3, the radii of plane (or circular) curvature, of torsion, and of tilt, of the globular centric at G. The direction-cosines of its tangent are $\lambda_1, \mu_1, \nu_1, \kappa_1$; of its principal normal are $\lambda_2, \mu_2, \nu_2, \kappa_2$; of its binormal are $\lambda_3, \mu_3, \nu_3, \kappa_3$; and of its trinormal are $\lambda_4, \mu_4, \nu_4, \kappa_4$. The Frenet equations for the globular centric are

$$\left.\begin{aligned} \frac{d\lambda_1}{ds_3} &= \frac{\lambda_2}{\rho_3} \\ \frac{d\lambda_2}{ds_3} &= -\frac{\lambda_1}{\rho_3} + \frac{\lambda_3}{\sigma_3} \\ \frac{d\lambda_3}{ds_3} &= -\frac{\lambda_2}{\sigma_3} + \frac{\lambda_4}{\tau_3} \\ \frac{d\lambda_4}{ds_3} &= -\frac{\lambda_3}{\tau_3} \end{aligned}\right\}.$$

But these equations can be written

$$\frac{l_3}{\rho_3} = \frac{1}{\theta}\frac{dl_4}{ds} = -\frac{1}{\theta}\frac{l_3}{\tau},$$

$$-\frac{l_4}{\rho_3} + \frac{l_2}{\sigma_3} = \frac{1}{\theta}\frac{dl_3}{ds} = \frac{1}{\theta}\left(-\frac{l_2}{\sigma} + \frac{l_4}{\tau}\right),$$

$$-\frac{l_3}{\sigma_3} + \frac{l_1}{\tau_3} = \frac{1}{\theta}\frac{dl_2}{ds} = \frac{1}{\theta}\left(-\frac{l_1}{\rho} + \frac{l_3}{\sigma}\right),$$

$$-\frac{l_2}{\tau_3} = \frac{1}{\theta}\frac{dl_1}{ds} = \frac{1}{\theta}\frac{l_2}{\rho};$$

and therefore we have

$$\left.\begin{aligned} \rho_3 &= -\theta\tau \\ \sigma_3 &= -\theta\sigma \\ \tau_3 &= -\theta\rho \end{aligned}\right\}.$$

We thus have the radii of circular curvature, of torsion, and of tilt, of the globular centric at the point G, which is the centre of globular curvature of the original curve at P.

The radius of spherical curvature R_3 is given by

$$R_3^2 = \rho_3^2 + \sigma_3^2\left(\frac{d\rho_3}{ds_3}\right)^2 = \rho_3^2 + \sigma^2\left(\frac{d\rho_3}{ds}\right)^2.$$

and the radius of globular curvature Γ_3 is given by

$$\begin{aligned} \Gamma_3^2 &= R_3^2 + \left\{\frac{d}{ds_3}\left(\frac{\tau_3}{\sigma_3}R_3\frac{dR_3}{d\rho_3}\right)\right\}^2 \\ &= R_3^2 + \frac{1}{\theta^2}\left\{\frac{d}{ds}\left(\frac{\rho}{\sigma}R_3\frac{R_3'}{\rho_3'}\right)\right\}^2. \end{aligned}$$

Further, if $d\epsilon_3, d\eta_3, d\omega_3$, are the angles of contingence, of torsion, and of tilt, respectively, for the globular centric, we have

$$\rho_3 = \frac{ds_3}{d\epsilon_3}, \quad \sigma_3 = \frac{ds_3}{d\eta_3}, \quad \tau_3 = \frac{ds_3}{d\omega_3};$$

and therefore

$$d\epsilon_3 = -d\omega, \quad d\eta_3 = -d\eta, \quad d\omega_3 = -d\epsilon.$$

Properties of the globular centric.

185. From the relations

$$\lambda_r = l_{5-r}, \quad \mu_r = m_{5-r}, \quad \nu_r = n_{5-r}, \quad \kappa_r = k_{5-r},$$

for $r = 1, 2, 3, 4$, various inferences can immediately be derived:

(i) the normal flat of the locus of P is the osculating flat of the locus of G, the globular centric;

(ii) the osculating flat of the locus of P and the normal flat of the locus of G are parallel;

(iii) the orthogonal plane of the locus of P and the osculating plane of the locus of G are parallel,

(iv) the osculating plane of the locus of P and the orthogonal plane of the locus of G are parallel.

All of them follow from the properties that

the tangent to the G-locus is parallel to the trinormal of the P-locus,
. principal normal .. . . binormal ....,
... binormal ....principal normal .. ..,
... trinormal .. ..tangent of the P-locus.

Again, let the globular centric of the G-locus, itself a new curve in the configuration, be called the second globular centric of the P-locus; let its radii of plane curvature, of torsion, and of tilt, be P, Σ, T; and let its element of arc be dS. Then

$$\frac{\mathrm{P}}{\rho} = \frac{\Sigma}{\sigma} = \frac{\mathrm{T}}{\tau} = \frac{dS}{ds}.$$

Manifestly the orthogonal frame of the second globular centric of the P-locus is similarly situated to that of the P-locus itself, with the tangents parallel, the principal normals parallel, the binormals parallel, and the trinormals parallel; and the two loci have the same angles of contingence, of torsion, and of tilt.

Further the developable region, which is the envelope of the normal flat of the P-locus and which may be called the *developable normal region* of the P-locus, is the s-eliminant of the two equations

$$\Sigma(\bar{x} - x)x' = 0, \quad \Sigma(\bar{x} - x)x'' = 1,$$

or, what is the same thing, of the two equations

$$\Sigma(\bar{x} - \xi_3)x' = 0, \quad \Sigma(\bar{x} - \xi_3)x'' = 0.$$

As these two equations, for parametric values of s, represent a plane parallel to the orthogonal plane of P (it does not pass through P, but through S and G), it follows that the developable normal region of the P-locus is planar.

Similarly, for the envelope of these planes, which may be regarded as the eliminant of the three equations

$$\Sigma(\overline{x}-\xi_3)\,l_1=0,\quad \Sigma(\overline{x}-\xi_3)\,l_2=0,\quad \Sigma(\overline{x}-\xi_3)\,l_3=0,$$

which represent a line: it is a ruled surface. As consecutive lines meet, this ruled surface is a developable surface.

Manifestly, the developable normal region, the foregoing developable surface, and its edge of regression, are the osculating developable region and the osculating developable surface of the G-locus, that is, of the globular centric of the original curve.

Ex Given a flat $Ax+By+Cz+Dv=E$, where A, B, C, D, E, are functions of a parameter t: prove that the normal flat at the edge of regression has for its equation

$$\begin{vmatrix} x, & y, & z, & v \\ A, & B, & C, & D \\ A', & B', & C', & D' \\ A'', & B'', & C'', & D'' \end{vmatrix}\begin{vmatrix} A, & B, & C, & D \\ A', & B', & C', & D' \\ A'', & B'', & C'', & D'' \\ A''', & B''', & C''', & D''' \end{vmatrix}=\begin{vmatrix} E, & E', & E'', & E''' \\ \Sigma A^2, & \Sigma AA', & \Sigma AA'', & \Sigma AA''' \\ \Sigma A'A, & \Sigma A'^2, & \Sigma A'A'', & \Sigma A'A''' \\ \Sigma A''A, & \Sigma A''A', & \Sigma A''^2, & \Sigma A''A''' \end{vmatrix},$$

where

$$E'=\frac{dE}{dt},\quad E''=\frac{d^2E}{dt^2},\quad \ldots,\quad \Sigma A^2=A^2+B^2+C^2+D^2,\quad \Sigma AA'=AA'+BB'+CC'+DD',$$

and so for the other magnitudes.

Rectifying flat: developable region: rectifying plane and line.

186. Of the four principal flats at any point of a curve, the two which have already been considered—the osculating flat, to which the trinormal is perpendicular: and the normal flat, to which the tangent is perpendicular—appear to be the most important intrinsically, when associated with their configurations of enveloping regions. One of the remaining two flats calls for some consideration: it is the flat, to which the principal normal is perpendicular; and, for a reason which will appear later, it is called the *rectifying flat.*

Its equation, with the preceding notation, is

$$\Sigma(\overline{x}-x)\,l_2=0.$$

To obtain the envelope of this flat, called the *rectifying developable region,* we begin in a manner different from that used to determine the regions that envelope the osculating flat and the normal flat. Let

$$X_r=\Sigma(\overline{x}-x)\,l_r,$$

for $r = 1, 2, 3, 4$. Then, proceeding along the curve and using the Frenet equations, we have

$$\left.\begin{aligned}
\frac{dX_1}{ds} &= \Sigma\left\{(\bar{x}-x)\frac{dl_1}{ds} - l_1^2\right\} = \frac{1}{\rho}X_2 - 1\\
\frac{dX_2}{ds} &= \Sigma\left\{(\bar{x}-x)\frac{dl_2}{ds} - l_1 l_2\right\} = -\frac{1}{\rho}X_1 + \frac{1}{\sigma}X_3\\
\frac{dX_3}{ds} &= \Sigma\left\{(\bar{x}-x)\frac{dl_3}{ds} - l_1 l_3\right\} = -\frac{1}{\sigma}X_2 + \frac{1}{\tau}X_4\\
\frac{dX_4}{ds} &= \Sigma\left\{(\bar{x}-x)\frac{dl_4}{ds} - l_1 l_4\right\} = -\frac{1}{\tau}X_3
\end{aligned}\right\}.$$

In the first place, the equation of the rectifying flat is

$$X_2 = 0.$$

When the rectifying flat at the consecutive point of the curve is associated with this flat, its equation, in combination with $X_2 = 0$, can be taken as

$$\frac{dX_2}{ds} = 0,$$

that is, in effect,

$$\sigma X_1 - \rho X_3 = 0.$$

The two equations

$$X_2 = 0, \quad \sigma X_1 - \rho X_3 = 0,$$

represent a plane: this plane is called the *rectifying plane*: and, as is easily verified, its two equations can be taken in the form

$$\left\|\begin{matrix}
\bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v\\
\rho l_1 + \sigma l_3, & \rho m_1 + \sigma m_3, & \rho n_1 + \sigma n_3, & \rho k_1 + \sigma k_3\\
l_4, & m_4, & n_4, & k_4
\end{matrix}\right\| = 0.$$

Similarly, when the rectifying flat at the next consecutive point of the curve is associated with these two flats, its equation, in combination with the equations of the two preceding rectifying flats, can be taken in the form

$$\sigma\frac{dX_1}{ds} - \rho\frac{dX_3}{ds} + X_1\sigma' - X_3\rho' = 0,$$

which, by the use of the two preceding equations, can be represented in the form

$$\left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho}\right)X_3 - \frac{1}{\tau}X_4 = \frac{\sigma}{\rho}.$$

The three equations

$$X_2 = 0, \quad \sigma X_1 - \rho X_3 = 0, \quad \left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho}\right)X_3 - \frac{1}{\tau}X_4 = \frac{\sigma}{\rho},$$

represent a line: this line is called the *rectifying line*. Let the direction-cosines of this line be denoted by $\lambda_1, \mu_1, \nu_1, \kappa_1$, so that this direction lies in each of these flats: then

$$\Sigma l_2 \lambda_1 = 0,$$

$$\Sigma (\sigma l_1 - \rho l_3) \lambda_1 = 0,$$

$$\Sigma \left\{ \left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho} \right) l_3 - \frac{1}{\tau} l_4 \right\} \lambda_1 = 0.$$

Thus, if

$$\lambda_1 = \alpha l_1 + \beta l_2 + \gamma l_3 + \delta l_4,$$

with corresponding expressions for μ_1, ν_1, κ_1, we have

$$\beta = 0,$$

$$\alpha\sigma - \gamma\rho = 0,$$

$$\gamma \left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho} \right) - \delta \frac{1}{\tau} = 0,$$

so that $\lambda_1, \mu_1, \nu_1, \kappa_1$, are proportional to

$$\frac{\rho}{\tau} (\rho c_1 + \sigma c_3) - (\sigma\rho' - \rho\sigma') c_4,$$

for $c = l, m, n, k$, in succession. The factor, to give the actual values of $\lambda_1, \mu_1, \nu_1, \kappa_1$, is determinable by the relation $\Sigma \lambda_1^2 = 1$. Later, we shall return to these values: and we shall express the equations of the rectifying line in the customary form of such equations.

The last of the four consecutive flats in this connection can have its equation, when combined with the equations of the preceding flats, taken in the form

$$\left\{ \frac{d}{ds} \left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho} \right) \right\} X_3 + \left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho} \right) \left(-\frac{1}{\sigma} X_2 + \frac{1}{\tau} X_4 \right) - \left\{ \frac{d}{ds} \left(\frac{1}{\tau} \right) \right\} X_4 + \frac{1}{\tau^2} X_3 = \frac{d}{ds} \left(\frac{\sigma}{\rho} \right),$$

which, by combination with the other equations, can be modified to

$$\left\{ \tau \frac{d}{ds} \left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho} \right) + \frac{1}{\tau} \right\} X_3 + \left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho} + \frac{\tau'}{\tau} \right) X_4 = \tau \frac{d}{ds} \left(\frac{\sigma}{\rho} \right).$$

Edge of regression of the rectifying developable region.

187. These four equations are not yet in the form convenient for subsequent purposes: they are to represent the four principal flats for the edge of regression of the rectifying developable region, which is the envelope of the flat $X_2 = 0$: and we therefore have to obtain, in place of them, the equations of those four flats, the actual forms being suggested by the results of the investigation in § 180. Further, in order to obtain the curvatures of this edge of regression, we shall need a knowledge of the element of arc of that edge, as well as the coordinates of a current point on the edge.

Accordingly, we proceed to obtain the equations of the four orthogonal flats. For the first of them, which is the rectifying flat of the given curve, and also is the osculating flat of the edge of regression of the rectifying developable region, we have $X_2 = 0$ as its equation; or, writing

$$\lambda_4,\ \mu_4,\ \nu_4,\ \kappa_4 = l_2,\ m_2,\ n_2,\ k_2,$$

we have

$$\Sigma\,(\bar{x} - x)\,\lambda_4 = 0$$

as the equation of the first flat.

For the second flat (§ 180), we have

$$\Sigma\left\{(\bar{x} - x)\frac{d\lambda_4}{ds} - x'\lambda_4\right\} = 0.$$

Now $x' = l_1$, $\lambda_4 = l_2$, and therefore $\Sigma x'\lambda_4 = 0$. Also

$$\frac{d\lambda_4}{ds} = \frac{dl_2}{ds} = -\frac{1}{\rho}l_1 + \frac{1}{\sigma}l_3,$$

so that

$$\alpha^2 = \Sigma\left(\frac{d\lambda_4}{ds}\right)^2 = \frac{1}{\rho^2} + \frac{1}{\sigma^2}.$$

We then take

$$\frac{d\lambda_4}{ds} = -\alpha\lambda_3,$$

so that, if

$$\frac{\sigma}{(\rho^2 + \sigma^2)^{\frac{1}{2}}} = \cos A, \quad \frac{\rho}{(\rho^2 + \sigma^2)^{\frac{1}{2}}} = \sin A, \quad \frac{\rho}{\sigma} = \tan A,$$

we have

$$\lambda_3 = l_1\cos A - l_3\sin A = \frac{1}{(\rho^2 + \sigma^2)^{\frac{1}{2}}}(\sigma l_1 - \rho l_3).$$

The second equation thus is

$$\Sigma\,(\bar{x} - x)\,\lambda_3 = 0.$$

Manifestly, we have

$$\Sigma\lambda_3^2 = 1, \quad \Sigma\lambda_3\lambda_4 = 0.$$

For the third flat (§ 180), we have

$$\Sigma\left\{(\bar{x} - x)\frac{d\lambda_3}{ds} - x'\lambda_3\right\} = 0.$$

But $x' = l_1$, and therefore $\Sigma x'\lambda_3 = \Sigma l_1\lambda_3 = \sigma(\rho^2 + \sigma^2)^{-\frac{1}{2}}$: so that the equation is

$$\Sigma\,(\bar{x} - x)\frac{d\lambda_3}{ds} = \frac{\sigma}{(\rho^2 + \sigma^2)^{\frac{1}{2}}}.$$

Because $\lambda_3 = l_1\cos A - l_3\sin A$, we have

$$\begin{aligned}\frac{d\lambda_3}{ds} &= \frac{l_2}{\rho}\cos A - \left(-\frac{l_2}{\sigma} + \frac{l_4}{\tau}\right)\sin A - (l_1\sin A + l_3\cos A)\frac{dA}{ds}\\ &= l_2\frac{(\rho^2 + \sigma^2)^{\frac{1}{2}}}{\rho\sigma} - \frac{l_1\rho + l_3\sigma}{(\rho^2 + \sigma^2)^{\frac{1}{2}}}\frac{dA}{ds} - \frac{\rho}{\tau(\rho^2 + \sigma^2)^{\frac{1}{2}}}l_4\\ &= \alpha\lambda_4 - \frac{\sigma\rho' - \rho\sigma'}{(\rho^2 + \sigma^2)^{\frac{3}{2}}}(\rho l_1 + \sigma l_3) - \frac{\rho}{\tau(\rho^2 + \sigma^2)^{\frac{1}{2}}}l_4.\end{aligned}$$

But from the general investigation, we are to have

$$\frac{d\lambda_3}{ds} = \alpha\lambda_4 - \beta\lambda_2;$$

and therefore, when we take

$$\Theta^2 = \frac{(\sigma\rho' - \rho\sigma')^2}{\rho^2 + \sigma^2} + \frac{\rho^2}{\tau^2}, \qquad \beta(\rho^2 + \sigma^2)^{\frac{1}{2}} = \Theta,$$

our third equation becomes

$$\Sigma(\bar{x} - x)\lambda_2 = -\frac{\sigma}{\Theta},$$

where λ_2 is given by

$$\Theta\lambda_2 = \frac{\sigma\rho' - \rho\sigma'}{\rho^2 + \sigma^2}(\rho l_1 + \sigma l_3) + \frac{\rho}{\tau} l_4,$$

with corresponding expressions for μ_2, ν_2, κ_2. Clearly

$$\Sigma\lambda_2^2 = 1, \quad \Sigma\lambda_2\lambda_3 = 0, \quad \Sigma\lambda_2\lambda_4 = 0.$$

It may be noted that the earlier form of the third equation, and the form just obtained, are equivalent to one another, when combined with the first two. For the earlier form was

$$\left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho}\right) X_3 - \frac{1}{\tau} X_4 = \frac{\sigma}{\rho},$$

that is,

$$\Sigma(\bar{x} - x)\left(\frac{\sigma\rho' - \rho\sigma'}{\rho\sigma} l_3 + \frac{1}{\tau} l_4\right) = -\frac{\sigma}{\rho}.$$

Now it is easy to verify that

$$\frac{\sigma\rho' - \rho\sigma'}{\rho\sigma} l_3 + \frac{1}{\tau} l_4 = \frac{\Theta}{\rho}\lambda_2 - \frac{\sigma\rho' - \rho\sigma'}{\sigma(\rho^2 + \sigma^2)^{\frac{1}{2}}}\lambda_3,$$

so that the earlier form is

$$\frac{\Theta}{\rho}\Sigma(\bar{x} - x)\lambda_2 - \frac{\sigma\rho' - \rho\sigma'}{\sigma(\rho^2 + \sigma^2)^{\frac{1}{2}}}\Sigma(\bar{x} - x)\lambda_3 = -\frac{\sigma}{\rho},$$

which, in association with $\Sigma(\bar{x} - x)\lambda_3 = 0$, becomes

$$\Sigma(\bar{x} - x)\lambda_2 = -\frac{\sigma}{\Theta},$$

the later form.

We have already (§ 186) obtained the direction-cosines of the line of intersection of these three perpendicular flats, that is, of a line which is perpendicular to the three directions $\lambda_4, \mu_4, \nu_4, \kappa_4$; $\lambda_3, \mu_3, \nu_3, \kappa_3$; $\lambda_2, \mu_2, \nu_2, \kappa_2$; themselves perpendicular to one another. Denoted by $\lambda_1, \mu_1, \nu_1, \kappa_1$, they are proportional to

$$\frac{\rho}{\tau}(\rho c_1 + \sigma c_3) - (\sigma\rho' - \rho\sigma') c_4,$$

for $c = l, m, n, k$. Now

$$\Sigma\left\{\frac{\rho}{\tau}(\rho c_1+\sigma c_3)-(\sigma\rho'-\rho\sigma')\,c_4\right\}^2=\frac{\rho^2}{\tau^2}(\rho^2+\sigma^2)+(\sigma\rho'-\rho\sigma')^2$$
$$=(\rho^2+\sigma^2)\,\Theta^2;$$

and therefore the direction-cosines are given by

$$(\rho^2+\sigma^2)^{\frac{1}{2}}\,\Theta\lambda_1=\frac{\rho}{\tau}(\rho l_1+\sigma l_3)-(\sigma\rho'-\rho\sigma')\,l_4,$$

with corresponding expressions for μ_1, ν_1, κ_1. It is easy to verify that

$$\Sigma\lambda_1^2=1,\quad \Sigma\lambda_1\lambda_2=0,\quad \Sigma\lambda_1\lambda_3=0,\quad \Sigma\lambda_1\lambda_4=0.$$

The fourth flat is represented by associating, with the other equations, the derivative of the third equation, viz.

$$\Sigma\left\{(\bar{x}-x)\frac{d\lambda_2}{ds}-x'\lambda_2\right\}=-\frac{d}{ds}\left(\frac{\sigma}{\Theta}\right).$$

But

$$\Sigma x'\lambda_2=\Sigma l_1\lambda_2=\frac{\rho\,(\sigma\rho'-\rho\sigma')}{\Theta\,(\rho^2+\sigma^2)};$$

and we are to have

$$\frac{d\lambda_2}{ds}=\beta\lambda_3-\gamma\lambda_1$$
$$=\frac{\Theta}{(\rho^2+\sigma^2)^{\frac{1}{2}}}\lambda_3-\gamma\lambda_1,$$

so that the fourth equation, in association with $\Sigma(\bar{x}-x)\lambda_3=0$, becomes

$$\gamma\Sigma\{(\bar{x}-x)\lambda_1\}=\frac{d}{ds}\left(\frac{\sigma}{\Theta}\right)-\frac{\rho\,(\sigma\rho'-\rho\sigma')}{\Theta\,(\rho^2+\sigma^2)}.$$

Thus it is necessary to determine γ.

From the equation

$$\Theta\lambda_2=\frac{\sigma\rho'-\rho\sigma'}{\rho^2+\sigma^2}(\rho l_1+\sigma l_3)+\frac{\rho}{\tau}l_4,$$

we have, on differentiating and substituting for the derivatives of l_1, l_3, l_4,

$$\Theta\frac{d\lambda_2}{ds}+\lambda_2\frac{d\Theta}{ds}=E,$$

where

$$E=l_1\frac{d}{ds}\left\{\frac{\rho\,(\sigma\rho'-\rho\sigma')}{\rho^2+\sigma^2}\right\}+l_3\left[\frac{d}{ds}\left\{\frac{\sigma\,(\sigma\rho'-\rho\sigma')}{\rho^2+\sigma^2}\right\}-\frac{\rho}{\tau^2}\right]+l_4\left\{\frac{d}{ds}\left(\frac{\rho}{\tau}\right)+\frac{\sigma\,(\sigma\rho'-\rho\sigma')}{\tau\,(\rho^2+\sigma^2)}\right\}.$$

Hence, as $\Sigma\lambda_1\lambda_2=0$, we have

$$\Theta\Sigma\lambda_1\frac{d\lambda_2}{ds}=\Sigma E\lambda_1,$$

or, on substituting $\beta\lambda_3-\gamma\lambda_1$ for $\frac{d\lambda_2}{ds}$ and using the relation $\Sigma\lambda_1\lambda_3=0$,

$$-\gamma\Theta=\Sigma E\lambda_1.$$

Consequently

$$\begin{aligned}
-(\rho^2+\sigma^2)^{\frac{1}{2}}\,\Theta^2\gamma &= \Sigma E\,(\rho^2+\sigma^2)^{\frac{1}{2}}\,\Theta\lambda_1 \\
&= \Sigma E\left\{\frac{\rho}{\tau}(\rho l_1+\sigma l_3)-(\sigma\rho'-\rho\sigma')\,l_4\right\} \\
&= \frac{\rho^2}{\tau}\frac{d}{ds}\left\{\frac{\rho(\sigma\rho'-\rho\sigma')}{\rho^2+\sigma^2}\right\}+\frac{\rho\sigma}{\tau}\left[\frac{d}{ds}\left\{\frac{\sigma(\sigma\rho'-\rho\sigma')}{\rho^2+\sigma^2}\right\}-\frac{\rho}{\tau^2}\right] \\
&\quad -(\sigma\rho'-\rho\sigma')\left\{\frac{d}{ds}\left(\frac{\rho}{\tau}\right)+\frac{\sigma}{\tau}\frac{\sigma\rho'-\rho\sigma'}{\rho^2+\sigma^2}\right\} \\
&= -\frac{\sigma}{\tau}\Theta^2-(\sigma\rho'-\rho\sigma')\frac{d}{ds}\left(\frac{\rho}{\tau}\right)+\frac{\rho}{\tau}\Omega,
\end{aligned}$$

where

$$\begin{aligned}
\Omega &= \rho\frac{d}{ds}\left\{\frac{\rho(\sigma\rho'-\rho\sigma')}{\rho^2+\sigma^2}\right\}+\sigma\frac{d}{ds}\left\{\frac{\sigma(\sigma\rho'-\rho\sigma')}{\rho^2+\sigma^2}\right\} \\
&= (\rho^2+\sigma^2)^{\frac{1}{2}}\frac{d}{ds}\left\{\frac{\sigma\rho'-\rho\sigma'}{(\rho^2+\sigma^2)^{\frac{1}{2}}}\right\}.
\end{aligned}$$

We thus have

$$\gamma\Theta^2=\frac{\sigma}{\tau}\frac{\Theta^2}{(\rho^2+\sigma^2)^{\frac{1}{2}}}+\frac{\sigma\rho'-\rho\sigma'}{(\rho^2+\sigma^2)^{\frac{1}{2}}}\frac{d}{ds}\left(\frac{\rho}{\tau}\right)-\frac{\rho}{\tau}\frac{d}{ds}\left\{\frac{\sigma\rho'-\rho\sigma'}{(\rho^2+\sigma^2)^{\frac{1}{2}}}\right\}.$$

Having regard to the value of Θ given by ,

$$\Theta^2=\frac{(\sigma\rho'-\rho\sigma')^2}{\rho^2+\sigma^2}+\frac{\rho^2}{\tau^2},$$

we write

$$\frac{\sigma\rho'-\rho\sigma'}{(\rho^2+\sigma^2)^{\frac{1}{2}}}=\Theta\sin\xi,\quad \frac{\rho}{\tau}=\Theta\cos\xi,$$

so that the last two terms in the expression for $\gamma\Theta^2$ are

$$=-\Theta^2\frac{d\xi}{ds}.$$

Thus

$$\gamma=\frac{\sigma}{\tau}\frac{1}{(\rho^2+\sigma^2)^{\frac{1}{2}}}-\frac{d\xi}{ds}=\frac{\cos A}{\tau}-\frac{d\xi}{ds};$$

and ξ is defined by the equation

$$\tan\xi=\frac{\tau(\sigma\rho'-\rho\sigma')}{\rho(\rho^2+\sigma^2)^{\frac{1}{2}}}.$$

Thus the fourth equation

$$\begin{aligned}
\gamma\Sigma\{(\bar{x}-x)\lambda_1\} &= \frac{d}{ds}\left(\frac{\sigma}{\Theta}\right)-\frac{\rho(\sigma\rho'-\rho\sigma')}{\Theta(\rho^2+\sigma^2)} \\
&= \frac{d}{ds}\left(\frac{\sigma}{\Theta}\right)-\sin A\sin\xi,
\end{aligned}$$

becomes determinate, because the value of γ is known.

Summarising, the equations of the four principal flats that are connected with the rectifying developable region are

$$\left.\begin{aligned}
&\Sigma(\bar{x}-x)\lambda_4=0\\
&\Sigma(\bar{x}-x)\lambda_3=0\\
&\Sigma(\bar{x}-x)\lambda_2=-\frac{\sigma}{\Theta}\\
&\Sigma(\bar{x}-x)\lambda_1=\frac{1}{\gamma}\left\{\frac{d}{ds}\left(\frac{\sigma}{\Theta}\right)-\sin A\sin\xi\right\}
\end{aligned}\right\},$$

where the quantities A, Θ, ξ, γ, are given by the relations

$$\frac{\rho}{\sin A}=\frac{\sigma}{\cos A}=(\rho^2+\sigma^2)^{\frac{1}{2}},$$

$$\Theta\sin\xi=\frac{\sigma\rho'-\rho\sigma'}{(\rho^2+\sigma^2)^{\frac{1}{2}}},\quad \Theta\cos\xi=\frac{\rho}{\tau},$$

$$\gamma=\frac{\cos A}{\tau}-\frac{d\xi}{ds},$$

and the four sets of direction-cosines λ_r, μ_r, ν_r, κ_r, for $r=1, 2, 3, 4$, (on pp. 319, 320), represent an orthogonal frame, with the successive relations between two sets to which we shall return immediately.

Curvatures of the edge of regression of the rectifying developable region.

188. The coordinates of the point on the edge of regression of the rectifying developable region, corresponding to the point P of the original curve, are the values of $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{v}$, which satisfy the preceding four equations taken simultaneously. Hence the *edge of regression is given by the equations*

$$\left.\begin{aligned}
X&=x+\frac{1}{\gamma}\left\{\frac{d}{ds}\left(\frac{\sigma}{\Theta}\right)-\sin A\sin\xi\right\}\lambda_1-\frac{\sigma}{\Theta}\lambda_2\\
Y&=y+\frac{1}{\gamma}\left\{\frac{d}{ds}\left(\frac{\sigma}{\Theta}\right)-\sin A\sin\xi\right\}\mu_1-\frac{\sigma}{\Theta}\mu_2\\
Z&=z+\frac{1}{\gamma}\left\{\frac{d}{ds}\left(\frac{\sigma}{\Theta}\right)-\sin A\sin\xi\right\}\nu_1-\frac{\sigma}{\Theta}\nu_2\\
V&=v+\frac{1}{\gamma}\left\{\frac{d}{ds}\left(\frac{\sigma}{\Theta}\right)-\sin A\sin\xi\right\}\kappa_1-\frac{\sigma}{\Theta}\kappa_2
\end{aligned}\right\}.$$

The tangent to this edge of regression has λ_1, μ_1, ν_1, κ_1, for its direction-cosines, and it is *the rectifying line of the original curve* which accordingly *is given by the equations*

$$\frac{\bar{x}-X}{\lambda_1}=\frac{\bar{y}-Y}{\mu_1}=\frac{\bar{z}-Z}{\nu_1}=\frac{\bar{v}-V}{\kappa_1}.$$

The principal normal of the edge of regression has λ_2, μ_2, ν_2, κ_2, for its direction-cosines; its binormal has λ_3, μ_3, ν_3, κ_3, for its direction-cosines, and its trinormal has λ_4, μ_4, ν_4, κ_4, that is, l_2, m_2, n_2, k_2, for its direction-cosines, so that it is parallel to the principal normal of the original curve. Also, the osculating flat of the edge of regression is the actual rectifying flat of the original curve.

As regards these sets of direction-cosines, we have

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1,$$

so that

$$\lambda_1 \frac{d\lambda_1}{ds} + \lambda_2 \frac{d\lambda_2}{ds} + \lambda_3 \frac{d\lambda_3}{ds} + \lambda_4 \frac{d\lambda_4}{ds} = 0,$$

hence, substituting for $\frac{d\lambda_2}{ds}$, $\frac{d\lambda_3}{ds}$, $\frac{d\lambda_4}{ds}$, the values already given, we find

$$\frac{d\lambda_1}{ds} = \gamma\lambda_2.$$

Thus the four relations are

$$\left.\begin{aligned} \frac{d\lambda_4}{ds} &= - \alpha\lambda_3 \\ \frac{d\lambda_3}{ds} &= \alpha\lambda_4 - \beta\lambda_2 \\ \frac{d\lambda_2}{ds} &= \beta\lambda_3 - \gamma\lambda_1 \\ \frac{d\lambda_1}{ds} &= \gamma\lambda_2 \end{aligned}\right\},$$

where

$$\alpha = \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)^{\frac{1}{2}}, \quad \beta = \frac{\Theta}{(\rho^2 + \sigma^2)^{\frac{1}{2}}},$$

and γ has the foregoing value: while the direction-cosines are

$$\left.\begin{aligned} \lambda_4 = l_2, \quad \lambda_3 &= l_1 \cos A - l_3 \sin A \\ \Theta\lambda_2 &= (\rho l_1 + \sigma l_3)\frac{dA}{ds} + \frac{\rho}{\tau} l_4 \\ (\rho^2 + \sigma^2)^{\frac{1}{2}}\,\Theta\lambda_1 &= \frac{\rho}{\tau}(\rho l_1 + \sigma l_3) - (\rho^2 + \sigma^2)\frac{dA}{ds} l_4 \end{aligned}\right\}.$$

The coordinates of a point on the edge of regression are

$$X = x + \lambda_1 E - \lambda_2 \frac{\sigma}{\Theta},$$

with like expressions for Y, Z, V, where

$$E = \frac{1}{\gamma}\left\{\frac{d}{ds}\left(\frac{\sigma}{\Theta}\right) - \sin A \sin \xi\right\};$$

hence, if dS be the element of arc of that edge of regression, the tangent to which has direction-cosines $\lambda_1, \mu_1, \nu_1, \kappa_1$, we have

$$\begin{aligned}\lambda_1 \frac{dS}{ds} &= \frac{dX}{dS}\frac{dS}{ds}\\ &= \frac{dX}{ds}\\ &= l_1 + \lambda_1\left(\frac{dE}{ds} - \gamma\frac{\sigma}{\Theta}\right) - \lambda_2 \sin A \sin \xi + \frac{\sigma}{\Theta}\beta\lambda_3.\end{aligned}$$

Now, from the values obtained for λ_2 and λ_3, we have

$$\begin{aligned}& l_1 - \lambda_2 \sin A \sin \xi + \frac{\sigma}{\Theta}\beta\lambda_3\\ &= l_1 - \lambda_2 \sin A \sin \xi + \lambda_3 \cos A\\ &= \sin A \left\{l_1 \sin A + l_3 \cos A - \lambda_2 \frac{\sigma\rho' - \rho\sigma'}{\Theta(\rho^2+\sigma^2)^{\frac{1}{2}}}\right\}\\ &= \lambda_1 \frac{\rho}{\Theta\tau} \sin A = \lambda_1 \frac{\rho^2}{\Theta\tau(\rho^2+\sigma^2)^{\frac{1}{2}}};\end{aligned}$$

hence

$$\frac{dS}{ds} = \frac{\rho^2}{\Theta\tau(\rho^2+\sigma^2)^{\frac{1}{2}}} - \frac{\gamma\sigma}{\Theta} + \frac{d}{ds}\left[\frac{1}{\gamma}\left\{\frac{d}{ds}\left(\frac{\sigma}{\Theta}\right) - \sin A \sin \xi\right\}\right],$$

which accordingly is the expression for the element of arc dS.

It is to be noticed that, incidentally, the analysis verifies the property that the tangent to the edge of regression has $\lambda_1, \mu_1, \nu_1, \kappa_1$, for its direction-cosines.

Let the radii of plane curvature, of torsion, and of tilt, for the edge of regression of the rectifying developable region, be denoted by ρ_r, σ_r, τ_r, respectively Then Frenet's equations for the edge of regression are

$$\begin{aligned}-\frac{\lambda_3}{\tau_r} &= \frac{d\lambda_4}{dS} = \frac{ds}{dS}\frac{d\lambda_4}{ds} = \frac{ds}{dS}(-\alpha\lambda_3),\\ -\frac{\lambda_2}{\sigma_r} + \frac{\lambda_4}{\tau_r} &= \frac{d\lambda_3}{dS} = \frac{ds}{dS}\frac{d\lambda_3}{ds} = \frac{ds}{dS}(\alpha\lambda_4 - \beta\lambda_2),\\ -\frac{\lambda_1}{\rho_r} + \frac{\lambda_3}{\sigma_r} &= \frac{d\lambda_2}{dS} = \frac{ds}{dS}\frac{d\lambda_2}{ds} = \frac{ds}{dS}(\beta\lambda_3 - \gamma\lambda_1),\\ \frac{\lambda_2}{\rho_r} &= \frac{d\lambda_1}{dS} = \frac{ds}{dS}\frac{d\lambda_1}{ds} = \frac{ds}{dS}(\gamma\lambda_2);\end{aligned}$$

and therefore the three curvatures are given by the relations

$$\left.\begin{aligned}\frac{dS}{ds}\frac{1}{\rho_r} &= \gamma = \frac{\sigma}{\tau(\rho^2+\sigma^2)^{\frac{1}{2}}} - \frac{d\xi}{ds}\\ \frac{dS}{ds}\frac{1}{\sigma_r} &= \beta = \frac{\Theta}{(\rho^2+\sigma^2)^{\frac{1}{2}}}\\ \frac{dS}{ds}\frac{1}{\tau_r} &= \alpha = \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)^{\frac{1}{2}}\end{aligned}\right\},$$

where it will be noticed that α is the curvature of screw (§ 131) of the original curve. Manifestly, the angles of contingence, of torsion, and of tilt, for the edge of regression respectively are

$$\left.\begin{aligned} d\epsilon_r &= \frac{\sigma\, ds}{\tau(\rho^2+\sigma^2)^{\frac{1}{2}}} - d\xi \\ d\eta_r &= \frac{\Theta\, ds}{(\rho^2+\sigma^2)^{\frac{1}{2}}} \\ d\omega_r &= \left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)^{\frac{1}{2}} ds = d\iota \end{aligned}\right\},$$

where $d\iota$ is (§ 131) the inclination of two consecutive radii of plane curvature of the original curve, obviously equal to the angle $d\omega_r$ between the two consecutive flats to which those radii are perpendicular.

Equations of the rectifying developable region and developable surface.

189. Manifestly, the equation of the rectifying developable region of the curve is the eliminant of the two equations

$$\Sigma(\bar{x}-x)\lambda_4 = 0, \quad \Sigma(\bar{x}-x)\lambda_3 = 0,$$

s being the parameter to be eliminated. As these two equations represent a rectifying plane for each particular value of the parameter, and as the equation of the region is satisfied by them whatever value the parameter may possess, it follows that the region is planar.

Again, the equation of the rectifying developable surface of the curve is the eliminant of the three equations

$$\Sigma(\bar{x}-x)\lambda_4 = 0, \quad \Sigma(\bar{x}-x)\lambda_3 = 0, \quad \Sigma(\bar{x}-x)\lambda_2 = -\frac{\sigma}{\Theta},$$

the said eliminant being constituted by two equations. These three equations represent a rectifying line, and the equations of the surface are satisfied by them whatever value the parameter may possess; also, consecutive rectifying lines meet; hence the surface is developable.

The three equations of the edge of regression are the eliminant of the four equations, made up of the preceding three equations and of a fourth equation

$$\begin{aligned} \Sigma(\bar{x}-x)\lambda_1 &= \frac{1}{\gamma}\left\{\frac{d}{ds}\left(\frac{\sigma}{\Theta}\right) - \sin\mathcal{A}\sin\xi\right\} \\ &= \frac{1}{\gamma}\left\{\frac{d}{ds}\left(\frac{\sigma}{\Theta}\right) - \frac{\rho(\sigma\rho'-\rho\sigma')}{\Theta(\rho^2+\sigma^2)}\right\}. \end{aligned}$$

The edge is also represented by the parametric expressions for the coordinates of a current point, which have already (§ 188) been obtained.

When the rectifying region is developed, the original curve becomes a straight line.

190. As regards the developable region, which is the envelope of the flat perpendicular to the radius of plane curvature of the original curve, the process of development is the same as for any other developable region. A beginning is made with a flat, tangential to the region and osculant to the edge of regression of the region: that flat is turned, through the small angle of tilt, about the osculating plane of the edge which is the plane of intersection with the next osculating flat, that is, the next flat tangential to the region: and thus the first flat is brought into coincidence with the second. This doubled flat is then brought into coincidence with a third consecutive flat, tangential to the region and osculant to the edge, by a rotation about the next osculating plane. In this operation, the first osculating plane (it is a tangent plane of the developable surface) is brought into coincidence with the second osculating plane, which remains fixed throughout the operation, by the small rotation round the line of intersection of the two osculating planes, that is, by the small rotation about the plane through this tangent line and the trinormal. And so on, from stage to stage. In every stage of the whole process, deformation occurs, and nothing but deformation: there is no rending, no tearing, no stretching, no compression, either of volume or of area or of length.

Such is the general process. In the instance of the rectifying developable region, when that region is developed into an ultimate flat, the successive flats which osculate the edge of regression of the region have been called the rectifying flats of the original curve. The successive planes which osculate the edge of regression of the region have been called the rectifying planes of the original curve. The successive tangents of the edge of regression of the region have been called the rectifying lines of the original curve. Throughout, there has been neither stretching nor compression, of any of the magnitudes: in particular, there has been neither stretching nor compression of any length, and therefore neither stretching nor compression of the length of the original curve. Now, before the deformation, any radius of the plane curvature of the curve is perpendicular to the corresponding plane, that is, to the plane tangential to the region: thus the radius of plane curvature of the curve coincides with the normal to the developable region in which it exists. The curve is therefore (§ 297, *post*) a geodesic in that region. As there is neither compression nor stretching in the development of the region into flatness, the curve remains a geodesic throughout · it therefore ends as a geodesic in the ultimate flat. But in a flat, being a three-dimensional homaloidal space, all geodesics are straight lines: the original curve thus ends as a straight line, that is, the original curve has been rectified. Hence the title *rectifying* is assigned to the flat, to the plane, and to the line, the use of which in the development of the region has led to the rectification of the curve into a straight line.

CHAPTER XI.

CURVES IN N-FOLD HOMALOIDAL SPACE.

In this chapter, the results relating to the principal homaloidal amplitudes connected with a skew curve in n-fold homaloidal space are merely stated, usually without the contributory analysis That analysis follows (with, of course, the necessary changes due to the fact that a point has n coordinates in the full space, instead of the four coordinates appertaining to homaloidal quadruple space) the course of the analysis in the three preceding chapters and the figure, representing the moving orthogonal frame of the curve, is merely an extension of the earlier figure (p 211) representing the moving orthogonal frame of the skew curve in homaloidal quadruple space

Reference may be made to the (posthumous) tract by Guichard "Sur les courbes de l'espace à n dimensions," *Mémorial des Sciences Mathématiques*, Fasc. XIX. (1928), pp. 64 (Paris, Gauthier-Villars)

Skew curves in general homaloidal space.

191. The preceding investigations and discussions have been concerned with a skew curve in homaloidal quadruple space. The same processes can be carried out for a skew curve in homaloidal n-fold space: and the following results, merely stated, can be established by the same kind of process that uses both the analysis and the diagram connected with the orthogonal frame at any point of the skew curve.

The coordinates of any point in the homaloidal n-fold space are denoted by $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$. The coordinates of a current point P on a curve in that space are denoted by $x_1, x_2, \ldots, x_n$. An arc-element of a skew curve in the space is given by

$$ds^2 = dx_1^2 + dx_2^2 + \ldots + dx_n^2,$$

so that

$$\Sigma\left(\frac{dx}{ds}\right)^2 = \left(\frac{dx_1}{ds}\right)^2 + \left(\frac{dx_2}{ds}\right)^2 + \ldots + \left(\frac{dx_n}{ds}\right)^2 = 1,$$

and the summation sign Σ here, as elsewhere in this connection, will denote summation over the n point-coordinates. The march of the curve is determinate, when $x_1, x_2, \ldots, x_n$, are given as functions of a single parameter; this parameter will be taken to be the arc s, measured along the curve from some initial point of reference. If the parameter were some other quantity t, the transformations would be made by the use of the relation

$$\frac{ds}{dt} = \left\{\Sigma\left(\frac{dx}{dt}\right)^2\right\}^{\frac{1}{2}}$$

Various homaloids occur in connection with the curve. Of these homaloids, two sets are more important than the others. One of these sets is constituted by contact homaloids, of successive degrees of contact and of corresponding dimensions. The other of the sets is constituted by normal homaloids, of successive degrees of normality and of corresponding dimensions.

Contact homaloids.

192. The simplest contact homaloid is that with one-fold contact and of one dimension. It passes through two consecutive points of the curve at P and is the tangent to the curve. Its equations, $n-1$ in number and independent of one another, are

$$\left\| \begin{matrix} \overline{x}_1 - x_1, & \overline{x}_2 - x_2, & \ldots\ldots, & \overline{x}_n - x_n \\ \dfrac{dx_1}{ds}, & \dfrac{dx_2}{ds}, & \ldots\ldots, & \dfrac{dx_n}{ds} \end{matrix} \right\| = 0.$$

We shall denote this contact homaloid by T_1.

There is a contact homaloid, with two-fold contact and of two dimensions. It passes through three consecutive points of the curve at P and is the osculating plane of the curve. Its equations, $n-2$ in number and independent of one another, are

$$\left\| \begin{matrix} \overline{x}_1 - x_1, & \overline{x}_2 - x_2, & \ldots\ldots, & \overline{x}_n - x_n \\ \dfrac{dx_1}{ds}, & \dfrac{dx_2}{ds}, & \ldots\ldots, & \dfrac{dx_n}{ds} \\ \dfrac{d^2x_1}{ds^2}, & \dfrac{d^2x_2}{ds^2}, & \ldots\ldots, & \dfrac{d^2x_n}{ds^2} \end{matrix} \right\| = 0.$$

We shall denote this contact homaloid by T_2.

There is a contact homaloid, with three-fold contact and of three dimensions It passes through four consecutive points of the curve at P and is the osculating flat of the curve. Its equations, $n-3$ in number and independent of one another, are

$$\left\| \begin{matrix} \overline{x}_1 - x_1, & \overline{x}_2 - x_2, & \ldots\ldots, & \overline{x}_n - x_n \\ \dfrac{dx_1}{ds}, & \dfrac{dx_2}{ds}, & \ldots\ldots, & \dfrac{dx_n}{ds} \\ \dfrac{d^2x_1}{ds^2}, & \dfrac{d^2x_2}{ds^2}, & \ldots\ldots, & \dfrac{d^2x_n}{ds^2} \\ \dfrac{d^3x_1}{ds^3}, & \dfrac{d^3x_2}{ds^3}, & \ldots\ldots, & \dfrac{d^3x_n}{ds^3} \end{matrix} \right\| = 0.$$

We shall denote this contact homaloid by T_3.

And so on, with increasing multiplicity of contact and of the same increasing multiplicity of dimensions. The contact homaloid, with m-fold contact and of m dimensions, passes through $(m+1)$ consecutive points of the curve at P. Its equations, $n-m$ in number and independent of one another, are

$$\left\| \begin{matrix} \bar{x}_1 - x_1, & \bar{x}_2 - x_2, & \ldots\ldots, & \bar{x}_n - x_n \\ \frac{dx_1}{ds}, & \frac{dx_2}{ds}, & \ldots\ldots, & \frac{dx_n}{ds} \\ \frac{d^2x_1}{ds^2}, & \frac{d^2x_2}{ds^2}, & \ldots\ldots, & \frac{d^2x_n}{ds^2} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{d^mx_1}{ds^m}, & \frac{d^mx_2}{ds^m}, & \ldots\ldots, & \frac{d^mx_n}{ds^m} \end{matrix} \right\| = 0.$$

We shall denote this contact homaloid by T_m.

The final contact homaloid, with $(n-1)$-fold contact and of $(n-1)$ dimensions, passes through n consecutive points of the curve at P: that is, through P and $(n-1)$ next consecutive points there. It is the homaloidal amplitude of closest possible contact with the curve in the homaloidal space of n dimensions. Its (single) equation is

$$\left| \begin{matrix} \bar{x}_1 - x_1, & \bar{x}_2 - x_2, & \ldots\ldots, & \bar{x}_n - x_n \\ \frac{dx_1}{ds}, & \frac{dx_2}{ds}, & \ldots\ldots, & \frac{dx_n}{ds} \\ \frac{d^2x_1}{ds^2}, & \frac{d^2x_2}{ds^2}, & \ldots\ldots, & \frac{d^2x_n}{ds^2} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{d^{n-1}x_1}{ds^{n-1}}, & \frac{d^{n-1}x_2}{ds^{n-1}}, & \ldots\ldots, & \frac{d^{n-1}x_n}{ds^{n-1}} \end{matrix} \right| = 0.$$

We shall denote this final contact homaloid by T_{n-1}.

Each contact homaloid exists completely within each contact homaloid of a greater number of dimensions—an inference obvious from the equations.

NOTE. In every instance, these equations are the initial forms of the equations of the respective contact homaloids. Alternative (and more useful) forms, equivalent to them, can be stated after the principal lines of the orthogonal frame of the curve at P have been established.

Normal homaloids.

193. Following the discrimination between full orthogonality and mere perpendicularity, which arises first in quadruple space in the instance of a couple of planes, we define two homaloids as orthogonal to one another when every direction in either of them is perpendicular to every other direction

The notion of the inclination, to one another, of two directions in n-fold space is only a renewed extension of the customary notion, already (§ 19) used for quadruple space. When two directions have direction-cosines $\xi_1, \xi_2, \ldots, \xi_n$, such that $\sum\limits_m \xi_m^2 = 1$, and $\eta_1, \eta_2, \ldots, \eta_n$, such that $\sum\limits_m \eta_m^2 = 1$, their inclination θ is given by the relation $\sum\limits_m \xi_m \eta_m = \cos\theta$; and they are said to be perpendicular when

$$\sum_m \xi_m \eta_m = 0.$$

With each contact homaloid we associate a complementary normal homaloid, the two homaloids being orthogonal. When one of them is of r dimensions, the other is of $n-r$ dimensions. Unlike the contact homaloids, no normal homaloid after the earliest (of $n-1$ dimensions) passes through the point P of the curve.

There is a normal homaloid, with one-fold normality and of $(n-1)$ dimensions, being the earliest normal homaloid to which allusion has just been made. It is orthogonal to the contact homaloid T_1 which has one-fold contact and is of one dimension; it passes through the point P of the curve and its single equation is

$$\Sigma(\bar{x}-x)\frac{dx}{ds} = 0.$$

It is convenient to have a symbol for $\Sigma x \frac{dx}{ds}$: we take

$$p = \Sigma x \frac{dx}{ds};$$

and then, for the sake of ulterior formal completeness, we take the equation of the normal homaloid with one-fold normality to be

$$\Sigma(\bar{x}-x)\frac{dx}{ds} = p - \Sigma x \frac{dx}{ds},$$

this particular right-hand side being zero. We shall denote this normal homaloid by N_1.

There is a normal homaloid, with two-fold normality and of $(n-2)$ dimensions: it is orthogonal to the contact homaloid T_2 which has two-fold contact and is of two dimensions. It is the amplitude common to the normal homaloid N_1 at P and to the normal homaloid N_1 at the point next consecutive to P (in which latter homaloid, the point P does not lie): and its two equations are

$$\left.\begin{aligned} \Sigma(\bar{x}-x)\frac{dx}{ds} &= p - \Sigma x \frac{dx}{ds} \\ \Sigma(\bar{x}-x)\frac{d^2x}{ds^2} &= \frac{dp}{ds} - \Sigma x \frac{d^2x}{ds^2} \end{aligned}\right\}.$$

(The right-hand side of the second equation is equal to $\Sigma\left(\frac{dx}{ds}\right)^2$, that is, to unity, so that the point P does not lie in this amplitude. The actual values

0, 1, of the two right-hand sides are irrelevant for the moment; the right-hand sides, here and later, remain unevaluated until the formation of the equations of all the normal homaloids.) We denote this normal homaloid by N_2.

There is a normal homaloid, with three-fold normality and of $(n-3)$ dimensions: it is orthogonal to the contact homaloid T_3 which has three-fold contact and is of three dimensions. It is the amplitude common to the normal homaloid N_1 at P and to the normal homaloids N_1 at the two points next consecutive to P: and its three equations are

$$\left.\begin{array}{l} \Sigma(\bar{x}-x)\dfrac{dx}{ds} = p - \Sigma x\dfrac{dx}{ds} \\ \Sigma(\bar{x}-x)\dfrac{d^2x}{ds^2} = \dfrac{dp}{ds} - \Sigma x\dfrac{d^2x}{ds^2} \\ \Sigma(\bar{x}-x)\dfrac{d^3x}{ds^3} = \dfrac{d^2p}{ds^2} - \Sigma x\dfrac{d^3x}{ds^3} \end{array}\right\}.$$

We shall denote this normal homaloid by N_3.

And so on. There is a normal homaloid, with m-fold normality and of $(n-m)$ dimensions: it is orthogonal to the contact homaloid T_m which has m-fold contact and is of m dimensions. It is the amplitude common to the normal homaloid N_1 at P and to the normal homaloids N_1 at the $(m-1)$ points next consecutive to P: and its m equations are

$$\left.\begin{array}{l} \Sigma(\bar{x}-x)\dfrac{dx}{ds} = p - \Sigma x\dfrac{dx}{ds} \\ \Sigma(\bar{x}-x)\dfrac{d^2x}{ds^2} = \dfrac{dp}{ds} - \Sigma x\dfrac{d^2x}{ds^2} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \Sigma(\bar{x}-x)\dfrac{d^mx}{ds^m} = \dfrac{d^{m-1}p}{ds^{m-1}} - \Sigma x\dfrac{d^mx}{ds^m} \end{array}\right\}.$$

We shall denote this normal homaloid by N_m.

The final normal homaloid, with $(n-1)$-fold normality and of one dimension—in fact, a line—is orthogonal to the contact homaloid T_{n-1} which has $(n-1)$-fold contact and is of $(n-1)$ dimensions, being the final contact homaloid. It is the (line) amplitude common to the normal homaloid N_1 at P and to the normal homaloids N_1 at the $(n-2)$ points next consecutive to P: and its $(n-1)$ equations are

$$\left.\begin{array}{l} \Sigma(\bar{x}-x)\dfrac{dx}{ds} = p - \Sigma x\dfrac{dx}{ds} \\ \Sigma(\bar{x}-x)\dfrac{d^2x}{ds^2} = \dfrac{dp}{ds} - \Sigma x\dfrac{d^2x}{ds^2} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \Sigma(\bar{x}-x)\dfrac{d^{n-1}x}{ds^{n-1}} = \dfrac{d^{n-2}p}{ds^{n-2}} - \Sigma x\dfrac{d^{n-1}x}{ds^{n-1}} \end{array}\right\}.$$

We shall denote this normal homaloid (a line) by N_{n-1}. Thus, in quadruple space, it is parallel to the trinormal, being the line SG in Fig. 12 of § 128.

Each normal homaloid exists completely within each normal homaloid of a greater number of dimensions—an inference obvious from the equations.

NOTE. When the principal lines of the orthogonal frame of the curve have been determined, these equations will be changed into equivalent forms which shall leave in evidence the orthogonality of N_m and T_m, after the equations of T_m have been modified.

The curvatures of tilt.

194. We have seen that each contact homaloid at P lies in every other contact homaloid there which is of any greater number of dimensions. Now, alike from the definitions and from the forms of the respective equations, it follows that each contact homaloid T_r at P contains the preceding contact homaloid T_{r-1} at P and also the contact homaloid T_{r-1} at the point next consecutive to P.

We denote by $d\epsilon_m$ the small angle of tilt between the contact homaloid T_m at P and the contact homaloid T_m at the point next consecutive to P; thus, as both these homaloids T_m lie in the homaloid T_{m+1} at P, the small tilt $d\epsilon_m$ is the small change between the consecutive homaloids that lie in T_{m+1}. Hence, along the curve, $\frac{d\epsilon_m}{ds}$ is the arc-rate of the specified tilt, and writing

$$\frac{1}{\rho_m} = \frac{d\epsilon_m}{ds},$$

we call $\frac{1}{\rho_m}$ the *m-ary curvature of tilt* or, more briefly, the m-ary tilt: thus $\frac{1}{\rho_1}$ is the primary tilt (it will bear an additional title, because of an additional property), $\frac{1}{\rho_2}$ is the secondary tilt, $\frac{1}{\rho_3}$ the tertiary tilt, and so on. In quadruple space, $\frac{1}{\rho_2}$ is what has been called torsion and the curvature of torsion, and $\frac{1}{\rho_3}$ is what has been called tilt and the curvature of tilt. The magnitude ρ_m is called *the radius of m-ary tilt of the curve.* Also, no one of these curvatures, after the primary tilt, has a centre; and no magnitude ρ_m has a direction, for values of $m > 1$.

The definition holds for all the possible values of m from unity onwards. The greatest value of m is $n-1$; for that value, $d\epsilon_{n-1}$ is the angle of tilt of T_{n-1}: that is, the angle between T_{n-1} at P and T_{n-1} at the point next consecutive to P, these two contact homaloids existing in the comprehensive homaloidal space of n dimensions.

Fuller analytical expressions for these curvatures of tilt of successive rank are as follows. We have

$$\frac{1}{\rho_1^2} = \Sigma \begin{vmatrix} \dfrac{d^2x_1}{ds^2}, & \dfrac{d^2x_2}{ds^2} \\[2ex] \dfrac{dx_1}{ds}, & \dfrac{dx_2}{ds} \end{vmatrix}^2,$$

where the sign Σ indicates summation over all the combinations of (two) variables x_1, x_2, from the full set of n variables. In a space of two dimensions,

$$-\frac{1}{\rho_1} = \begin{vmatrix} x_1'', & x_2'' \\ x_1', & x_2' \end{vmatrix}.$$

We have

$$\frac{1}{\rho_1^4\rho_2^2} = \Sigma \begin{vmatrix} \dfrac{d^3x_1}{ds^3}, & \dfrac{d^3x_2}{ds^3}, & \dfrac{d^3x_3}{ds^3} \\[2ex] \dfrac{d^2x_1}{ds^2}, & \dfrac{d^2x_2}{ds^2}, & \dfrac{d^2x_3}{ds^2} \\[2ex] \dfrac{dx_1}{ds}, & \dfrac{dx_2}{ds}, & \dfrac{dx_3}{ds} \end{vmatrix}^2,$$

where again the sign Σ indicates summation over all the combinations of (three) variables x_1, x_2, x_3, from the full set of n variables In a space of three dimensions,

$$-\frac{1}{\rho_1^2\rho_2} = \begin{vmatrix} x_1''', & x_2''', & x_3''' \\ x_1'', & x_2'', & x_3'' \\ x_1', & x_2', & x_3' \end{vmatrix}$$

in such a space, $1/\rho_2$ is the torsion.

We have

$$\frac{1}{\rho_1^6\rho_2^4\rho_3^2} = \Sigma \begin{vmatrix} \dfrac{d^4x_1}{ds^4}, & \dfrac{d^4x_2}{ds^4}, & \dfrac{d^4x_3}{ds^4}, & \dfrac{d^4x_4}{ds^4} \\[2ex] \dfrac{d^3x_1}{ds^3}, & \dfrac{d^3x_2}{ds^3}, & \dfrac{d^3x_3}{ds^3}, & \dfrac{d^3x_4}{ds^3} \\[2ex] \dfrac{d^2x_1}{ds^2}, & \dfrac{d^2x_2}{ds^2}, & \dfrac{d^2x_3}{ds^2}, & \dfrac{d^2x_4}{ds^2} \\[2ex] \dfrac{dx_1}{ds}, & \dfrac{dx_2}{ds}, & \dfrac{dx_3}{ds}, & \dfrac{dx_4}{ds} \end{vmatrix}^2,$$

with the corresponding signification for the sign Σ. In a space of four dimensions,

$$-\frac{1}{\rho_1^3\rho_2^2\rho_3} = \begin{vmatrix} x_1^{\text{iv}}, & x_2^{\text{iv}}, & x_3^{\text{iv}}, & x_4^{\text{iv}} \\ x_1''', & x_2''', & x_3''', & x_4''' \\ x_1'', & x_2'', & x_3'', & x_4'' \\ x_1', & x_2', & x_3', & x_4' \end{vmatrix}.$$

in such a space, $1/\rho_3$ is the tilt as defined in § 145.

And so on, for the ranks in successive order. The curvature of tilt of highest rank $n-1$ in the space of n dimensions is given by

$$\frac{-1}{\rho_1^{n-1}\rho_2^{n-2}\cdots\rho_{n-2}^{2}\rho_{n-1}} = \begin{vmatrix} \frac{d^n x_1}{ds^n}, & \frac{d^n x_2}{ds^n}, & \ldots\ldots, & \frac{d^n x_n}{ds^n} \\ \frac{d^{n-1} x_1}{ds^{n-1}}, & \frac{d^{n-1} x_2}{ds^{n-1}}, & \ldots\ldots, & \frac{d^{n-1} x_n}{ds^{n-1}} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{d^2 x_1}{ds^2}, & \frac{d^2 x_2}{ds^2}, & \ldots\ldots, & \frac{d^2 x_n}{ds^2} \\ \frac{dx_1}{ds}, & \frac{dx_2}{ds}, & \ldots\ldots, & \frac{dx_n}{ds} \end{vmatrix}$$

In making the estimates of tilt, there are conventions as to the sign of the angle of tilt of each contact homaloid T_m within the contact homaloid, it is taken to be positive when it moves the contact homaloid T_m at P into coincidence of orientation with the contact homaloid T_m at the point next consecutive to P.

Orbiculate amplitudes of contact.

195. The amplitudes, next in simplicity to those that are homaloidal, are called *orbiculate*. the most frequent instances are circles, spheres, and globes, as defined for quadruple space. The characteristic property of an orbiculate amplitude is that all points, which it contains, are at the same distance as one another from a centre: thus a circle (i.e. its circumference) is an amplitude of one dimension, a sphere (i.e. its surface) is an amplitude of two dimensions, a globe is an amplitude of three dimensions. In each instance, the distance of the contained points of the amplitude from its centre is called its radius.

In a contact homaloid T_m at a point P, the homaloid being of m dimensions, there lies an orbiculate amplitude of $m-1$ dimensions which passes through $m+1$ consecutive points of the curve: that is, through the point P and the next m immediately consecutive points of the curve. This orbiculate amplitude we shall denote by O_{m-1}; it lies in the contact homaloid T_m. The radius of the orbiculate amplitude O_{m-1} will be denoted by R_{m-1}: and the radius will be called *the radius of* $(m-1)$*-fold orbiculate curvature.*

Thus there is no orbiculate amplitude O_0. for the contact homaloid T_1 is merely the (linear) tangent of the curve at P, and such a region, being of only one dimension, contains no dimensional region of lower rank.

The orbiculate amplitude O_1, of one dimension, is a circle: it lies in the contact homaloid T_2 which has double contact with the curve and is therefore the osculating plane. Thus O_1 is the circle of curvature of the curve (in earlier sections, this curvature was spoken of also as plane curvature). Its radius, in accordance with the specification of the radius of an orbiculate amplitude, is denoted by R_1. Now this radius of the circle, measuring the angle of tilt of the tangent T_1 to the curve in the osculating plane, is the quantity previously denoted by ρ_1; so we have

$$R_1 = \rho_1.$$

Thus it appears that ρ_1 at once is the radius of primary tilt of the curve and is the radius of one-fold orbiculate curvature of the curve. Both titles will be retained for ρ_1. It appears that, for calculations connected with the curve, the radii of tilt are more convenient than the radii of orbiculate curvature: for descriptions and diagrams, the radii of orbiculate curvature are more convenient than the radii of tilt which, having neither associated centre nor associated direction, have no position.

To formulate the difference-relations between the radii R of successive ranks of curvature, it is convenient to introduce certain quantities D, defined as follows. We take

$$D_1 = \rho_2 \frac{d\rho_1}{ds} = \rho_2 \rho_1';$$

and we find

$$R_2^2 = R_1^2 + D_1^2.$$

We take

$$D_2 = \rho_3 \frac{R_2 R_2'}{D_1},$$

which is equal to $\rho_3 \dfrac{R_2 R_2'}{\rho_2 \rho_1'}$, and we find

$$R_3^2 = R_2^2 + D_2^2.$$

The general expression for these quantities D is

$$D_m = \rho_{m+1} \frac{R_m R_m'}{D_{m-1}},$$

and we find

$$R_{m+1}^2 = R_m^2 + D_m^2.$$

The general expression holds for all values of m from $m = 1$ onwards up to $m = n - 2$. The orbiculate region O_m of radius R_m, which is of m dimensions and has $(m+1)$-fold contact with the curve, lies in the contact homaloid T_{m+1}. In quadruple space, R_1 (or ρ_1) is the radius of circular curvature, R_2 (previously denoted by R) is the radius of spherical curvature, and R_3 (previously denoted by Γ) is the radius of globular curvature.

These results can be derived from a consideration of the orthogonal frame of the curve at a point P and the orthogonal frame at a point next consecutive to P, when once the principal lines of the curve that constitute the frame have been determined: or they can be derived, rather laboriously, by analysis similar to that used, in the first instance, for a skew curve in quadruple space.

These quantities D_m are subject to the mixed-difference relation

$$\frac{dD_m}{ds} = \frac{D_{m+1}}{\rho_{m+2}} - \frac{D_{m-1}}{\rho_{m+1}},$$

for $m > 1$, while, for $m = 1$,

$$\frac{dD_1}{ds} = \frac{D_2}{\rho_3} - \frac{\rho_1}{\rho_2}.$$

The result proves useful in considering the loci of the centres of curvature of various ranks.

Principal lines.

196. The principal lines of the curve at a point P constitute the orthogonal frame that moves along the curve with P. With the exception of the first two, they are lines drawn through P parallel to axial lines which arise organically in the construction of the successive orbiculate regions O_m associated with the curve; and these first two lines, the tangent and the primary (or principal) normal, themselves arise organically in this process.

I. The *first principal line is the tangent* at P. It constitutes the homaloid T_1 of one-fold contact, as the homaloid is of only one dimension, it cannot contain an orbiculate region of lower dimensions. We denote* its direction-cosines by

$$(l_1)_1,\ (l_1)_2,\ \ldots\ldots,\ (l_1)_n,$$

where

$$\sum_r \{(l_1)_r\}^2 = 1.$$

II. The *second principal line is the primary normal* (usually called the principal normal) of the curve. It lies within the contact homaloid T_2 of two-fold contact, and it is the intersection of that homaloid T_2 with the normal homaloid N_1 of one-fold normality. We denote its direction-cosines by

$$(l_2)_1,\ (l_2)_2,\ \ldots\ldots,\ (l_2)_n,$$

where

$$\sum_r \{(l_2)_r\}^2 = 1,$$

while there is the relation of perpendicularity

$$\sum_r \{(l_1)_r\}\{(l_2)_r\} = 0.$$

Further, the *centre of primary curvature* C_1 lies on this principal normal: this point is the intersection of the contact homaloid T_2 of two-fold contact with the normal homaloid N_2 of two-fold normality. The principal normal is the line PC_1. We denote* the coordinates of C_1 by

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* Here, as with the notation for later sets of direction-cosines, the subscript r outside the bracket indicates the inclination to the axis of the variable x_r, for $r=1, \quad , n$; and the subscript within the bracket, here unity, is the ordinal number of the principal line having the specified set of direction-cosines.

$$(\xi_1)_1,\ (\xi_1)_2,\ \ldots\ldots,\ (\xi_1)_n.$$

Thus the orbiculate amplitude O_1 (ordinarily the circumference of the circle of curvature) lies in the homaloid T_2 of two-fold contact; its centre is the point C_1; its radius is R_1 (or ρ_1), and it passes through P.

III. The *third principal line of the curve is the secondary normal* at P (usually called the binormal in space of three dimensions) It is a line drawn through P, parallel to the axial line which is the intersection of the contact homaloid T_3 of three-fold contact with the normal homaloid N_2 of two-fold normality. We denote its direction-cosines by

$$(l_3)_1,\ (l_3)_2,\ \ldots\ldots,\ (l_3)_n,$$

where

$$\sum_r \{(l_3)_r\}^2 = 1,$$

while there are the relations of perpendicularity

$$\sum_r \{(l_1)_r (l_3)_r\} = 0, \quad \sum_r \{(l_2)_r (l_3)_r\} = 0.$$

Further, the *centre of secondary curvature* C_2 lies on this axial line; this point C_2 is the intersection of the contact homaloid T_3 of three-fold contact with the normal homaloid N_3 of three-fold normality. The axial line, to which the binormal is parallel, is the line joining C_1 and C_2. We denote the coordinates of C_2 by

$$(\xi_2)_1,\ (\xi_2)_2,\ \ldots\ldots\ (\xi_2)_n.$$

Thus the orbiculate amplitude O_2 (ordinarily the surface of the sphere of curvature) lies in the homaloid T_3 of three-fold contact, O_2 being a two-dimensional amplitude and T_3 a three-dimensional amplitude. The centre of the orbiculate region O_2 is the point C_2; the radius of the region is R_2, and it passes through P.

IV. The *mth principal line of the curve* is the normal of rank $m-1$. It is a line drawn through P, parallel to the axial line which is the intersection of the contact homaloid T_m of m-fold contact with the normal homaloid N_{m-1} of $(m-1)$-fold normality. We denote its direction-cosines by

$$(l_m)_1,\ (l_m)_2,\ \ldots\ldots,\ (l_m)_n,$$

where

$$\sum_r \{(l_m)_r\}^2 = 1,$$

while there are the relations of perpendicularity

$$\sum_r \{(l_1)_r\} \{(l_m)_r\} = 0,$$

---

* As with the notation for the direction-cosines, the subscript r outside the bracket indicates the x_r-coordinate of the centre of curvature, for $r=1, \ldots, n$; and the subscript within the bracket, here unity, is the ordinal number of the centre of curvature with the specified coordinates as well as the ordinal number of the rank of curvature with that centre.

for $i = 1, 2, \ldots, m-1$. Further, the centre of $(m-1)$-ary curvature C_{m-1} lies on this axial line; this point C_{m-1} is the intersection of the contact homaloid T_m of m-fold contact with the normal homaloid N_m of m-fold normality. The axial line, to which the mth principal line is parallel, is the line joining C_{m-2} and C_{m-1}. We denote the coordinates of C_{m-1} by

$$(\xi_{m-1})_1,\ (\xi_{m-1})_2,\ \ldots\ldots,\ (\xi_{m-1})_n.$$

Thus the orbiculate amplitude O_{m-1}, being an amplitude of $m-1$ dimensions, lies in the contact homaloid T_m with m-fold contact and of m dimensions, and it has the same degree of contact with the curve as T_m. The centre of the orbiculate amplitude O_{m-1} is the point C_{m-1}; its radius is R_{m-1}, and it passes through P.

V. The *nth* (*and final*) *principal line of the curve* is the normal of rank $n-1$. It is the line drawn through P parallel to the normal homaloid N_{n-1} of $(n-1)$-fold normality: this homaloid N_{n-1} being the line which is the intersection of the $n-1$ normal homaloids N_1, each of one-fold normality, at P and at $n-2$ points next consecutive to P. We denote its direction-cosines by

$$(l_n)_1,\ (l_n)_2,\ \ldots\ldots,\ (l_n)_n,$$

where

$$\sum_r \{(l_n)_r\}^2 = 1,$$

while there are the relations of perpendicularity

$$\sum_i \{(l_i)_r\}\{(l_n)_r\} = 0,$$

for $i = 1, 2, \ldots, n-1$. Further, the centre of $(n-1)$-ary curvature C_{n-1}, the last in the succession of centres of curvature for a curve in space of n dimensions, lies on this normal homaloid N_{n-1}, this point C_{n-1} is the intersection of the n normal homaloids N_1, each of one-fold normality, at P and at $(n-1)$ points next consecutive to P. We denote the coordinates of C_{n-1} by

$$(\xi_{n-1})_1,\ (\xi_{n-1})_2,\ \ldots\ldots,\ (\xi_{n-1})_n.$$

Thus there is the (final) orbiculate amplitude O_{n-1}, being an amplitude of $(n-1)$ dimensions. It lies in the general space of n dimensions; it has n-fold contact with the curve, contact of one order greater than that of T_{n-1}, the final contact homaloid. The centre of the orbiculate amplitude O_{n-1} is the point C_{n-1}, its radius is R_{n-1}; and it passes through P.

We thus have *the orthogonal frame for the curve* at the point P. It is constituted by the foregoing n principal lines at P which are an orthogonal system, every pair of lines in the system being perpendicular to one another.

Frenet equations for the principal lines and tilts.

197. The direction-cosines of the n lines in the orthogonal frame are

$$(l_m)_1,\ (l_m)_2,\ \ldots\ldots,\ (l_m)_n,$$

for $m = 1, 2, \ldots, n$. We denote any member of the set of direction-cosines of the mth principal line by l_m; and thus $l_1, l_2, \ldots, l_n$, taken together, will denote the aggregate of $(l_1)_r, (l_2)_r, \ldots, (l_n)_r$, taken together, being the cosines of the inclinations of the n principal lines to the axis of x_r, for all the values $r = 1, \ldots, n$, in succession.

Among these direction-cosines $l_1, l_2, \ldots, l_n$, there exist the equations which are the extension of the Frenet equations for homaloidal three-dimensional space. These *extended Frenet equations* are

$$\left.\begin{aligned}
\frac{dl_1}{ds} &= \qquad\qquad\quad \frac{1}{\rho_1} l_2 \\
\frac{dl_2}{ds} &= -\frac{1}{\rho_1} l_1 + \frac{1}{\rho_2} l_3 \\
\frac{dl_3}{ds} &= -\frac{1}{\rho_2} l_2 + \frac{1}{\rho_3} l_4 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{dl_{n-1}}{ds} &= -\frac{1}{\rho_{n-2}} l_{n-2} + \frac{1}{\rho_{n-1}} l_n \\
\frac{dl_n}{ds} &= -\frac{1}{\rho_{n-1}} l_{n-1}
\end{aligned}\right\}$$

These equations allow the formation of the expressions for the sets of direction-cosines in terms of the derivatives of the coordinates $x_1, x_2, \ldots, x_n$, of the point P on the curve. In particular, $(l_1)_1, (l_1)_2, \ldots, (l_1)_n$, are the direction-cosines of the tangent: we have

$$(l_1)_1 = \frac{dx_1}{ds}, \quad (l_1)_2 = \frac{dx_2}{ds}, \quad \ldots\ldots, \quad (l_1)_n = \frac{dx_n}{ds}.$$

The direction-cosines of the second principal line of the curve (its principal normal, and its radius of circular curvature) are

$$(l_2)_1 = \rho_1 \frac{d^2x_1}{ds^2}, \quad (l_2)_2 = \rho_1 \frac{d^2x_2}{ds^2}, \quad \ldots\ldots, \quad (l_2)_n = \rho_1 \frac{d^2x_n}{ds^2}.$$

The direction-cosines of the third principal line of the curve (its binormal, which is parallel to the axial line joining C_1 and C_2) are

$$(l_3)_1 = \frac{\rho_2}{\rho_1}(x_1' + \rho_1\rho_1' x_1'' + \rho_1^2 x_1'''), \quad (l_3)_2 = \frac{\rho_2}{\rho_1}(x_2' + \rho_1\rho_1' x_2'' + \rho_1^2 x_2'''), \quad \ldots\ldots,$$

$$(l_3)_n = \frac{\rho_2}{\rho_1}(x_n' + \rho_1\rho_1' x_n'' + \rho_1^2 x_n''').$$

And so on, for the principal lines in succession.

It is to be noted that two sets of expressions can be obtained for the direction-cosines of the nth (the final) principal line. This line is perpendicular to the contact homaloid T_{n-1}, with $(n-1)$-fold contact and of $n-1$ dimensions, having

$$\left|\begin{array}{cccc} \bar{x}_1 - x_1, & \bar{x}_2 - x_2, & \ldots\ldots, & \bar{x}_n - x_n \\ \frac{dx_1}{ds}, & \frac{dx_2}{ds}, & \ldots\ldots, & \frac{dx_n}{ds} \\ \frac{d^2x_1}{ds^2}, & \frac{d^2x_2}{ds^2}, & \ldots\ldots, & \frac{d^2x_n}{ds^2} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{d^{n-1}x_1}{ds^{n-1}}, & \frac{d^{n-1}x_2}{ds^{n-1}}, & \ldots\ldots, & \frac{d^{n-1}x_n}{ds^{n-1}} \end{array}\right| = 0$$

for its equation. Let this equation be written

$$\sum_m (\bar{x}_m - x_m) J_m = 0,$$

where

$$J_m = (-1)^{(n-1)(m-1)} \left|\begin{array}{cccc} \frac{dx_{m+1}}{ds}, & \frac{dx_{m+2}}{ds}, & \ldots\ldots, & \frac{dx_{m-1}}{ds} \\ \frac{d^2x_{m+1}}{ds^2}, & \frac{d^2x_{m+2}}{ds^2}, & \ldots\ldots, & \frac{d^2x_{m-1}}{ds^2} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{d^{n-1}x_{m+1}}{ds^{n-1}}, & \frac{d^{n-1}x_{m+2}}{ds^{n-1}}, & \ldots\ldots, & \frac{d^{n-1}x_{m-1}}{ds^{n-1}} \end{array}\right|,$$

the sequence of the columns in J_m is for the subscripts 1, 2, ..., n, taken in cyclical order, beginning with $m+1$ and omitting m. Now

$$\sum_m J_m^2 = \frac{1}{(\rho_1^{n-2}\rho_2^{n-3} \ldots \rho_{n-3}^2 \rho_{n-2})^2};$$

so that the direction-cosines of the line perpendicular to the contact homaloid T_{n-1} are

$$\rho_1^{n-2}\rho_2^{n-3} \ldots \rho_{n-3}^2 \rho_{n-2} J_m,$$

for $m = 1, 2, \ldots, n$. But this perpendicular to T_{n-1} is the (line) normal homaloid N_{n-1}: that is, it is parallel to the nth principal line of the curve, the direction-cosines of which are $(l_n)_1, (l_n)_2, \ldots, (l_n)_n$. Hence

$$(l_n)_m = \rho_1^{n-2}\rho_2^{n-3} \ldots \rho_{n-3}^2 \rho_{n-2} J_m.$$

Coordinates of the centres of (orbiculate) curvature.

198. The coordinates of the centre of curvature C_m have been denoted by $(\xi_m)_1, (\xi_m)_2, \ldots, (\xi_m)_n$: any selected one of these coordinates will be denoted by ξ_m. When a set of selected coordinates $\xi_1, \xi_2, \xi_3, \ldots$ is considered, it represents a set $(\xi_1)_r, (\xi_2)_r, (\xi_3)_r, \ldots$ of coordinates corresponding to the coordinate x_r; and this convention holds for all the values $r = 1, 2, \ldots, n$. Under this same convention, we shall use x to denote x_r simultaneously with the use of ξ_m to denote $(\xi_m)_r$. Equally under that convention, we shall use

$l_1, l_2, l_3, \ldots$ to represent a set of direction-cosines $(l_1)_r, (l_2)_r, (l_3)_r, \ldots$, being the cosines of the inclinations of the successive principal lines to the axis of the x_r-coordinate.

The coordinates of the centre of curvature C_1, stated in full, are

$$(\xi_1)_1 - x_1 = \rho_1 (l_2)_1, \quad (\xi_1)_2 - x_2 = \rho_1 (l_2)_2, \quad \ldots\ldots, \quad (\xi_1)_n - x_n = \rho_1 (l_2)_n.$$

Under the convention for abbreviation, the whole set is to be represented by the single equation

$$\xi_1 - x = \rho_1 l_2,$$

and, when this equation is interpreted, we take ξ_1, x, l_2, respectively, to mean $(\xi_1)_r, x_r, (l_2)_r$, for each of the n values of $r, = 1, 2, \ldots, n$.

The *coordinates of all the centres of curvature* are then expressible in the following form, for the respective centres:

$$\left.\begin{array}{llcl}
\text{for } C_1, & \xi_1 \quad - x & = & \rho_1 l_2 \\
\ldots\ C_2, & \xi_2 \quad - \xi_1 & = & D_1 l_3 \\
\ldots\ C_3, & \xi_3 \quad - \xi_2 & = & D_2 l_4 \\
\ldots\ldots\ldots & \ldots\ldots\ldots & & \ldots\ldots \\
\ldots\ C_{n-2}, & \xi_{n-2} - \xi_{n-3} & = & D_{n-3} l_{n-1} \\
\ldots\ C_{n-1}, & \xi_{n-1} - \xi_{n-2} & = & D_{n-2} l_n
\end{array}\right\}.$$

Tangents to the loci of these centres.

199. These results can be stated in another way; and, as they stand, they provide further information concerning the organic construction of the framework of the curve in relation to its orbiculate regions.

(A) The coordinates of the centre C_m of curvature of the orbiculate region O_m of $(m+1)$-fold curvature, lying within the contact homaloid T_{m+1} also of $(m+1)$-fold curvature, are represented by the equation

$$\xi_m = x + \rho_1 l_2 + D_1 l_3 + D_2 l_4 + \ldots + D_{m-1} l_{m+1},$$

with the interpretation of this equation under the convention as stated (§ 198). Accordingly, the coordinates

$$\xi_m, \; = (\xi_m)_1, \; (\xi_m)_2, \; \ldots\ldots, \; (\xi_m)_n,$$

regarded as functions of the parameter s of the original curve, give the locus of the centre C_m of $(m+1)$-fold curvature.

Moreover, the elements of arc of these successive loci are easily derived. We have

$$\frac{d\xi_1}{ds} = \frac{1}{\rho_2}(\rho_1 l_3 + D_1 l_2),$$

$$\frac{d\xi_2}{ds} = \frac{1}{\rho_3}(D_1 l_4 + D_2 l_3),$$

$$\frac{d\xi_3}{ds} = \frac{1}{\rho_4}(D_2 l_5 + D_3 l_4),$$

and, generally,

$$\frac{d\xi_m}{ds} = \frac{1}{\rho_{m+1}}(D_{m-1} l_{m+2} + D_m l_{m+1}).$$

If then ds_m be the element of arc of the locus of C_m at the point C_m, corresponding to the element of arc ds of the original curve at P,

$$\left(\frac{ds_m}{ds}\right)^2 = \frac{1}{\rho_{m+1}^2}(D_{m-1}^2 + D_m^2);$$

and therefore

$$\frac{d\xi_m}{ds_m} = \frac{D_{m-1} l_{m+2} + D_m l_{m+1}}{(D_{m-1}^2 + D_m^2)^{\frac{1}{2}}}.$$

These results give the direction-cosines of the tangent to the locus of the centre C_m; and they shew that the tangent to the locus lies in a plane which has the $(m+1)$th and the $(m+2)$th principal directions for its guiding lines. The further curvatures of that curve can be derived by the use of the formulæ given for the curvatures of the locus of P.

(B) The expressions for the coordinates of the centres of successive curvatures, as given (§ 198), indicate the axial lines to which the successive principal lines of the curve are parallel. They indicate also the length along each such line between the two successive centres of curvature which lie upon a line.

Thus the axial line $C_1 C_2$ is given by the representative equation

$$\xi_2 - \xi_1 = D_1 l_3,$$

the length $C_1 C_2$ being D_1, and its direction-cosines being $(l_3)_1, (l_3)_2, \ldots, (l_3)_n$. The axial line $C_2 C_3$ is given by the similar equation

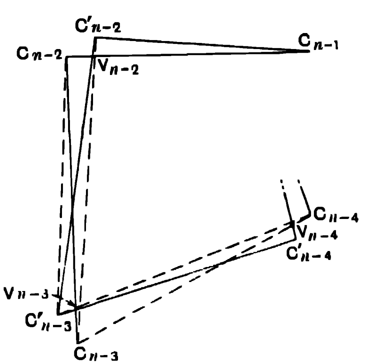
$$\xi_3 - \xi_2 = D_2 l_4,$$

the length $C_2 C_3$ being D_2, and its direction-cosines being $(l_4)_1, (l_4)_2, \ldots, (l_4)_n$. And so for the axial lines in succession, the last of them being the final normal (line) amplitude with $(n-1)$-fold normality.

Organic (orthogonal) frame.

200. The whole organic framework of the curve, with its principal lines and its centres of curvature at the point P, and at a consecutive point P', are indicated on the accompanying diagram. The lower part of the diagram shews the initial principal lines and centres of curvature, from P inwards and upwards and onwards through the successive amplitudes. The upper part of the diagram shews the axial lines, to which the principal lines of the curve are parallel: these pass from centre to centre, culminating in C_{n-1}, the centre of orbiculate amplitude O_{n-1} of (greatest) $(n+1)$-fold contact. In the succession of lines PC_1, C_1C_2, C_2C_3, C_3C_4, …, $C_{n-4}C_{n-3}$, $C_{n-3}C_{n-2}$, $C_{n-2}C_{n-1}$, each is perpendicular not merely to its predecessor but to all the other lines. The point P' is consecutive to P: the orthogonal frame for P' is shewn in the same manner as the orthogonal frame for P. The limiting position of PP' is the tangent at P, the earliest principal line of the curve: and all the principal lines are perpendicular to the tangent.

Again, $PP'C_1$ is the two-fold contact homaloid T_2 at P, and $PP'C_1'$ is the two-fold contact homaloid T_2 at P', their intersection is the tangent PP'. The primary normal at P is PC_1: the primary normal at P' is $P'C_1'$. The angle $C_1'PP'$ in the plane $C_1'PP'$ is a right angle, and the angle C_1PP' in the plane C_1PP' also is a right angle; thus the angle $C_1'PC_1$ is the angle between the two successive osculating planes, that is, it is the angle of tilt $d\epsilon_2$. (The angle of tilt $d\epsilon_1$ is the angle between the tangent at P and the tangent at P', both lying in the osculating plane C_1PP'.)



The arc C_1C_1' lies in the plane through the two lines C_1P and C_1C_2 to which the second and the third principal lines respectively are parallel. The angle PC_1C_2 is a right angle: the angle $C_2C_1'P$ also is a right angle: and the four points P, C_1, C_1', C_2, lie in the one plane of which C_1P and C_1C_2 can be taken as the guiding lines. Hence these four points lie on a circle, a property leading to several results: the only result to be noted at the moment is

$$\text{angle } C_1'C_2C_1 = \text{angle } C_1'PC_1$$
$$= \text{angle of tilt } d\epsilon_2.$$

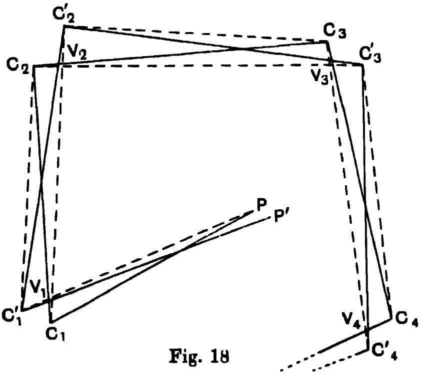


Fig. 18

The arc C_2C_2' lies in the plane through the two lines C_2C_1 and C_2C_3, to which the third and the fourth principal lines respectively are parallel. The angle $C_1C_2C_3$ is a right angle and the angle $C_3C_2'C_1$ also is a right angle: and the four points C_1, C_2, C_2', C_3, lie in the one plane of which C_2C_1 and C_2C_3 are the guiding lines. Hence these four points lie on a circle; and therefore

$$\text{angle } C_2'C_3C_2 = \text{angle } C_2'C_1C_2 = \text{angle of tilt } d\epsilon_3.$$

And so on, in succession; the full set of results is

$$\begin{array}{lll}
\text{angle } C_1'C_2C_1 & = \text{angle } C_1'PC_1 & = \text{angle of tilt } d\epsilon_2, \\
\ldots\ldots\ C_2'C_3C_2 & = \ldots\ldots\ C_2'C_1C_2 & = \ldots\ldots\ldots\ldots\ldots\ d\epsilon_3, \\
\ldots\ldots\ C_3'C_4C_3 & = \ldots\ldots\ C_3'C_2C_3 & = \ldots\ldots\ldots\ldots\ldots\ d\epsilon_4, \\
\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ C'_{n-3}C_{n-2}C_{n-3} & = \ldots\ldots\ C'_{n-3}C_{n-4}C_{n-3} & = \ldots\ldots\ldots\ldots\ldots\ d\epsilon_{n-2}, \\
\ldots\ldots\ C'_{n-2}C_{n-1}C_{n-2} & = \ldots\ldots\ C'_{n-2}C_{n-3}C_{n-2} & = \ldots\ldots\ldots\ldots\ldots\ d\epsilon_{n-1}.
\end{array}$$

Certain magnitudes are easily obtained from the properties of the two orthogonal frames, at P and at P' respectively, when taken in combination.

In the plane $PC_1C_1'C_2$, the point V_1 is the intersection of C_1C_2 and $C_1'P$: the length of C_1P is ρ_1: and the angle C_1PC_1' is $d\epsilon_2$, while C_2C_1P and $C_2C_1'P$ are right angles. Then

$$\begin{aligned} d\rho_1 = C'V_1 &= C_1C_2 . \text{angle } C_1C_2C_1' \text{ to the first order} \\ &= C_1C_2 . d\epsilon_2, \end{aligned}$$

and therefore

$$D_1 = C_1C_2 = \frac{d\rho_1}{d\epsilon_2} = \rho_2\rho_1'.$$

Also, we have $C_1'V_1 = \rho_1 d\epsilon_2$, to the first order of small quantities.

In the plane $C_1C_2C_2'C_3$, the point V_2 is the intersection of C_2C_3 and $C_2'C_1$: the length of C_1C_2 is D_1 and that of $C_1'C_2'$ is $D_1 + D_1'ds$: and the angle $C_2C_1C_2'$ is $d\epsilon_3$, while the angles $C_3C_2C_1$ and $C_3C_2'C_1$ are right angles. Then

$$\begin{aligned} C_2'V_2 &= C_2'C_1' - C_2C_1' \\ &= C_2'C_1' - (C_2C_1 - C_1V_1) \\ &= (C_2'C_1' - C_2C_1) + C_1V_1 \\ &= D_1'ds + \rho_1 d\epsilon_2; \end{aligned}$$

and also

$$\begin{aligned} C_2'V_2 &= C_2C_3 . \text{angle } C_2C_3C_2' \text{ to the first order} \\ &= D_2 d\epsilon_3. \end{aligned}$$

Hence

$$D_2 d\epsilon_3 = D_1'ds + \rho_1 d\epsilon_2,$$

and therefore

$$\frac{D_2}{\rho_3} = D_1' + \frac{\rho_1}{\rho_2}.$$

Also, we have $C_2'V_2 = D_1 d\epsilon_3$, to the first order of small quantities.

Similarly, in the plane $C_2C_3C_3'C_4$, we have

$$\begin{aligned} C_3'V_3 &= C_3'C_2' - C_3C_2' \\ &= C_3'C_2' - (C_3C_2 - C_2V_2) \\ &= (C_3'C_2' - C_3C_2) + C_2V_2 \\ &= D_2'ds + D_1 d\epsilon_3, \end{aligned}$$

and

$$C_3'V_3 = C_3C_4 \,.\, \text{angle } C_3C_4C_3' \text{ to the first order}$$
$$= D_3 d\epsilon_4.$$

Hence

$$D_3 d\epsilon_4 = D_2' ds + D_1 d\epsilon_3,$$

and therefore

$$\frac{D_3}{\rho_4} = D_2' + \frac{D_1}{\rho_3}.$$

And so on for each plane, containing the directions of two principal lines $C_m C_{m-1}$ and $C_m C_{m+1}$, for all the values of m. In the final plane $C_{n-3}C_{n-2}C'_{n-2}C_{n-1}$ (which four specified points lie on a circle), we have

$$\begin{aligned} C'_{n-2}V_{n-2} &= C'_{n-2}C'_{n-3} - (C_{n-2}C_{n-3} - C_{n-3}V_{n-3}) \\ &= (C'_{n-2}C'_{n-3} - C_{n-2}C_{n-3}) + C_{n-3}V_{n-3} \\ &= D'_{n-3}ds + D_{n-4}d\epsilon_{n-2}. \end{aligned}$$

Also

$$C'_{n-2}V_{n-2} = C_{n-2}C_{n-1} \,.\, d\epsilon_{n-1} = D_{n-2}d\epsilon_{n-1};$$

thus

$$D_{n-2}d\epsilon_{n-1} = D'_{n-3}ds + D_{n-4}d\epsilon_{n-2},$$

and therefore

$$\frac{D_{n-2}}{\rho_{n-1}} = D'_{n-3} + \frac{D_{n-4}}{\rho_{n-2}}.$$

All these formulæ involve the various tilts. For the various radii of curvature, we have

$$R_1 = C_1P = \rho_1;$$

then

$$\begin{aligned} R_2^2 = C_2P^2 &= C_2C_1^2 + C_1P^2 \\ &= D_1^2 + R_1^2, \\ R_3^2 = C_3P^2 &= C_3C_2^2 + C_2P^2 \\ &= D_2^2 + R_2^2, \end{aligned}$$

and so on, down to

$$\begin{aligned} R_{n-1}^2 = C_{n-1}P^2 &= C_{n-1}C_{n-2}^2 + C_{n-2}P^2 \\ &= D_{n-2}^2 + R_{n-2}^2. \end{aligned}$$

The results are immediate consequences of the complete orthogonality of the frame of lines PC_1, C_1C_2, C_2C_3, ..., $C_{n-3}C_{n-2}$, $C_{n-2}C_{n-1}$.

Finally, it may be remarked that the one-dimensional orbiculate amplitude O_1 (the circle of curvature, with centre C_1) is the section of the two-dimensional orbiculate amplitude O_2 (the sphere of curvature, with centre C_2) by the two-fold contact homaloid T_2, which is the osculating plane. Similarly, the two-dimensional orbiculate amplitude O_2 is the section of the three-dimensional orbiculate amplitude O_3 (called the globe of curvature in a four-dimensional space, with centre C_3) by the three-fold contact homaloid T_3, which is called the osculating flat in a four-dimensional space. And so on, in the succession of the orbiculate amplitudes $O_1, O_2, \ldots, O_{n-2}$. the final amplitude O_{n-1} requires the full n-fold space for its existence.

Modified equations of contact homaloids and normal homaloids.

201. The equations of the contact homaloids T_m, of successive orders of contact, and the normal homaloids N_m, of successive grades of normality, can be changed so as to become expressible in terms of the direction-cosines of the principal lines.

The equations of the contact homaloid T_m, with m-fold contact and of m dimensions, are

$$\left\Vert \begin{matrix} \bar{x}_1 - x_1, & \bar{x}_2 - x_2, & \ldots\ldots, & \bar{x}_n - x_n \\ (l_1)_1\ , & (l_1)_2\ , & \ldots\ldots, & (l_1)_n \\ (l_2)_1\ , & (l_2)_2\ , & \ldots\ldots, & (l_2)_n \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ (l_m)_1\ , & (l_m)_2\ , & \ldots\ldots, & (l_m)_n \end{matrix} \right\Vert = 0,$$

being a set of $n-m$ independent equations. The $n-m$ equations of the same contact homaloid T_m with m-fold contact can also be taken in the form

$$\left. \begin{array}{l} \Sigma\,\{(\bar{x}_r - x_r)\,(l_{m+1})_r\} = 0 \\ \Sigma\,\{(\bar{x}_r - x_r)\,(l_{m+2})_r\} = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Sigma\,\{(\bar{x}_r - x_r)\,(l_n)_r\} \quad = 0 \end{array} \right\}.$$

Both forms are valid for the values $m = 1, 2, \ldots, n-1$.

The equations of the normal homaloid N_m, with m-fold normality and of $n-m$ dimensions, are

$$\left. \begin{array}{l} \Sigma\,[\{\bar{x}_r - (\xi_m)_r\}\,(l_1)_r\,] = 0 \\ \Sigma\,[\{\bar{x}_r - (\xi_m)_r\}\,(l_2)_r\,] = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Sigma\,[\{\bar{x}_r - (\xi_m)_r\}\,(l_m)_r] = 0 \end{array} \right\};$$

and these equations can be expressed in the equivalent form

$$\left\Vert \begin{matrix} \bar{x}_1 - (\xi_m)_1, & \bar{x}_2 - (\xi_m)_2, & \ldots\ldots, & \bar{x}_n - (\xi_m)_n \\ (l_{m+1})_1\ , & (l_{m+1})_2\ , & \ldots\ldots, & (l_{m+1})_n \\ (l_{m+2})_1\ , & (l_{m+2})_2\ , & \ldots\ldots, & (l_{m+2})_n \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ (l_n)_1\ , & (l_n)_2\ , & \ldots\ldots, & (l_n)_n \end{matrix} \right\Vert = 0.$$

The coordinates of any point Q in the contact homaloid T_m are expressible in the form

$$\bar{x}_r - x_r = t_1\,(l_1)_r + t_2\,(l_2)_r + \ldots\ldots + t_m\,(l_m)_r,$$

for $r = 1, 2, \ldots, n$, where $t_1, t_2, \ldots, t_m$, are m parameters for the homaloid and manifestly represent the projections of the distance QP upon the first m principal lines, these being the guiding lines of that contact homaloid.

Similarly, the coordinates of any point K in the normal homaloid N_m are expressible in the form

$$\bar{x}_r - (\xi_m)_r = \theta_{m+1}(l_{m+1})_r + \theta_{m+2}(l_{m+2})_r + \ldots\ldots + \theta_n (l_n)_r,$$

for $r = 1, 2, \ldots, n$, where $\theta_{m+1}, \theta_{m+2}, \ldots, \theta_n$ are $n - m$ parameters for the homaloid and manifestly represent the projections of the distance KC_m upon the last $n - m$ principal lines, these being taken as the guiding lines of that normal homaloid.

A homaloid of $n-1$ dimensions, involving one parameter, as the fundamental element for a curve.

202. Finally, as for three-dimensional space and for four-dimensional space, it is possible to take a homaloid of $n - 1$ dimensions* as the fundamental element for a skew curve in n-dimensional homaloidal space, provided its equation involves a single parameter t. As in § 180 for a skew curve in quadruple space, we work back from the given $(n-1)$-fold homaloidal amplitude as the osculating contact homaloid T_{n-1} of the curve, and a poin on the curve, which is the edge of regression on the $(n-1)$-fold amplitud enveloping that given contact homaloid T_{n-1}, is determined as the inter section of n consecutive homaloids T_{n-1}. The following are the successiv stages.

The homaloid of $n - 1$ dimensions in the n-fold space is represented by an equation

$$\sum_{r=1}^{n} L_r \bar{x}_r = P.$$

If the quantities L_r in this equation are not direction-cosines, we divide the equation throughout by $(\Sigma L_r^2)^{\frac{1}{2}}$; we write

$$L_r (\Sigma L_r^2)^{-\frac{1}{2}} = (l_n)_r, \quad P(\Sigma L_r^2)^{-\frac{1}{2}} = p_n;$$

and the equation becomes

$$\sum_{r=1}^{n} (l_n)_r \bar{x}_r = p_n,$$

which, in accordance with the propounded notation, can be written

$$\Sigma l_n \bar{x} = p_n.$$

The equation of this homaloid of $n - 1$ dimensions is to involve a single parameter: the parameter will be denoted by t.

Let

$$\sum_r \left\{\frac{d(l_n)_r}{dt}\right\}^2 = \Sigma \left(\frac{dl_n}{dt}\right)^2 = \alpha_1^2,$$

so that α_1 is a known function of t: let

$$\frac{dl_n}{dt} = -\alpha_1 l_{n-1},$$

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* Hereafter (§ 422) called *primary*.

that is, we take

$$\frac{d(l_n)_r}{dt} = -\alpha_1 (l_{n-1})_r,$$

for $r = 1, \ldots, n$: and let

$$-\frac{1}{\alpha_1}\frac{dp_n}{dt} = p_{n-1}.$$

Then the second equation is

$$\Sigma l_{n-1}\bar{x} = p_{n-1};$$

and the quantities l_{n-1} are such that

$$\Sigma l_{n-1}^2 = 1, \quad \Sigma l_n l_{n-1} = 0.$$

Next, let

$$\Sigma\left(\frac{dl_{n-1}}{dt}\right)^2 = \alpha_1^2 + \alpha_2^2,$$

so that α_2 is a known function of t; we take

$$\frac{dl_{n-1}}{dt} = \alpha_1 l_n - \alpha_2 l_{n-2}, \quad \frac{1}{\alpha_2}\left(-\frac{dp_{n-1}}{dt} + \alpha_1 p_n\right) = p_{n-2};$$

and then the third equation is

$$\Sigma l_{n-2}\bar{x} = p_{n-2},$$

where

$$\Sigma l_{n-2}^2 = 1, \quad \Sigma l_n l_{n-2} = 0, \quad \Sigma l_n l_{n-1} = 0.$$

And so on, in succession. At the last stage but one, giving the $(n-1)$th equation, we have

$$\Sigma l_2\bar{x} = p_2 = \frac{1}{\alpha_{n-2}}\left(-\frac{dp_3}{dt} + \alpha_{n-3}p_4\right).$$

Let

$$\left(\frac{dl_2}{dt}\right)^2 = \alpha_{n-2}^2 + \alpha_{n-1}^2:$$

we take

$$\frac{dl_2}{dt} = \alpha_{n-2}l_3 - \alpha_{n-1}l_1, \quad \frac{1}{\alpha_{n-1}}\left(-\frac{dp_2}{dt} + \alpha_{n-2}p_3\right) = p_1;$$

and then the nth equation, being the last in the sequence, is

$$\Sigma l_1\bar{x} = p_1,$$

while

$$\Sigma l_1^2 = 1, \quad \Sigma l_1 l_2 = 0, \quad \Sigma l_1 l_3 = 0, \quad \ldots, \quad \Sigma l_1 l_n = 0.$$

The lines l_m, that is, $(l_m)_1, (l_m)_2, \ldots, (l_m)_n$, for $m = 1, \ldots, n$, form an orthogonal system: and so

$$(l_1)_r^2 + (l_2)_r^2 + \ldots + (l_n)_r^2 = 1,$$

for $r = 1, \ldots, n$: hence

$$\sum_m (l_m)_r \frac{d(l_m)_r}{dt} = 0,$$

and therefore

$$\frac{dl_1}{dt} = l_2 a_{n-1}.$$

There now are n equations, viz.

$$\Sigma l_n \bar{x} = p_n, \quad \Sigma l_{n-1} \bar{x} = p_{n-1}, \quad \ldots\ldots, \quad \Sigma l_2 \bar{x} = p_2, \quad \Sigma l_1 \bar{x} = p_1,$$

which, together, represent n successive contact-homaloids of the edge of regression. Their common point of intersection is a point on that edge: its coordinates are given by

$$\bar{x}_r = (l_n)_r p_n + (l_{n-1})_r p_{n-1} + \ldots\ldots + (l_2)_r p_2 + (l_1)_r p_1,$$

for $r = 1, \ldots, n$.

The curvatures of the curve, thus derived.

203. Let s be an arc of this curve, the edge of regression of the developable amplitude which envelopes T_{n-1} and arises as the t-eliminant of the two equations

$$\Sigma l_n \bar{x} = p_n, \quad \Sigma l_{n-1} \bar{x} = p_{n-1}:$$

and write

$$\theta = \frac{ds}{dt}.$$

Now the direction-cosines of the tangent to the edge of regression are $(l_1)_1, (l_1)_2, \ldots, (l_1)_n$: and they also are

$$\frac{d\bar{x}_1}{ds}, \quad \frac{d\bar{x}_2}{ds}, \quad \ldots, \quad \frac{d\bar{x}_n}{ds},$$

that is, they are

$$\frac{1}{\theta}\frac{d\bar{x}_1}{dt}, \quad \frac{1}{\theta}\frac{d\bar{x}_2}{dt}, \quad \ldots, \quad \frac{1}{\theta}\frac{d\bar{x}_n}{dt}.$$

Hence

$$(l_1)_r = \frac{1}{\theta}\frac{d\bar{x}_r}{dt}.$$

Now, from the final equation

$$\Sigma l_1 \bar{x} = p_1,$$

we have

$$\Sigma l_1 \frac{d\bar{x}}{dt} + \Sigma \bar{x} \frac{dl_1}{dt} = \frac{dp_1}{dt},$$

that is,

$$\theta + a_{n-1} \Sigma l_2 \bar{x} = \frac{dp_1}{dt},$$

so that

$$\theta = \frac{dp_1}{dt} - a_{n-1} p_2.$$

These results implicitly contain the values of the curvatures of tilt of the curve. The successive sets of direction-cosines are given by the relations

$$\begin{aligned}
\frac{dl_n}{dt} &= \qquad\quad - \alpha_1 l_{n-1}, \\
\frac{dl_{n-1}}{dt} &= \alpha_1 l_n \quad - \alpha_2 l_{n-2}, \\
\frac{dl_{n-2}}{dt} &= \alpha_2 l_{n-1} - \alpha_3 l_{n-3}, \\
&\dots\dots\dots\dots\dots\dots \\
\frac{dl_3}{dt} &= \alpha_{n-3} l_4 - \alpha_{n-2} l_2, \\
\frac{dl_2}{dt} &= \alpha_{n-2} l_3 - \alpha_{n-1} l_1, \\
\frac{dl_1}{dt} &= \alpha_{n-1} l_2,
\end{aligned}$$

hence

$$\begin{aligned}
\frac{1}{\theta}(-\alpha_1 l_{n-1}) &= \frac{1}{\theta}\frac{dl_n}{dt} = \frac{dl_n}{ds} = \qquad\quad - \frac{l_{n-1}}{\rho_{n-1}}, \\
\frac{1}{\theta}(\alpha_1 l_n - \alpha_2 l_{n-2}) &= \frac{1}{\theta}\frac{dl_{n-1}}{dt} = \frac{dl_{n-1}}{ds} = \frac{l_n}{\rho_{n-1}} - \frac{l_{n-2}}{\rho_{n-2}}, \\
\frac{1}{\theta}(\alpha_2 l_{n-1} - \alpha_3 l_{n-3}) &= \frac{1}{\theta}\frac{dl_{n-2}}{dt} = \frac{dl_{n-2}}{ds} = \frac{l_{n-1}}{\rho_{n-2}} - \frac{l_{n-3}}{\rho_{n-3}}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\frac{1}{\theta}(\alpha_{n-2} l_3 - \alpha_{n-1} l_1) &= \frac{1}{\theta}\frac{dl_2}{dt} = \frac{dl_2}{ds} = \frac{l_3}{\rho_2} - \frac{l_1}{\rho_1}, \\
\frac{1}{\theta}\alpha_{n-1} l_2 &= \frac{1}{\theta}\frac{dl_1}{dt} = \frac{dl_1}{ds} = \frac{1}{\rho_1} l_2.
\end{aligned}$$

Consequently the successive curvatures of tilt are given by the equations

$$\frac{1}{\rho_1} = \frac{\alpha_{n-1}}{\theta}, \quad \frac{1}{\rho_2} = \frac{\alpha_{n-2}}{\theta}, \quad \dots\dots, \quad \frac{1}{\rho_{n-2}} = \frac{\alpha_2}{\theta}, \quad \frac{1}{\rho_{n-1}} = \frac{\alpha_1}{\theta}.$$

The successive radii of the orbiculate amplitudes of successive degrees of contact are derived from these curvatures of tilt. Thus, as

$$\theta\frac{d}{ds} = \frac{d}{dt},$$

we have

$$\theta\rho_1' = \theta\frac{d\rho_1}{ds} = \frac{d\rho_1}{dt} = \frac{d}{dt}\left(\frac{\theta}{\alpha_{n-1}}\right).$$

Therefore

$$R_1^2 = \rho_1^2 + \rho_2^2\left(\frac{d\rho_1}{ds}\right)^2 = \frac{\theta^2}{\alpha_{n-1}^2} + \frac{1}{\alpha_{n-2}^2}\left\{\frac{d}{dt}\left(\frac{\theta}{\alpha_{n-1}}\right)\right\}^2;$$

and similarly for the quantities $R_2, \dots, R_{n-1}$, in succession.

CHAPTER XII.

SURFACES IN HOMALOIDAL QUADRUPLE SPACE.

Analytical representation of surfaces.

204. We pass to the consideration of surfaces in quadruple space, which have been defined as amplitudes of two dimensions in that space. We now consider only such amplitudes as are not homaloidal.

There are various forms of data by which a surface can be represented analytically.

A surface may be regarded as the range common to two regions which are represented by two equations

$$\Theta(x, y, z, v) = 0, \quad \Phi(x, y, z, v) = 0.$$

In this form of expression there is a disadvantage, analogous to that which occurs when a skew curve in three dimensions is represented as the intersection of two surfaces expressed by their equations. The disadvantage arises from the possibility (which frequently also is fact) that the skew curve in triple space and our two-dimensional amplitude in quadruple space do not represent the whole of the amplitude of intersection; e.g. the whole of that amplitude of intersection may consist of two. or of more than two, portions which are not geometrically continuous with one another.

A surface may be represented by means of equations which express the coordinates of any point in the surface in terms of two independent parameters, in a form

$$x = x(p, q), \quad y = y(p, q), \quad z = z(p, q), \quad v = v(p, q),$$

as in Gauss's geometry of surfaces in triple space. When this form is given, it is always possible to derive a partial representation of the surface in the preceding form: because it is always possible to eliminate p and q between these equations, and thus to obtain two equations

$$f(x, y, z) = 0, \quad g(x, y, v) = 0,$$

independent of one another. This deduced representation suffers from another kind of disadvantage. The equation $f(x, y, z) = 0$ is derived without regard to the value of v, and is the projection of the surface into a flat parallel to $v = 0$: while, similarly, the equation $g(x, y, v) = 0$ is derived without regard to the value of z, and is the projection of the surface into a flat parallel to $z = 0$. Each of these projections in itself suppresses curvature properties of the surface.

There is yet a third mode of representing a surface it may be an amplitude within a region. The region is defined by coordinates expressible in terms of three parameters p, q, r, independent of one another, in forms

$$x = x(p, q, r), \quad y = y(p, q, r), \quad z = z(p, q, r), \quad v = v(p, q, r):$$

the amplitude is defined by a functional relation $\theta(p, q, r) = 0$. If the essential character of such a representation is to be used, a prior knowledge of the properties of the containing region is both desirable and necessary: while, if the former two-parameter representation is initially propounded, we are not in a position to determine an appropriate region containing the surface. Surfaces organically connected with the region will, of course, occur: they belong to a different range of enquiry.

Now, when a surface is given in free space without any of the intrinsic restrictions of a containing region, itself enclosed in that space, all external measurements have to be made relative solely to the tangential or normal homaloids of various types—line, plane, flat. When a surface is given in a region, itself enclosed in the quadruple space, the external measurements are of two categories. The first of these includes measurements made relative solely to the enclosing region or its organic elements; the second category includes those which may belong to the quadruple space and are affected, directly or indirectly, by the properties of the region itself relative to the space. Thus the intrinsic properties of a surface in free space are less detailed than those of a surface as a configuration in a region.

Here it may be remarked initially, and it will appear abundantly in the course of investigations, that, in any amplitude of more than one dimension, the geodesics in the amplitude (and also geodesics in any containing amplitude of more dimensions) are of fundamental importance In fact, to all such amplitudes, geodesics bear a relation as fully organic as is borne by straight lines to homaloidal space in which, indeed, they are the geodesics.

Thus the discussion of a surface in a region must be deferred until the intrinsic geometry of the region itself has been considered. We shall accordingly proceed to the consideration of a surface, propounded solely as a configuration in free space.

The coordinates of a point on a surface are therefore expressible, for our purpose, in terms of two parameters p and q. Much of the initial analysis in the Gauss theory of surfaces appears to recur formally. It must be remembered that the Gauss surfaces exist in homaloidal triple space and possess properties which are related to that spatial existence; indeed, these properties (such as the customary measure of curvature of a surface) are actually established by essential dependence upon the three-coordinate representation of a surface-point in the homaloidal space. The surfaces now to be considered exist in homaloidal quadruple space. The properties to be deduced are established through their existence in that space; and though a number of analytical results will be found to have the same formal expression for quadruple space as for triple space, it is not to be assumed that we can ignore the kind of homaloidal space in which the surface exists. Nevertheless, for the sake of comparison, there is convenience in using the same symbol for the same kind of magnitude wherever this may be possible.

Primary magnitudes.

205. Derivatives of the variables x, y, z, v, with regard to p will be denoted by a suffix, such as in x_1, y_1, z_1, v_1, x_{11}, and the like: derivatives with regard to q by a different suffix, such as in x_2, y_2, z_2, v_2, x_{22}, and the like: and derivatives with regard to p and q by a combination of these suffixes, such as in x_{12}, y_{12}, z_{12}, v_{12}. Certain combinations of these derivatives occur: the most frequent are

$$\left.\begin{aligned} E &= x_1^2 + y_1^2 + z_1^2 + v_1^2 = \Sigma x_1^2 \\ F &= x_1 x_2 + y_1 y_2 + z_1 z_2 + v_1 v_2 = \Sigma x_1 x_2 \\ G &= x_2^2 + y_2^2 + z_2^2 + v_2^2 = \Sigma x_2^2 \end{aligned}\right\},$$

where the sign Σ of summation denotes summation over the four variables x, y, z, v. Also, it is convenient to use a quantity V, where

$$V^2 = EG - F^2 = \Sigma (x_1 y_2 - y_1 x_2)^2,$$

the positive root of V^2 being selected as the value of V.

The locus on the surface, represented by $q =$ constant, is a curve along which p is parametric: and the locus, represented by $p =$ constant, is a curve along which q is parametric. On the surface, there are the two families of curves, $q =$ constant, $p =$ constant.

The element of arc ds, measured in the surface existing in the homaloidal quadruple space, is given by

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 + dv^2 \\ &= \Sigma (x_1 dp + x_2 dq)^2 \\ &= E dp^2 + 2F dp\, dq + G dq^2. \end{aligned}$$

Thus the element of arc along the curve $q =$ constant is $E^{\frac{1}{2}} dp$: the element of arc along the curve $p =$ constant is $G^{\frac{1}{2}} dq$.

The direction of the tangent along the curve $q =$ constant is given by $x_1 dp$, $y_1 dp$, $z_1 dp$, $v_1 dp$; thus the direction-cosines of the tangent to that curve are

$$x_1 E^{-\frac{1}{2}}, \quad y_1 E^{-\frac{1}{2}}, \quad z_1 E^{-\frac{1}{2}}, \quad v_1 E^{-\frac{1}{2}}.$$

Similarly, the direction-cosines of the tangent to the curve $p =$ constant at any point are

$$x_2 G^{-\frac{1}{2}}, \quad y_2 G^{-\frac{1}{2}}, \quad z_2 G^{-\frac{1}{2}}, \quad v_2 G^{-\frac{1}{2}}.$$

Let ω denote the angle between the tangents to the curves $q=$ constant, $p=$ constant, passing through any point: then (§ 19)

$$\cos\omega = \Sigma\,(x_1 E^{-\frac{1}{2}})\,(x_2 G^{-\frac{1}{2}}) = \Sigma x_1 x_2\,(EG)^{-\frac{1}{2}} = \frac{F}{(EG)^{\frac{1}{2}}},$$

and therefore

$$\sin\omega = \frac{V}{(EG)^{\frac{1}{2}}}.$$

The expression for ds^2 must be unaltered in value by a change of parameters, so that ds^2 is a covariant: thus

$$Edp^2 + 2Fdp\,dq + Gdq^2 = E'dp'^2 + 2F'dp'dq' + G'dq'^2.$$

It possesses one invariant: we have, in fact,

$$E'G' - F'^2 = (EG - F^2)\left\{J\left(\frac{p,\ q}{p',\ q'}\right)\right\}^2,$$

where J cannot vanish so long as the parameters remain independent.

Inclinations of directions on surfaces.

206. The following results are easily established; they are placed on record for subsequent use.

(1) When a direction OA on the surface is represented by p' and q', where

$$Ep'^2 + 2Fp'q' + Gq'^2 = 1,$$

the angles ϕ and ψ which it makes with the parametric curves in the figure (Op, for $p=$ *constant*, and Oq, for $q=$ *constant*) are given by

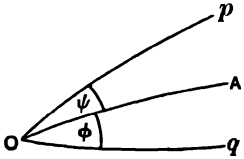

Fig. 19.

$$\left.\begin{aligned} E^{\frac{1}{2}}\cos\phi &= Ep' + Fq', & E^{\frac{1}{2}}\sin\phi &= Vq' \\ G^{\frac{1}{2}}\cos\psi &= Fp' + Gq', & G^{\frac{1}{2}}\sin\psi &= Vp' \end{aligned}\right\},$$

with the necessary relation $\phi + \psi = \omega$.

(2) When the curve OA is represented by an equation $\theta(p, q) = c$, where c is parametric, then

$$\theta_1 p' + \theta_2 q' = 0,$$

so that

$$\frac{p'}{\theta_2} = \frac{q'}{-\theta_1} = \frac{1}{(E\theta_2^2 - 2F\theta_2\theta_1 + G\theta_1^2)^{\frac{1}{2}}};$$

and now

$$\left.\begin{aligned} \frac{\cos\phi}{E\theta_2 - F\theta_1} &= \frac{\sin\phi}{-V\theta_1} = \frac{1}{\{E\,(E\theta_2^2 - 2F\theta_2\theta_1 + G\theta_1^2)\}^{\frac{1}{2}}} \\ \frac{\cos\psi}{F\theta_2 - G\theta_1} &= \frac{\sin\psi}{V\theta_2} = \frac{1}{\{G\,(E\theta_2^2 - 2F\theta_2\theta_1 + G\theta_1^2)\}^{\frac{1}{2}}} \end{aligned}\right\}.$$

The angle at which two curves on the surface $\theta(p, q) = c$, $\vartheta(p, q) = k$, cut at a point of intersection, c and k being parametric, is given by

$$\frac{\cos\chi}{E\theta_2\vartheta_2 - F(\theta_2\vartheta_1 + \theta_1\vartheta_2) + G\theta_1\vartheta_1} = \frac{\sin\chi}{V(\theta_2\vartheta_1 - \theta_1\vartheta_2)}$$

$$= (E\theta_2^2 - 2F\theta_2\theta_1 + G\theta_1^2)^{-\frac{1}{2}}(E\vartheta_2^2 - 2F\vartheta_2\vartheta_1 + G\vartheta_1^2)^{-\frac{1}{2}}.$$

(3) Any direction in the plane

$$\left\| \begin{array}{cccc} \bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\ x_1\ , & y_1\ , & z_1\ , & v_1 \\ x_2\ , & y_2\ , & z_2\ , & v_2 \end{array} \right\| = 0,$$

through the point O and containing the tangents at O to the parametric lines at O, is given by the direction-cosines

$$l = \lambda x_1 + \mu x_2, \quad m = \lambda y_1 + \mu y_2, \quad n = \lambda z_1 + \mu z_2, \quad k = \lambda v_1 + \mu v_2,$$

where

$$E\lambda^2 + 2F\lambda\mu + G\mu^2 = 1.$$

When this direction l, m, n, k, is that of the tangent at O to the curve OA in the foregoing figure, λ and μ satisfy equations

$$E\lambda + F\mu = E^{\frac{1}{2}}\cos\phi, \quad V\lambda = G^{\frac{1}{2}}\sin\psi,$$

$$F\lambda + G\mu = G^{\frac{1}{2}}\cos\psi, \quad V\mu = E^{\frac{1}{2}}\sin\phi,$$

with the relation $\phi + \psi = \omega$.

(4) The angle δ between two directions p_1' and q_1', p_2' and q_2', is given by

$$\frac{\cos\delta}{Ep_1'p_2' + 2F(p_1'q_2' + q_1'p_2') + Gq_1'q_2'} = \frac{\sin\delta}{V(p_1'q_2' - q_1'p_2')}$$

$$= (Ep_1'^2 + 2Fp_1'q_1' + Gq_1'^2)^{-\frac{1}{2}}(Ep_2'^2 + 2Fp_2'q_2' + Gq_2'^2)^{-\frac{1}{2}},$$

with a convention as to the positive direction for measurement of δ.

The angle δ between two directions given by an equation

$$Ap'^2 + 2Bp'q' + Cq'^2 = 0$$

is given by

$$\frac{\cos\delta}{EC - 2FB + GA} = \frac{\sin\delta}{2V(B^2 - AC)^{\frac{1}{2}}} = \frac{1}{I^{\frac{1}{2}}},$$

where

$$I = (EC - 2FB + GA)^2 - 4(EG - F^2)(AC - B^2).$$

(5) The angles β and γ, which a direction P' and Q' makes with the two directions given by an equation

$$Ap'^2 + 2Bp'q' + Cq'^2 = 0$$

are such that

$$I^{\frac{1}{2}} \sin\beta \sin\gamma = V^2 \frac{AP'^2 + 2BP'Q' + CQ'^2}{EP'^2 + 2FP'Q' + GQ'^2},$$

$$I^{\frac{1}{2}} \cos\beta \cos\gamma = \frac{A(FP' + GQ')^2 - 2B(EP' + FQ')(FP' + GQ') + C(EP' + FQ')^2}{EP'^2 + 2FP'Q' + GQ'^2},$$

$$I^{\frac{1}{2}} \sin(\beta + \gamma) = \frac{2V}{EP'^2 + 2FP'Q' + GQ'^2} \begin{vmatrix} EP' + FQ', & FP' + GQ' \\ AP' + BQ', & BP' + CQ' \end{vmatrix}$$

(6) The directions of the lines bisecting the angles between directions

$$Ap'^2 + 2Bp'q' + Cq'^2 = 0$$

are given by the equation

$$\begin{vmatrix} AP' + BQ', & BP' + CQ' \\ EP' + FQ', & FP' + GQ' \end{vmatrix} = 0,$$

these two directions being perpendicular to one another. Thus the directions bisecting the angles between the curves of reference are given by the equation

$$EP'^2 - GQ'^2 = 0.$$

Derivatives of the primary magnitudes.

207. The quantities E, F, G, occurring in the expression for the element of arc on the surface, are the primary magnitudes of the surface: they involve the first derivatives (and only the first derivatives) of the four coordinates x, y, z, v, with regard to the parameters p and q. Subsequent investigations demand the use of certain magnitudes derived from E, F, G, and they require also some relations among these derived magnitudes. For the purpose of comparison with the Gauss theory of surfaces in homaloidal triple space, the Gauss notation for the most part is adopted: but, here, the magnitudes are concerned with amplitudes which occur in homaloidal quadruple space, and thus they implicitly involve the four coordinates of a point in that space, though variation is restricted to the two-fold dimensional surface.

We write

$$\alpha = \Sigma x_1 x_{11} = \tfrac{1}{2} E_1, \quad \alpha' = \Sigma x_1 x_{12} = \tfrac{1}{2} E_2, \quad \alpha'' = \Sigma x_1 x_{22} = F_2 - \tfrac{1}{2} G_1,$$

$$\beta = \Sigma x_2 x_{11} = F_1 - \tfrac{1}{2} E_2, \quad \beta' = \Sigma x_2 x_{12} = \tfrac{1}{2} G_1, \quad \beta'' = \Sigma x_2 x_{22} = \tfrac{1}{2} G_2,$$

and define six quantities Γ, Γ', Γ''; Δ, Δ', Δ''; by the relations

$$\left.\begin{aligned} V^2\Gamma &= \alpha G - \beta F \\ V^2\Gamma' &= \alpha' G - \beta' F \\ V^2\Gamma'' &= \alpha'' G - \beta'' F \end{aligned}\right\}, \quad \left.\begin{aligned} V^2\Delta &= -\alpha F + \beta E \\ V^2\Delta' &= -\alpha' F + \beta' E \\ V^2\Delta'' &= -\alpha'' F + \beta'' E \end{aligned}\right\},$$

so that, reciprocally,

$$\left.\begin{aligned} \alpha &= E\Gamma + F\Delta \\ \alpha' &= E\Gamma' + F\Delta' \\ \alpha'' &= E\Gamma'' + F\Delta'' \end{aligned}\right\}, \quad \left.\begin{aligned} \beta &= F\Gamma + G\Delta \\ \beta' &= F\Gamma' + G\Delta' \\ \beta'' &= F\Gamma'' + G\Delta'' \end{aligned}\right\}.$$

Ex. Verify the relations, invariantive in form:

$$
\begin{gathered}
E\beta^2 - 2F\alpha\beta + G\alpha^2 = V^2(E\Gamma^2 + 2F\Gamma\Delta + G\Delta^2),\\
E\beta\beta' - F(\alpha\beta' + \beta\alpha') + G\alpha\alpha' = V^2\{E\Gamma\Gamma' + F(\Gamma\Delta' + \Delta\Gamma') + G\Delta\Delta'\},\\
E\beta'^2 - 2F\alpha'\beta' + G\alpha'^2 = V^2(E\Gamma'^2 + 2F\Gamma'\Delta' + G\Delta'^2),\\
E\beta\beta'' - F(\alpha\beta'' + \beta\alpha'') + G\alpha\alpha'' = V^2\{E\Gamma\Gamma'' + F(\Gamma\Delta'' + \Delta\Gamma'') + G\Delta\Delta''\},\\
E\beta'\beta'' - F(\alpha'\beta'' + \beta'\alpha'') + G\alpha'\alpha'' = V^2\{E\Gamma'\Gamma'' + F(\Gamma'\Delta'' + \Delta'\Gamma'') + G\Delta'\Delta''\},\\
E\beta''^2 - 2F\alpha''\beta'' + G\alpha''^2 = V^2(E\Gamma''^2 + 2F\Gamma''\Delta'' + G\Delta''^2).
\end{gathered}
$$

We have, at once,

$$V_1 = V(\Gamma + \Delta'), \quad V_2 = V(\Gamma' + \Delta'');$$

and therefore

$$\Gamma_2 + \Delta_2' = \frac{\partial^2 \log V}{\partial p \partial q} = \Gamma_1' + \Delta_1''.$$

Again, we have

$$E\Gamma + F\Delta = \alpha = \tfrac{1}{2}E_1, \quad F\Gamma + G\Delta = \beta = \tfrac{1}{2}E_2,$$

and therefore

$$\frac{\partial}{\partial q}(E\Gamma + F\Delta) = \frac{\partial}{\partial p}(F\Gamma + G\Delta),$$

leading to a relation

$$\frac{1}{E}\{(\Delta_2 + \Delta'\Gamma + \Delta''\Delta) - (\Delta_1' + \Gamma'\Delta + \Delta'^2)\} = -\frac{1}{F}\{(\Gamma_2 + \Gamma''\Delta) - (\Gamma_1' + \Gamma'\Delta')\} = K',$$

where, momentarily, K' denotes the common value of the expressions. Similarly, from the two equations

$$F\Gamma' + G\Delta' = \beta' = \tfrac{1}{2}G_1, \quad F\Gamma'' + G\Delta'' = \beta'' = \tfrac{1}{2}G_2,$$

we are led to a relation

$$-\frac{1}{F}\{(\Delta_1'' + \Gamma''\Delta) - (\Delta_2' + \Delta'\Gamma')\} = \frac{1}{G}\{(\Gamma_1'' + \Gamma'\Delta'' + \Gamma\Gamma'') - (\Gamma_2' + \Gamma'\Delta'' + \Gamma'^2)\} = K'',$$

K'' similarly denoting the common value of the expressions. The relation

$$\Gamma_2 + \Delta_2' = \Gamma_1' + \Delta_1''$$

then leads to the property

$$K' = K'',$$

or, denoting the common value of K' and K'' by K, we have

$$
\left.
\begin{array}{rl}
(\Delta_2 + \Delta'\Gamma + \Delta''\Delta) - (\Delta_1' + \Gamma'\Delta + \Delta'^2) = & EK\\
(\Gamma_2 + \Gamma''\Delta) - (\Gamma_1' + \Gamma'\Delta') = & -FK\\
(\Delta_1'' + \Gamma''\Delta) - (\Delta_2' + \Delta'\Gamma') = & -FK\\
(\Gamma_1'' + \Gamma'\Delta'' + \Gamma\Gamma'') - (\Gamma_2' + \Gamma'\Delta'' + \Gamma'^2) = & GK
\end{array}
\right\},
$$

the significance of K being deferred for the present.

Proceeding similarly from the relations

$$E\Gamma' + F\Delta' + F\Gamma + G\Delta = \alpha' + \beta = F_1, \quad E\Gamma'' + F\Delta'' + F\Gamma' + G\Delta' = \alpha'' + \beta' = F_2,$$

we are led to a relation which is satisfied identically in virtue of the results already established.

Finally for the present, we have

$$\tfrac{1}{2}(E_{22} - 2F_{12} + G_{11}) = \frac{\partial \alpha'}{\partial q} - \frac{\partial \alpha''}{\partial p} = \frac{\partial \beta'}{\partial p} - \frac{\partial \beta''}{\partial q}:$$

on substituting for α' and α'', and for β' and β'', and using the relations already established, we find

$$\begin{aligned}&\tfrac{1}{2}(E_{22} - 2F_{12} + G_{11}) + V^2K \\ &\qquad = \{E\Gamma'^2 + 2F\Gamma'\Delta' + G\Delta'^2\} - \{E\Gamma\Gamma'' + F(\Gamma\Delta'' + \Delta\Gamma'') + G\Delta\Delta''\}.\end{aligned}$$

As yet, no significance has been attached to K. The relation, in form, is the same as the Gauss characteristic equation for surfaces in homaloidal triple space; in that space, K denotes the Gauss measure of superficial curvature.

It may be pointed out that, in quadruple space, as in triple space, because

$$\tfrac{1}{2}E_2 = \Sigma x_1 x_{12}, \quad F_2 - \tfrac{1}{2}G_1 = \Sigma x_1 x_{22},$$

or because

$$\tfrac{1}{2}G_1 = \Sigma x_2 x_{12}, \quad F_1 - \tfrac{1}{2}E_2 = \Sigma x_2 x_{11},$$

we have

$$\tfrac{1}{2}(E_{22} - 2F_{12} + G_{11}) = \Sigma x_{12}^2 - \Sigma x_{11} x_{22}.$$

In triple space, the summation Σ extends over the three coordinates x, y, z; in quadruple space, it extends over the four coordinates x, y, z, v. For a surface in a homaloidal space of n dimensions, with the correspondingly extended significance of E, F, G, the same relation holds, the summation extending over the n coordinates specifying the position of a point in the space.

Some relations involving third-order derivatives of the point-coordinates.

208. Certain relations, not complete in ultimate significance, can be deduced affecting derivatives of the third order. Hereafter, some combinations affecting derivatives of only the second order, will arise in the forms

$$\begin{aligned}\mathbf{a} &= \Sigma x_{11}^2 &&- \{E\Gamma^2 + 2F\Gamma\Delta + G\Delta^2\},\\ \mathbf{h} &= \Sigma x_{11} x_{12} &&- \{E\Gamma\Gamma' + F(\Gamma\Delta' + \Delta\Gamma') + G\Delta\Delta'\},\\ \mathbf{g} &= \Sigma x_{11} x_{22} &&- \{E\Gamma\Gamma'' + F(\Gamma\Delta'' + \Delta\Gamma'') + G\Delta\Delta''\},\\ \mathbf{b} &= \Sigma x_{12}^2 &&- \{E\Gamma'^2 + 2F\Gamma'\Delta' + G\Delta'^2\},\\ \mathbf{f} &= \Sigma x_{12} x_{22} &&- \{E\Gamma'\Gamma'' + F(\Gamma'\Delta'' + \Delta'\Gamma'') + G\Delta'\Delta''\},\\ \mathbf{c} &= \Sigma x_{22}^2 &&- \{E\Gamma''^2 + 2F\Gamma''\Delta'' + G\Delta''^2\}.\end{aligned}$$

The indicated relations affecting the derivatives of x, y, z, v, of the third order occur as follows. Differentiating

$$\Sigma x_1 x_{11} = \alpha = E\Gamma + F\Delta$$

with respect to p, we have

$$\begin{aligned}\Sigma x_1 x_{111} + \Sigma x_{11}^2 &= E\Gamma_1 + F\Delta_1 + \Gamma E_1 + \Delta F_1\\ &= E\Gamma_1 + F\Delta_1 + 2\Gamma(E\Gamma + F\Delta) + \Delta(F\Gamma + G\Delta + E\Gamma' + F\Delta'),\end{aligned}$$

and therefore

$$\Sigma x_1 x_{111} + \mathbf{a} = E(\Gamma_1 + \Gamma^2 + \Delta\Gamma') + F(\Delta_1 + \Gamma\Delta + \Delta\Delta').$$

We proceed similarly, differentiating the six quantities α, α', α'', β, β', β'', with respect to p and to q separately. The full aggregate of results, in addition to the preceding result, is as follows:

$$\Sigma x_2 x_{111} + \mathbf{h} = F(\Gamma_1 + \Gamma^2 + \Delta\Gamma') + G(\Delta_1 + \Gamma\Delta + \Delta\Delta'),$$

from the value of β_1; there are two results

$$\begin{aligned} \Sigma x_1 x_{112} + \mathbf{h} &= E(\Gamma_2 + \Gamma\Gamma' + \Delta\Gamma'') + F(\Delta_2 + \Gamma\Delta' + \Delta\Delta''), \\ &= E(\Gamma_1' + \Gamma'\Gamma + \Delta'\Gamma') + F(\Delta_1' + \Gamma'\Delta + \Delta'^2), \end{aligned}$$

arising from α_2 and α_1' respectively; there are two results

$$\Sigma x_2 x_{112} + \mathbf{g} = F(\Gamma_2 + \Gamma\Gamma' + \Delta\Gamma'') + G(\Delta_2 + \Gamma\Delta' + \Delta\Delta''),$$

$$\Sigma x_2 x_{112} + \mathbf{b} = F(\Gamma_1' + \Gamma'\Gamma + \Delta'\Gamma') + G(\Delta_1' + \Gamma'\Delta + \Delta'^2),$$

arising from β_2 and β_1' respectively; there are two results

$$\Sigma x_1 x_{122} + \mathbf{g} = E(\Gamma_1'' + \Gamma''\Gamma + \Delta''\Gamma') + F(\Delta_1'' + \Gamma''\Delta + \Delta''\Delta'),$$

$$\Sigma x_1 x_{122} + \mathbf{b} = E(\Gamma_2' + \Gamma'^2 + \Delta'\Gamma'') + F(\Delta_2' + \Gamma'\Delta' + \Delta'\Delta''),$$

arising from α_1'' and α_2' respectively; there are two results

$$\begin{aligned} \Sigma x_2 x_{122} + \mathbf{f} &= F(\Gamma_2' + \Gamma'^2 + \Delta'\Gamma'') + G(\Delta_2' + \Gamma'\Delta' + \Delta'\Delta''), \\ &= F(\Gamma_1'' + \Gamma''\Gamma + \Delta''\Gamma') + G(\Delta_1'' + \Gamma''\Delta + \Delta''\Delta'), \end{aligned}$$

arising from β_2' and β_1'' respectively, there is the result

$$\Sigma x_1 x_{222} + \mathbf{f} = E(\Gamma_2'' + \Gamma''\Gamma' + \Delta''\Gamma'') + F(\Delta_2'' + \Gamma''\Delta' + \Delta''^2),$$

arising from α_2''; and there is the result

$$\Sigma x_2 x_{222} + \mathbf{c} = F(\Gamma_2'' + \Gamma''\Gamma' + \Delta''\Gamma'') + G(\Delta_2'' + \Gamma''\Delta' + \Delta''^2),$$

arising from β_2''.

The two expressions for $\Sigma x_1 x_{112} + \mathbf{h}$ are equivalent to one another, owing to the relations in § 207; likewise the two expressions for $\Sigma x_2 x_{122} + \mathbf{f}$. In virtue of the same relations, the two expressions for $\Sigma x_2 x_{112}$ are equivalent to one another; and the two expressions for $\Sigma x_1 x_{122}$ also are equivalent to one another, in virtue of a single relation

$$\mathbf{g} - \mathbf{b} = V^2 K,$$

where K has the same (still unspecified) significance as before.

Tangent plane · orthogonal plane.

209. The equations of the tangent plane at any point O of the surface are derived at once from the property that it is the locus of the tangent to any curve on the surface passing through O. For any such curve, the direction-cosines of the tangent are

$$x' = x_1 p' + x_2 q', \quad y' = y_1 p' + y_2 q', \quad z' = z_1 p' + z_2 q', \quad v' = v_1 p' + v_2 q',$$

whatever be the definition of the curve; and thus the tangent to the curve is given by the equations

$$\frac{\bar{x}-x}{x'}=\frac{\bar{y}-y}{y'}=\frac{\bar{z}-z}{z'}=\frac{\bar{v}-v}{v'},=\Omega,$$

so that, along the tangent,

$$\bar{x}-x=x_1p'\Omega+x_2q'\Omega,\quad \bar{y}-y=y_1p'\Omega+y_2q'\Omega,$$
$$\bar{z}-z=z_1p'\Omega+z_2q'\Omega,\quad \bar{v}-v=v_1p'\Omega+v_2q'\Omega.$$

Consequently, the whole line for given values of p' and q', and all lines for different values of p' and q', lie in the plane

$$\left\|\begin{array}{cccc} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ x_1\ , & y_1\ , & z_1\ , & v_1 \\ x_2\ , & y_2\ , & z_2\ , & v_2 \end{array}\right\|=0,$$

which accordingly are the equations of the tangent plane. The guiding lines, in this form of equation, are the tangents to the parametric curves through O.

Any direction lying in this plane is given by direction-cosines

$$l=\lambda x_1+\mu x_2,\quad m=\lambda y_1+\mu y_2,\quad n=\lambda z_1+\mu z_2,\quad k=\lambda v_1+\mu v_2,$$

where

$$E\lambda^2+2F\lambda\mu+G\mu^2=1$$

Any point in this tangent plane is

$$\bar{x}-x=rx_1+tx_2,\quad \bar{y}-y=ry_1+ty_2,\quad \bar{z}-z=rz_1+tz_2,\quad \bar{v}-v=rv_1+tv_2,$$

where r and t are parameters in the plane, and the point can be reached in the plane by measuring a length $rE^{\frac{1}{2}}$ along the tangent to $q=$ *constant*, a length $tG^{\frac{1}{2}}$ along the tangent to $p=$ *constant*, and completing the parallelogram of which these two lengths are adjacent sides

When the surface is given as the amplitude common to the two regions $\Theta(x, y, z, v)=0$ and $\Phi(x, y, z, v)=0$, we have $\Theta=0$ and $\Phi=0$ as identities when the parametric values of x, y, z, v, are substituted. Hence

$$\Sigma\frac{\partial\Theta}{\partial x}x_1=0,\quad \Sigma\frac{\partial\Theta}{\partial x}x_2=0,\quad \Sigma\frac{\partial\Phi}{\partial x}x_1=0,\quad \Sigma\frac{\partial\Phi}{\partial x}x_2=0;$$

and therefore

$$\Sigma\frac{\partial\Theta}{\partial x}(rx_1+tx_2)=0,\quad \Sigma\frac{\partial\Phi}{\partial x}(rx_1+tx_2)=0.$$

Consequently the coordinates of every point in the tangent plane at the point x, y, z, v, satisfy the two equations

$$\Sigma(\bar{x}-x)\frac{\partial\Theta}{\partial x}=0,\quad \Sigma(\bar{x}-x)\frac{\partial\Phi}{\partial x}=0.$$

These equations are, in fact, the equations of the tangent plane; and they represent the tangent plane as the intersection of the two flats which respectively are tangential to the two regions $\Theta = 0$ and $\Phi = 0$ defining the surface.

The plane through the point O, orthogonal to the tangent plane to the surface, has for its equations

$$\Sigma (\bar{x} - x) x_1 = 0, \quad \Sigma (\bar{x} - x) x_2 = 0,$$

(§ 67) when its orthogonality to the tangent plane is to be in evidence in relation to the equations

$$\left\| \begin{matrix} \bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\ x_1\ , & y_1\ , & z_1\ , & v_1 \\ x_2\ , & y_2\ , & z_2\ , & v_2 \end{matrix} \right\| = 0;$$

and it has

$$\left\| \begin{matrix} \bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\ \dfrac{\partial \Theta}{\partial x}, & \dfrac{\partial \Theta}{\partial y}, & \dfrac{\partial \Theta}{\partial z}, & \dfrac{\partial \Theta}{\partial v} \\ \dfrac{\partial \Phi}{\partial x}, & \dfrac{\partial \Phi}{\partial y}, & \dfrac{\partial \Phi}{\partial z}, & \dfrac{\partial \Phi}{\partial v} \end{matrix} \right\| = 0,$$

for an equivalent form of equations. (See also § 217.) The latter form shews that any direction, with direction-cosines

$$\lambda \frac{\partial \Theta}{\partial x} + \mu \frac{\partial \Phi}{\partial x}, \quad \lambda \frac{\partial \Theta}{\partial y} + \mu \frac{\partial \Phi}{\partial y}, \quad \lambda \frac{\partial \Theta}{\partial z} + \mu \frac{\partial \Phi}{\partial z}, \quad \lambda \frac{\partial \Theta}{\partial v} + \mu \frac{\partial \Phi}{\partial v},$$

is perpendicular to the tangent plane.

Ex. Later, we shall require a direction, which lies in the tangent plane and is perpendicular to a direction x', y', z', v', in that plane, where

$$x' = x_1 p' + x_2 q', \quad y' = y_1 p' + y_2 q', \quad z' = z_1 p' + z_2 q', \quad v' = v_1 p' + v_2 q'.$$

Let it be denoted by λ, μ, ν, κ, where

$$\lambda = x_1 P' + x_2 Q', \quad \mu = y_1 P' + y_2 Q', \quad \nu = z_1 P' + z_2 Q', \quad \kappa = v_1 P' + v_2 Q'.$$

The condition of perpendicularity of the two directions is

$$\Sigma \lambda x' = 0,$$

that is,

$$Ep'P' + F(p'Q' + q'P') + Gq'Q' = 0.$$

Hence

$$\frac{P'}{-(Fp' + Gq')} = \frac{Q'}{Ep' + Fq'} = \frac{1}{\mu},$$

using μ to express the common value. But

$$EP'^2 + 2FP'Q' + GQ'^2 = 1;$$

and therefore

$$\begin{aligned} \mu^2 &= E(Fp' + Gq')^2 - 2F(Ep' + Fq')(Fp' + Gq') + G(Ep' + Fq')^2 \\ &= (EG - F^2)(Ep'^2 + 2Fp'q' + Gq'^2) = V^2, \end{aligned}$$

so that we may take

Hence the required direction in the tangent plane, perpendicular to the direction p' and q', is given by

$$\left.\begin{aligned} V\lambda &= p'(x_2E - x_1F) + q'(x_2F - x_1G) \\ V\mu &= p'(y_2E - y_1F) + q'(y_2F - y_1G) \\ V\nu &= p'(z_2E - z_1F) + q'(z_2F - z_1G) \\ V\kappa &= p'(v_2E - v_1F) + q'(v_2F - v_1G) \end{aligned}\right\}.$$

Normal plane, and normal section, through a direction.

210. Although, at any point in a plane, there is no unique direction perpendicular to the plane, (for the locus of the perpendicular directions through such a point is the orthogonal plane), there is a unique perpendicular from an external point to a plane. It is the minimum distance of the external point from the plane; it also is the one line which, among all those in a direction perpendicular to the plane, actually intersects the plane. We proceed to obtain this perpendicular on the tangent plane.

Let Q be a point $\xi, \eta, \zeta, \upsilon$, on the surface in the near vicinity of O. From Q let the perpendicular be drawn to the tangent plane at O, this perpendicular being of length D and having direction-cosines l, m, n, k, measured from $\overline{X}, \overline{Y}, \overline{Z}, \overline{V}$, its foot in the plane, towards Q. Thus

$$\xi - \overline{X} = lD, \quad \eta - \overline{Y} = mD, \quad \zeta - \overline{Z} = nD, \quad \upsilon - \overline{V} = kD.$$

As the point $\overline{X}, \overline{Y}, \overline{Z}, \overline{V}$, lies in the tangent plane at O, there are parameters λ and μ such that

$$\overline{X} - x = \lambda x_1 + \mu x_2, \quad \overline{Y} - y = \lambda y_1 + \mu y_2, \quad \overline{Z} - z = \lambda z_1 + \mu z_2, \quad \overline{V} - v = \lambda v_1 + \mu v_2,$$

and, in connection with the perpendicular from Q, the particular values of λ and μ are determined so that the quantity

$$D^2, \ = \Sigma(\xi - \overline{X})^2, \ = \Sigma(\xi - x - \lambda x_1 - \mu x_2)^2,$$

is a minimum for all values of λ and μ. The conditions, necessary and sufficient to secure the minimum, are

$$\frac{\partial D^2}{\partial \lambda} = 0, \quad \frac{\partial D^2}{\partial \mu} = 0, \quad \frac{\partial^2 D^2}{\partial \lambda^2} > 0, \quad \frac{\partial^2 D^2}{\partial \mu^2} > 0, \quad \frac{\partial^2 D^2}{\partial \lambda^2}\frac{\partial^2 D^2}{\partial \mu^2} - \left(\frac{\partial^2 D^2}{\partial \lambda \, \partial \mu}\right)^2 > 0.$$

The last three of these equations become

$$E > 0, \quad G > 0, \quad EG - F^2 > 0,$$

all of which are satisfied. The first two of the conditions become

$$\Sigma x_1(\xi - x - \lambda x_1 - \mu x_2) = 0, \quad \Sigma x_2(\xi - x - \lambda x_1 - \mu x_2) = 0,$$

which are two equations for the specific determination of λ and μ.

These two equations can be written in the form

$$\Sigma x_1(\xi - \overline{X}) = 0. \quad \Sigma x_2(\xi - \overline{X}) = 0,$$

and therefore

$$\Sigma x_1 l = 0, \quad \Sigma x_2 l = 0,$$

with the further inference that, whatever be the tangential direction x', y', z', v',

$$\Sigma x' l = \Sigma \left(x_1 p' + x_2 q'\right) l = 0.$$

Thus the direction of the perpendicular from Q on the tangent plane is (as always with the perpendicular to a plane) at right angles to every direction in that plane.

Take the plane through O, rendered determinate by this direction l, m, n, k, and by the tangent x', y', z', v', as guiding lines; its equations are

$$\left\| \begin{matrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ x', & y', & z', & v' \\ l, & m, & n, & k \end{matrix} \right\| = 0,$$

while those of the tangent plane are

$$\left\| \begin{matrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \end{matrix} \right\| = 0.$$

The two planes are not orthogonal to one another, but they are perpendicular for, with the notation used in § 99, we have

$$\cos\theta_{13} = E^{-\frac{1}{2}}\left(Ep' + Fq'\right), \quad \cos\theta_{14} = G^{-\frac{1}{2}}\left(Fp' + Gq'\right),$$
$$\cos\theta_{23} = 0 \qquad\qquad \cos\theta_{24} = 0$$

so that the relation

$$\cos\theta_{13}\cos\theta_{24} - \cos\theta_{14}\cos\theta_{23} = 0$$

is satisfied, while not all the four cosines vanish. Accordingly, we call this plane through QO and the tangent, a *normal plane*: and its curve of section on the surface we call a *normal section* of the surface. Moreover, the tangent plane and the normal plane both contain the tangent to the normal section, which therefore is a line of intersection of the planes; consequently both planes exist in one and the same flat, and the equation of the flat is

$$\begin{vmatrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ l, & m, & n, & k \end{vmatrix} = 0$$

Perpendicular from a neighbouring point upon the tangent plane; curvature of normal section.

211. Again, the two equations for the determination of λ and μ can be taken in the form

$$\{\Sigma x_1(\xi - x)\} - E\lambda - F\mu = 0, \quad \{\Sigma x_2(\xi - x)\} - F\lambda - G\mu = 0;$$

and therefore λ and μ are determinable in the form

$$V^2\lambda = \Sigma\left\{(Gx_1 - Fx_2)(\xi - x)\right\}, \quad V^2\mu = \Sigma\left\{(-Fx_1 + Ex_2)(\xi - x)\right\}.$$

This appears also from the fact that the line x', y', z', v', lies in both planes, so that they cannot be orthogonal.

Now let δ denote the arc-distance OQ measured in the surface along the normal section through O and Q; we have

$$\xi - x = x'\delta + \frac{x''}{2!}\delta^2 + \frac{x'''}{3!}\delta^3 + \ldots,$$

$$\eta - y = y'\delta + \frac{y''}{2!}\delta^2 + \frac{y'''}{3!}\delta^3 + \ldots,$$

$$\zeta - z = z'\delta + \frac{z''}{2!}\delta^2 + \frac{z'''}{3!}\delta^3 + \ldots,$$

$$\upsilon - v = v'\delta + \frac{v''}{2!}\delta^2 + \frac{v'''}{3!}\delta^3 + \ldots,$$

where the quantities x'', y'', z'', v'', ... are magnitudes belonging to the curve of normal section. We have

$$x' = x_1p' + x_2q', \quad y' = y_1p' + y_2q', \quad z' = z_1p' + z_2q', \quad v' = v_1p' + v_2q';$$

hence

$$\Sigma\{(Gx_1 - Fx_2)x'\} = V^2p', \quad \Sigma\{(-Fx_1 + Ex_2)x'\} = V^2q',$$

and therefore

$$\lambda - p'\delta = \frac{1}{2}\frac{\delta^2}{V^2}\Sigma\{(Gx_1 - Fx_2)x''\} + \frac{1}{6}\frac{\delta^3}{V^2}\Sigma\{(Gx_1 - Fx_2)x'''\} + \ldots,$$

$$\mu - q'\delta = \frac{1}{2}\frac{\delta^2}{V^2}\Sigma\{(-Fx_1 + Ex_2)x''\} + \frac{1}{6}\frac{\delta^3}{V^2}\Sigma\{(-Fx_1 + Ex_2)x'''\} + \ldots.$$

Along any curve lying in the surface,

$$x'' = x_1p'' + x_2q'' + x_{11}p'^2 + 2x_{12}p'q' + x_{22}q'^2,$$

with corresponding expressions for y'', z'', v''; hence

$$\begin{aligned}\Sigma x_1x'' &= Ep'' + Fq'' + \alpha p'^2 + 2\alpha'p'q' + \alpha''q'^2\\ &= E(p'' + \Gamma p'^2 + 2\Gamma'p'q' + \Gamma''q'^2) + F(q'' + \Delta p'^2 + 2\Delta'p'q' + \Delta''q'^2),\\ \Sigma x_2x'' &= F(p'' + \Gamma p'^2 + 2\Gamma'p'q' + \Gamma''q'^2) + G(q'' + \Delta p'^2 + 2\Delta'p'q' + \Delta''q'^2),\end{aligned}$$

and therefore

$$\Sigma\{(Gx_1 - Fx_2)x''\} = (p'' + \Gamma p'^2 + 2\Gamma'p'q' + \Gamma''q'^2)V^2,$$

$$\Sigma\{(-Fx_1 + Ex_2)x''\} = (q'' + \Delta p'^2 + 2\Delta'p'q' + \Delta''q'^2)V^2,$$

on substitution and reduction. Thus the expressions for λ and μ become

$$\left.\begin{aligned}\lambda &= p'\delta + \frac{1}{2}(p'' + \Gamma p'^2 + 2\Gamma'p'q' + \Gamma''q'^2)\delta^2 + \frac{1}{6}\frac{\delta^3}{V^2}\Sigma\{(Gx_1 - Fx_2)x'''\} + \ldots\\ \mu &= q'\delta + \frac{1}{2}(q'' + \Delta p'^2 + 2\Delta'p'q' + \Delta''q'^2)\delta^2 + \frac{1}{6}\frac{\delta^3}{V^2}\Sigma\{(-Fx_1 + Ex_2)x'''\} + \ldots\end{aligned}\right\}.$$

With these values, we have

$$\begin{aligned}\xi - \overline{X} &= \xi - x - \lambda x_1 - \mu x_2\\ &= x'\delta + \tfrac{1}{2}x''\delta^2 + \tfrac{1}{6}x'''\delta^3 + \ldots - (\lambda x_1 + \mu x_2)\\ &= \tfrac{1}{2}\delta^2\{x'' - x_1(p'' + \Gamma p'^2 + 2\Gamma'p'q' + \Gamma''q'^2)\\ &\qquad\qquad - x_2(q'' + \Delta p'^2 + 2\Delta'p'q' + \Delta''q'^2)\} + \ldots\\ &= \tfrac{1}{2}\delta^2\{(x_{11} - x_1\Gamma - x_2\Delta)p'^2 + 2(x_{12} - x_1\Gamma' - x_2\Delta')p'q'\\ &\qquad\qquad + (x_{22} - x_1\Gamma'' - x_2\Delta'')q'^2\} + \ldots,\end{aligned}$$

where the unspecified terms involve third and higher powers of δ. Also we have

$$\xi - \overline{X} = lD,$$

and so for the other variables. Hence, for sufficiently small values of δ that allow powers of δ higher than the second to be regarded as negligible compared with δ, we have

$$\frac{2D}{\delta^2}\, l = (x_{11} - x_1\Gamma - x_2\Delta)\, p'^2 + 2\,(x_{12} - x_1\Gamma' - x_2\Delta')\, p'q' + (x_{22} - x_1\Gamma'' - x_2\Delta'')\, q'^2,$$

with similar expressions for m, n, k, thus giving expressions for the direction-cosines l, m, n, k, of the perpendicular from Q on the tangent plane at O. Also, for $\overline{X}$, $\overline{Y}$, $\overline{Z}$, $\overline{V}$, the foot of this perpendicular (say the point N) on the plane, we have

$$\begin{aligned}\overline{X} - x &= \xi - x - (\xi - \overline{X})\\ &= \xi - x - lD,\end{aligned}$$

and similarly for $\overline{Y} - y$, $\overline{Z} - z$, $\overline{V} - v$; hence, taking x', y', z', v', to be the tangent in the normal section, we have

$$\begin{aligned}ON &= \text{projection of } OQ \text{ on the tangent}\\ &= \Sigma\,(\overline{X} - x)\, x'\\ &= \Sigma\,\{(\xi - x)\, x'\} - D\Sigma l x'\\ &= \Sigma x'\left(x'\delta + \frac{1}{2}x''\delta^2 + \frac{1}{6}x'''\delta^3 + \dots\right)\\ &= \delta + \frac{\delta^3}{6}\Sigma x'x''' + \dots,\end{aligned}$$

the unspecified powers of δ being of order higher than the third. We denote ON by T, so that, up to the second order of small quantities inclusive,

$$T = \delta.$$

Now in this plane curve of normal section through O and Q, the length QN is perpendicular to the tangent at O; and therefore, if ρ be the radius of circular curvature of the normal section of the surface,

$$\frac{1}{\rho} = 2\frac{QN}{ON^2} = 2\frac{D}{T^2} = 2\frac{D}{\delta^2}.$$

And we now have

$$\left.\begin{aligned}\frac{l}{\rho} &= (x_{11} - x_1\Gamma - x_2\Delta)\, p'^2 + 2\,(x_{12} - x_1\Gamma' - x_2\Delta')\, p'q' + (x_{22} - x_1\Gamma'' - x_2\Delta'')\, q'^2\\ \frac{m}{\rho} &= (y_{11} - y_1\Gamma - y_2\Delta)\, p'^2 + 2\,(y_{12} - y_1\Gamma' - y_2\Delta')\, p'q' + (y_{22} - y_1\Gamma'' - y_2\Delta'')\, q'^2\\ \frac{n}{\rho} &= (z_{11} - z_1\Gamma - z_2\Delta)\, p'^2 + 2\,(z_{12} - z_1\Gamma' - z_2\Delta')\, p'q' + (z_{22} - z_1\Gamma'' - z_2\Delta'')\, q'^2\\ \frac{k}{\rho} &= (v_{11} - v_1\Gamma - v_2\Delta)\, p'^2 + 2\,(v_{12} - v_1\Gamma' - v_2\Delta')\, p'q' + (v_{22} - v_1\Gamma'' - v_2\Delta'')\, q'^2\end{aligned}\right\},$$

which, accordingly, give the direction-cosines l, m, n, k, of the perpendicular from a point Q, near to O, upon the tangent plane at O, while p' and q' give the direction at O of the tangent in the normal plane, which passes through O and contains the perpendicular from Q on the tangent plane at O.

212 The direction-cosines of this perpendicular to the tangent plane, which (as being unique) may be called the *normal associated with the direction* p' and q', may also be obtained from a definition that the normal is the line which, being perpendicular to every direction in the plane, actually meets the plane. Their determination is as follows.

The condition that a line through ξ, η, ζ, υ, with the equations

$$\frac{\bar{x}-\xi}{l}=\frac{\bar{y}-\eta}{m}=\frac{\bar{z}-\zeta}{n}=\frac{\bar{v}-\upsilon}{k},$$

shall meet the plane

$$\begin{vmatrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ x_1\ , & y_1\ , & z_1\ , & v_1 \\ x_2\ , & y_2\ , & z_2\ , & v_2 \end{vmatrix}=0$$

is the relation

$$\begin{vmatrix} l, & \xi-x, & x_1, & x_2 \\ m, & \eta-y, & y_1, & y_2 \\ n, & \zeta-z, & z_1, & z_2 \\ k, & \upsilon-v, & v_1, & v_2 \end{vmatrix}=0,$$

briefly represented by

$$l\Theta+m\Phi+n\Psi+kX=0.$$

Further, if the line be perpendicular to every direction in the tangent plane, we have

$$lx_1+my_1+nz_1+kv_1=0,$$

$$lx_2+my_2+nz_2+kv_2=0.$$

Hence there are three homogeneous linear relations satisfied by l, m, n, k; and thus there is some quantity H such that

$$Hl=\begin{vmatrix} \Phi, & \Psi, & X \\ y_1, & z_1, & v_1 \\ y_2, & z_2, & v_2 \end{vmatrix},$$

with corresponding values for Hm, Hn, Hk.

Now Θ, Φ, Ψ, X, are linear in $\xi-x$, $\eta-y$, $\zeta-z$, $\upsilon-v$. Hence, in Hl, the coefficient of $\xi-x$ is

$$\begin{aligned} &=-(z_1v_2-v_1z_2)^2-(y_1v_2-v_1y_2)^2-(y_1z_2-z_1y_2)^2 \\ &=-\{(y_1^2+z_1^2+v_1^2)(y_2^2+z_2^2+v_2^2)-(y_1y_2+z_1z_2+v_1v_2)^2\} \\ &=-V^2+Gx_1^2-2Fx_1x_2+Ex_2^2, \end{aligned}$$

the coefficient of $\eta - y$

$$\begin{aligned} &= -(v_1 y_2 - y_1 v_2)(v_1 x_2 - x_1 v_2) - (y_1 z_2 - z_1 y_2)(z_1 x_2 - x_1 z_2) \\ &= \quad G x_1 y_1 - F(x_1 y_2 + y_1 x_2) + E x_2 y_2\,; \end{aligned}$$

the coefficient of $\zeta - z$, similarly,

$$= \quad G x_1 z_1 - F(x_1 z_2 + z_1 x_2) + E x_2 z_2\,;$$

and the coefficient of $\upsilon - v$, similarly,

$$= \quad G x_1 v_1 - F(x_1 v_2 + v_1 x_2) + E x_2 v_2 .$$

Hence

$$Hl = -V^2(\xi - x) + (Gx_1 - Fx_2)\Sigma x_1(\xi - x) + (-Fx_1 + Ex_2)\Sigma x_2(\xi - x).$$

But

$$\xi - x = \delta(x_1 p' + x_2 q') + \frac{1}{2}\delta^2(x_{11}p'^2 + 2x_{12}p'q' + x_{22}q'^2) + \dots,$$

with similar expressions for $\eta - y$, $\zeta - z$, $\upsilon - v$; consequently

$$\Sigma x_1(\xi - x) = \delta(Ep' + Fq') + \frac{1}{2}\delta^2(\alpha p'^2 + 2\alpha' p'q' + \alpha'' q'^2) + \dots,$$

$$\Sigma x_2(\xi - x) = \delta(Fp' + Gq') + \frac{1}{2}\delta^2(\beta p'^2 + 2\beta' p'q' + \beta'' q'^2) + \dots,$$

and therefore

$$\begin{aligned} &(Gx_1 - Fx_2)\Sigma x_1(\xi - x) + (-Fx_1 + Ex_2)\Sigma x_2(\xi - x) \\ &\quad = \delta V^2(x_1 p' + x_2 q') \\ &\quad + \frac{1}{2}\delta^2 V^2\{(x_1\Gamma + x_2\Delta)p'^2 + 2(x_1\Gamma' + x_2\Delta')p'q' + (x_1\Gamma'' + x_2\Delta'')q'^2\} + \dots, \end{aligned}$$

after reduction. Thus

$$\begin{aligned} Hl = -\frac{1}{2}\delta^2 V^2\{(x_{11} - x_1\Gamma - x_2\Delta)p'^2 &+ 2(x_{12} - x_1\Gamma' - x_2\Delta')p'q' \\ &+ (x_{22} - x_1\Gamma'' - x_2\Delta'')q'^2\} + \dots, \end{aligned}$$

with corresponding expressions for Hm, Hn, Hk.

As, for the moment, our sole concern is with the ratios $l : m : n : k$, of the direction-cosines of the normal, it appears that the values of the ratios, which have just been obtained, agree with the values obtained in the earlier investigation.

Direction-cosines of the normal associated with a surface-direction: secondary quantities **a, b, c, f, g, h.**

213. It will be convenient to use abbreviations for quantities that occur in these expressions for l, m, n, k, and there are certain combinations of these quantities which frequently recur. We write

$$\left.\begin{aligned} \xi_{11} &= x_{11} - x_1\Gamma - x_2\Delta, & \xi_{12} &= x_{12} - x_1\Gamma' - x_2\Delta', & \xi_{22} &= x_{22} - x_1\Gamma'' - x_2\Delta'' \\ \eta_{11} &= y_{11} - y_1\Gamma - y_2\Delta, & \eta_{12} &= y_{12} - y_1\Gamma' - y_2\Delta', & \eta_{22} &= y_{22} - y_1\Gamma'' - y_2\Delta'' \\ \zeta_{11} &= z_{11} - z_1\Gamma - z_2\Delta, & \zeta_{12} &= z_{12} - z_1\Gamma' - z_2\Delta', & \zeta_{22} &= z_{22} - z_1\Gamma'' - z_2\Delta'' \\ \upsilon_{11} &= v_{11} - v_1\Gamma - v_2\Delta, & \upsilon_{12} &= v_{12} - v_1\Gamma' - v_2\Delta', & \upsilon_{22} &= v_{22} - v_1\Gamma'' - v_2\Delta'' \end{aligned}\right\}.$$

It is easy to verify the relations

$$\begin{aligned}\Sigma x_1 \xi_{11} &= 0, & \Sigma x_2 \xi_{11} &= 0,\\ \Sigma x_1 \xi_{12} &= 0, & \Sigma x_2 \xi_{12} &= 0,\\ \Sigma x_1 \xi_{22} &= 0, & \Sigma x_2 \xi_{22} &= 0,\end{aligned}$$

and thus incidentally to verify the former relations

$$\Sigma l x_1 = 0, \quad \Sigma l x_2 = 0.$$

Next, we introduce quantities $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, according to the definitions

$$\left.\begin{aligned}\mathbf{a} &= \Sigma\xi_{11}^2 &&= \Sigma x_{11}^2 &&- (E, F, G \between \Gamma, \Delta)^2\\ \mathbf{b} &= \Sigma\xi_{12}^2 &&= \Sigma x_{12}^2 &&- (E, F, G \between \Gamma', \Delta')^2\\ \mathbf{c} &= \Sigma\xi_{22}^2 &&= \Sigma x_{22}^2 &&- (E, F, G \between \Gamma'', \Delta'')^2\\ \mathbf{f} &= \Sigma\xi_{12}\xi_{22} &&= \Sigma x_{12}x_{22} &&- (E, F, G \between \Gamma', \Delta' \between \Gamma'', \Delta'')\\ \mathbf{g} &= \Sigma\xi_{11}\xi_{22} &&= \Sigma x_{11}x_{22} &&- (E, F, G \between \Gamma'', \Delta'' \between \Gamma, \Delta)\\ \mathbf{h} &= \Sigma\xi_{11}\xi_{12} &&= \Sigma x_{11}x_{12} &&- (E, F, G \between \Gamma, \Delta \between \Gamma', \Delta')\end{aligned}\right\};$$

and we write

$$\mathbf{g} + 2\mathbf{b} = 3\mathbf{k}.$$

Thus there are six magnitudes, manifestly secondary as involving second derivatives of the point-variables in parametric form. they are independent of direction through the point, and they appertain to the whole surface at the point. The direction-cosines of the perpendicular upon the tangent plane at O from a neighbouring point Q are now given by

$$\left.\begin{aligned}\frac{l}{\rho} &= \xi_{11}p'^2 + 2\xi_{12}p'q' + \xi_{22}q'^2\\ \frac{m}{\rho} &= \eta_{11}p'^2 + 2\eta_{12}p'q' + \eta_{22}q'^2\\ \frac{n}{\rho} &= \zeta_{11}p'^2 + 2\zeta_{12}p'q' + \zeta_{22}q'^2\\ \frac{k}{\rho} &= \upsilon_{11}p'^2 + 2\upsilon_{12}p'q' + \upsilon_{22}q'^2\end{aligned}\right\}.$$

When these four equations are squared and added, we obtain a first expression giving the magnitude of the radius of circular curvature ρ of the normal section of the surface: and l, m, n, k, are the direction-cosines of this radius of curvature. The expression is

$$\frac{1}{\rho^2} = \mathbf{a}p'^4 + 4\mathbf{h}p'^3q' + (2\mathbf{g} + 4\mathbf{b})\,p'^2q'^2 + 4\mathbf{f}p'q'^3 + \mathbf{c}q'^4 = (\mathbf{a}, \mathbf{h}, \mathbf{k}, \mathbf{f}, \mathbf{c} \between p', q')^4.$$

Note may be taken of a divergence in the properties of a surface, according as it lies in triple space or in quadruple space. In triple space, the direction of the radius of curvature is the same for all normal sections of the surface through the point. In quadruple space, the direction of the radius of curvature of normal sections of a surface varies from section to section.

All these directions in quadruple space lie in the plane through the point orthogonal to the tangent plane; for any one of the directions determines the line

$$\frac{\bar{x}-x}{l}=\frac{\bar{y}-y}{m}=\frac{\bar{z}-z}{n}=\frac{\bar{v}-v}{k}$$

lying in the plane

$$\Sigma\,(\bar{x}-x)\,x_1=0,\quad \Sigma\,(\bar{x}-x)\,x_2=0,$$

which is the plane orthogonal to the tangent plane.

Fundamental relation among the secondary quantities.

214. We return to the consideration of the quantities ξ_{ij}, η_{ij}, ζ_{ij}, υ_{ij}, for $i, j, = 1, 2$. Let the determinants

$$\left\|\begin{matrix} \xi_{11}, & \eta_{11}, & \zeta_{11}, & \upsilon_{11} \\ \xi_{12}, & \eta_{12}, & \zeta_{12}, & \upsilon_{12} \\ \xi_{22}, & \eta_{22}, & \zeta_{22}, & \upsilon_{22} \end{matrix}\right\|$$

be denoted by D_ξ, D_η, D_ζ, D_υ, respectively. From the equations

$$\Sigma x_1\xi_{11}=0,\quad \Sigma x_1\xi_{12}=0,\quad \Sigma x_1\xi_{22}=0,$$

we have

$$\frac{x_1}{D_\xi}=\frac{y_1}{D_\eta}=\frac{z_1}{D_\zeta}=\frac{v_1}{D_\upsilon};$$

and from the equations

$$\Sigma x_2\xi_{11}=0,\quad \Sigma x_2\xi_{12}=0,\quad \Sigma x_2\xi_{22}=0,$$

we have

$$\frac{x_2}{D_\xi}=\frac{y_2}{D_\eta}=\frac{z_2}{D_\zeta}=\frac{v_2}{D_\upsilon}.$$

Now x, y, z, v, the coordinates of a general point on the surface, are functions of the two parameters p and q, so that not all the relations

$$\left\|\begin{matrix} x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \end{matrix}\right\|=0$$

can be satisfied, and therefore we must have

$$D_\xi=0,\quad D_\eta=0,\quad D_\zeta=0,\quad D_\upsilon=0.$$

Consequently

$$D_\xi^2+D_\eta^2+D_\zeta^2+D_\upsilon^2=0,$$

that is,

$$\left|\begin{matrix} \Sigma\xi_{11}^2\;, & \Sigma\xi_{11}\xi_{12}, & \Sigma\xi_{11}\xi_{22} \\ \Sigma\xi_{11}\xi_{12}, & \Sigma\xi_{12}^2\;, & \Sigma\xi_{12}\xi_{22} \\ \Sigma\xi_{11}\xi_{22}, & \Sigma\xi_{12}\xi_{22}, & \Sigma\xi_{22}^2 \end{matrix}\right|=0,$$

or

$$\left|\begin{matrix} \mathbf{a}, & \mathbf{h}, & \mathbf{g} \\ \mathbf{h}, & \mathbf{b}, & \mathbf{f} \\ \mathbf{g}, & \mathbf{f}, & \mathbf{c} \end{matrix}\right|=0.$$

This relation is fundamental. It can be written in any of the three ways

$$(\mathbf{ab}-\mathbf{h}^2)(\mathbf{ac}-\mathbf{g}^2)=(\mathbf{gh}-\mathbf{af})^2,$$
$$(\mathbf{bc}-\mathbf{f}^2)(\mathbf{ba}-\mathbf{h}^2)=(\mathbf{hf}-\mathbf{bg})^2,$$
$$(\mathbf{ca}-\mathbf{g}^2)(\mathbf{cb}-\mathbf{f}^2)=(\mathbf{fg}-\mathbf{ch})^2,$$

three relations each of which is equivalent to the determinantal form of relation.

Again, the vanishing of the four determinants D_ξ, D_η, D_ζ, D_ν, can be expressed in the form

$$\left\|\begin{matrix}\xi_{11}, & \eta_{11}, & \zeta_{11}, & \nu_{11}\\ \xi_{12}, & \eta_{12}, & \zeta_{12}, & \nu_{12}\\ \xi_{22}, & \eta_{22}, & \zeta_{22}, & \nu_{22}\end{matrix}\right\|=0$$

Consequently, there are quantities ϵ and ω, such that

$$\xi_{12}=\epsilon\xi_{11}+\omega\xi_{22},$$
$$\eta_{12}=\epsilon\eta_{11}+\omega\eta_{22},$$
$$\zeta_{12}=\epsilon\zeta_{11}+\omega\zeta_{22},$$
$$\nu_{12}=\epsilon\nu_{11}+\omega\nu_{22}.$$

Let these equations be multiplied by ξ_{11}, η_{11}, ζ_{11}, ν_{11}, and the results be added; by ξ_{12}, η_{12}, ζ_{12}, ν_{12}, and the results be added; and by ξ_{22}, η_{22}, ζ_{22}, ν_{22}, and the results be added: then the successive relations

$$\mathbf{h}=\epsilon\mathbf{a}+\omega\mathbf{g},$$
$$\mathbf{b}=\epsilon\mathbf{h}+\omega\mathbf{f},$$
$$\mathbf{f}=\epsilon\mathbf{g}+\omega\mathbf{c},$$

are simultaneously satisfied in virtue of the determinantal relation.

By means of these relations, we find

$$\mathbf{h}=\epsilon\mathbf{a}+\omega\mathbf{g}$$
$$\mathbf{f}=\epsilon\mathbf{g}+\omega\mathbf{c},$$
$$\mathbf{b}=\epsilon^2\mathbf{a}+2\epsilon\omega\mathbf{g}+\omega^2\mathbf{c};$$

and therefore

$$\mathbf{ab}-\mathbf{h}^2=\omega^2(\mathbf{ac}-\mathbf{g}^2),\quad \mathbf{bc}-\mathbf{f}^2=\epsilon^2(\mathbf{ac}-\mathbf{g}^2),\quad \mathbf{fh}-\mathbf{bg}=\epsilon\omega(\mathbf{ac}-\mathbf{g}^2),$$
$$\mathbf{gh}-\mathbf{af}=-\omega(\mathbf{ac}-\mathbf{g}^2),\quad \mathbf{fg}-\mathbf{ch}=-\epsilon(\mathbf{ac}-\mathbf{g}^2).$$

In defining the magnitudes

$$(\mathbf{bc}-\mathbf{f}^2)^{\frac{1}{2}},\quad (\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}},\quad (\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}},$$

we assign a positive sign to each of the radicals. Consequently, we have

$$\left.\begin{aligned}(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}(\mathbf{ac}-\mathbf{g}^2)^{\frac{1}{2}}&=-(\mathbf{gh}-\mathbf{af})\\ (\mathbf{bc}-\mathbf{f}^2)^{\frac{1}{2}}(\mathbf{ba}-\mathbf{h}^2)^{\frac{1}{2}}&=+(\mathbf{fh}-\mathbf{bg})\\ (\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}(\mathbf{cb}-\mathbf{f}^2)^{\frac{1}{2}}&=-(\mathbf{fg}-\mathbf{ch})\end{aligned}\right\},$$

which are three equivalent forms of the fundamental relation connecting **a**, **b**, **c**, **f**, **g**, **h**. They will frequently recur.

Later (§ 232), we shall write

$$R = V(\mathbf{ab} - \mathbf{h}^2)^{\frac{1}{2}}, \quad S = V(\mathbf{ca} - \mathbf{g}^2)^{\frac{1}{2}}, \quad T = V(\mathbf{bc} - \mathbf{f}^2)^{\frac{1}{2}}$$

being magnitudes of the second order; and we infer

$$RS = V^2(\mathbf{af} - \mathbf{gh}), \quad TR = -V^2(\mathbf{bg} - \mathbf{fh}), \quad ST = V^2(\mathbf{ch} - \mathbf{fg}).$$

Also, we have

$$\left.\begin{aligned} S\mathbf{h} &= T\mathbf{a} + R\mathbf{g} \\ S\mathbf{b} &= T\mathbf{h} + R\mathbf{f} \\ S\mathbf{f} &= T\mathbf{g} + R\mathbf{c} \end{aligned}\right\},$$

relations which are required subsequently.

Secondary magnitudes L, M, N.

215. The equations giving the direction-cosines of the radius of curvature of the normal section are

$$\begin{aligned} \frac{l}{\rho} &= \xi_{11}p'^2 + 2\xi_{12}p'q' + \xi_{22}q'^2, \\ \frac{m}{\rho} &= \eta_{11}p'^2 + 2\eta_{12}p'q' + \eta_{22}q'^2, \\ \frac{n}{\rho} &= \zeta_{11}p'^2 + 2\zeta_{12}p'q' + \zeta_{22}q'^2, \\ \frac{k}{\rho} &= v_{11}p'^2 + 2v_{12}p'q' + v_{22}q'^2. \end{aligned}$$

In connection with these expressions, we introduce new magnitudes L, M, N, according to the definitions

$$\left.\begin{aligned} L &= l\xi_{11} + m\eta_{11} + n\zeta_{11} + kv_{11} = lx_{11} + my_{11} + nz_{11} + kv_{11} \\ M &= l\xi_{12} + m\eta_{12} + n\zeta_{12} + kv_{12} = lx_{12} + my_{12} + nz_{12} + kv_{12} \\ N &= l\xi_{22} + m\eta_{22} + n\zeta_{22} + kv_{22} = lx_{22} + my_{22} + nz_{22} + kv_{22} \end{aligned}\right\}.$$

Then, multiplying the above equations by $x_{11}, y_{11}, z_{11}, v_{11}$, and adding: by $x_{12}, y_{12}, z_{12}, v_{12}$, and adding: and by $x_{22}, y_{22}, z_{22}, v_{22}$, and adding: we obtain successively

$$\left.\begin{aligned} \frac{L}{\rho} &= \mathbf{a}p'^2 + 2\mathbf{h}p'q' + \mathbf{g}q'^2 \\ \frac{M}{\rho} &= \mathbf{h}p'^2 + 2\mathbf{b}p'q' + \mathbf{f}q'^2 \\ \frac{N}{\rho} &= \mathbf{g}p'^2 + 2\mathbf{f}p'q' + \mathbf{c}q'^2 \end{aligned}\right\};$$

and, multiplying by l, m, n, k, and adding, we obtain

$$\frac{1}{\rho} = Lp'^2 + 2Mp'q' + Nq'^2.$$

The substitution, in the last equation, of the values just obtained for L, M, N leads to the expression for $1/\rho^2$ given in § 213.

Meusnier's theorem: circular curvature of a curve on the surface.

216. Any number of curves on the surface can be drawn through the point O touching the same tangent line at O as the foregoing normal section: and each curve will have its own osculating plane. For any one such curve, let χ be the inclination of its osculating plane to the foregoing normal plane, χ being estimated as in § 90 because the two planes intersect in the tangent line; and let ρ_0 be its radius of circular curvature of the curve at O, so that the direction-cosines of that radius are $\rho_0 x''$, $\rho_0 y''$, $\rho_0 z''$, $\rho_0 v''$ Then

$$\cos\chi = l\,.\,\rho_0 x'' + m\,.\,\rho_0 y'' + n\,.\,\rho_0 z'' + k\,.\,\rho_0 v''$$
$$= \rho\rho_0 \Sigma x'' \frac{l}{\rho}.$$

But

$$x'' = x_1 p'' + x_2 q'' + x_{11} p'^2 + 2x_{12} p' q' + x_{22} q'^2;$$

and

$$\Sigma x_1 l = 0,$$
$$\Sigma x_2 l = 0,$$
$$\Sigma x_{11} \frac{l}{\rho} = \mathbf{a} p'^2 + 2\mathbf{h} p' q' + \mathbf{g} q'^2,$$
$$\Sigma x_{12} \frac{l}{\rho} = \mathbf{h} p'^2 + 2\mathbf{b} p' q' + \mathbf{f} q'^2,$$
$$\Sigma x_{22} \frac{l}{\rho} = \mathbf{g} p'^2 + 2\mathbf{f} p' q' + \mathbf{c} q'^2,$$

so that

$$\Sigma x'' \frac{l}{\rho} = \mathbf{a} p'^4 + 4\mathbf{h} p'^3 q' + (2\mathbf{g} + 4\mathbf{b}) p'^2 q'^2 + 4\mathbf{f} p' q'^3 + \mathbf{c} q'^4$$
$$= \frac{1}{\rho^2}.$$

Consequently,

$$\cos\chi = \frac{\rho_0}{\rho},$$

or

$$\frac{\cos\chi}{\rho_0} = \frac{1}{\rho},$$

which is Meusnier's theorem, pertaining to sections of a surface in triple space, and thus pertaining to sections of a surface in quadruple space.

This relation, however, gives only one equation towards the determination of ρ_0 and χ. To obtain a second relation to this end, let

$$\theta(p, q) = 0$$

be an equation giving the curve on the surface, so that θ is to be regarded as a known function. Along the curve, we have

$$p'\theta_1 + q'\theta_2 = 0,$$

so that

$$\frac{p'}{\theta_2} = \frac{q'}{-\theta_1} = \frac{1}{\Theta^{\frac{1}{2}}},$$

where

$$\Theta = E\theta_2^2 - 2F\theta_2\theta_1 + G\theta_1^2.$$

Again, along the curve

$$\begin{aligned} p''\theta_1 + q''\theta_2 &= -(\theta_{11}p'^2 + 2\theta_{12}p'q' + \theta_{22}q'^2) \\ &= -\frac{1}{\Theta}(\theta_{11}\theta_2^2 - 2\theta_{12}\theta_2\theta_1 + \theta_{22}\theta_1^2) = -\frac{1}{\Theta}C, \end{aligned}$$

say: while always, on the surface,

$$\frac{d}{ds}(Ep'^2 + 2Fp'q' + Gq'^2) = 0,$$

that is,

$$\begin{aligned} &(Ep' + Fq')p'' + (Fp' + Gq')q'' \\ &\qquad = -p'(E_1p'^2 + 2F_1p'q' + G_1q'^2) - q'(E_2p'^2 + 2F_2p'q' + G_2q'^2) \\ &\qquad = -(Ep' + Fq')A - (Fp' + Gq')B, \end{aligned}$$

where

$$A = \Gamma p'^2 + 2\Gamma' p'q' + \Gamma'' q'^2 = \frac{1}{\Theta}(\Gamma\theta_2^2 - 2\Gamma'\theta_2\theta_1 + \Gamma''\theta_1^2),$$

$$B = \Delta p'^2 + 2\Delta' p'q' + \Delta'' q'^2 = \frac{1}{\Theta}(\Delta\theta_2^2 - 2\Delta'\theta_2\theta_1 + \Delta''\theta_1^2).$$

From this last equation, we have

$$\frac{p'' + A}{Fp' + Gq'} = \frac{q'' + B}{-(Ep' + Fq')} = \Xi,$$

and then, substituting in $\Theta(p''\theta_1 + q''\theta_2) = -C$, we find

$$\begin{aligned} \Xi\Theta^{\frac{3}{2}} &= \theta_{11}\theta_2^2 - 2\theta_{12}\theta_2\theta_1 + \theta_{22}\theta_1^2 - \theta_1(\Gamma\theta_2^2 - 2\Gamma'\theta_2\theta_1 + \Gamma''\theta_1^2) \\ &\qquad - \theta_2(\Delta\theta_2^2 - 2\Delta'\theta_2\theta_1 + \Delta''\theta_1^2) \\ &= (\theta_{11} - \theta_1\Gamma - \theta_2\Delta)\theta_2^2 - 2(\theta_{12} - \theta_1\Gamma' - \theta_2\Delta')\theta_2\theta_1 + (\theta_{22} - \theta_1\Gamma'' - \theta_2\Delta'')\theta_1^2. \end{aligned}$$

Now we have, identically,

$$(x_{11} - \xi_{11})p'^2 + 2(x_{12} - \xi_{12})p'q' + (x_{22} - \xi_{22})q'^2 = x_1A + x_2B,$$

so that

$$x_{11}p'^2 + 2x_{12}p'q' + x_{22}q'^2 = \frac{l}{\rho} + x_1A + x_2B.$$

Hence

$$x_1p'' + x_2q'' + x_{11}p'^2 + 2x_{12}p'q' + x_{22}q'^2 = \frac{l}{\rho} + x_1(p'' + A) + x_2(q'' + B),$$

and therefore

$$\begin{aligned} \frac{1}{\rho_0^2} &= \Sigma(x_1p'' + x_2q'' + x_{11}p'^2 + 2x_{12}p'q' + x_{22}q'^2)^2 \\ &= \Sigma\left\{\frac{l}{\rho} + x_1(p'' + A) + x_2(q'' + B)\right\}^2. \end{aligned}$$

But $\Sigma l x_1 = 0$, $\Sigma l x_2 = 0$; and therefore

$$\frac{1}{{\rho_0}^2} - \frac{1}{\rho^2} = E(p'' + A)^2 + 2F(p'' + A)(q'' + B) + G(q'' + B)^2$$
$$= V^2 \Xi^2.$$

Consequently, as

$$\frac{\cos\chi}{\rho_0} = \frac{1}{\rho},$$

we have

$$\frac{\sin\chi}{\rho_0} = V\Xi$$
$$= V\frac{(\theta_{11} - \theta_1\Gamma - \theta_2\Delta){\theta_2}^2 - 2(\theta_{12} - \theta_1\Gamma' - \theta_2\Delta')\theta_2\theta_1 + (\theta_{22} - \theta_1\Gamma'' - \theta_2\Delta''){\theta_1}^2}{(E{\theta_2}^2 - 2F\theta_2\theta_1 + G{\theta_1}^2)^{\frac{1}{2}}},$$

which thus is a second relation connecting χ and ρ_0, on the one hand with the equation of the curve, on the other hand with the magnitudes of the surface.

Later (§ 223), the expression on the right-hand side will be identified with the geodesic curvature of the curve $\theta(p, q) = 0$.

NOTE. Meusnier's theorem holds for surfaces in n-fold space, a surface being an amplitude of two dimensions.

Geodesics on a surface: fundamental property.

217. Among the organic curves of a surface, geodesics are of prime importance. Their rudimentary properties will be obtained at once, as ancillary to the establishment of the curvature properties of the surface.

As usual, a geodesic is defined to be the curve of shortest distance, measured in the surface between two points. We are mainly concerned with the current properties of the curve along its course, and are less concerned with the relative positions of conjugate positions on the curve determining a (Jacobi) range within which it possesses the minimum property. The curve satisfies the (Legendre) condition of providing a minimum and not merely a stationary value; it satisfies also the (Weierstrass) condition of providing a minimum for strong variations as well as for weak variations. Moreover, as an inference from the analysis, it follows that, instead of being required to be a superficial arc joining two assigned points, the curve is uniquely defined in position and range on the surface by the assignment of an initial point and of a direction through that point.

We deal, first, with geodesics upon a surface when the equations of the surface are given by means of equations

$$\Theta(x, y, z, v) = 0, \quad \Phi(x, y, z, v) = 0.$$

Along the unknown geodesic curve, we take a current variable t; the element of arc ds is given by the usual form; and therefore the property, by which a geodesic is defined, requires that the integral

$$\int \frac{ds}{dt}\,dt, \quad =\int\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2+\left(\frac{dv}{dt}\right)^2\right\}^{\frac{1}{2}} dt, \quad =\int U\,dt,$$

shall be a minimum, the variables x, y, z, v, being subject to the two permanent relations $\Theta=0$, $\Phi=0$. The characteristic equations*, which specify the nature of the curve, though without regard to its range and without prejudice to the other tests, are four in number, each of the type

$$\frac{d}{dt}\left(\frac{\partial U}{\partial x_t}\right)=\lambda_t\frac{\partial\Theta}{\partial x}+\mu_t\frac{\partial\Phi}{\partial x},$$

where x_t denotes $\frac{dx}{dt}$, while λ_t and μ_t are multipliers that are not explicitly determinate at this stage. Now

$$\frac{\partial U}{\partial x_t}=\frac{x_t}{(x_t^2+y_t^2+z_t^2+v_t^2)^{\frac{1}{2}}}=\frac{dx}{ds},$$

and

$$\frac{d}{dt}\left(\frac{\partial U}{\partial x_t}\right)=\frac{d^2x}{ds^2}\frac{ds}{dt}.$$

If then we write $\lambda_t=\lambda\frac{ds}{dt}$, $\mu_t=\mu\frac{ds}{dt}$, the four characteristic equations of the curve are

$$x''=\lambda\frac{\partial\Theta}{\partial x}+\mu\frac{\partial\Phi}{\partial x},\quad y''=\lambda\frac{\partial\Theta}{\partial y}+\mu\frac{\partial\Phi}{\partial y},\quad z''=\lambda\frac{\partial\Theta}{\partial z}+\mu\frac{\partial\Phi}{\partial z},\quad v''=\lambda\frac{\partial\Theta}{\partial v}+\mu\frac{\partial\Phi}{\partial v}.$$

But the direction-cosines of the principal normal to the curve are $\rho x''$, $\rho y''$, $\rho z''$, $\rho v''$, where ρ is its radius of circular curvature: the quantities, such as $\lambda\frac{\partial\Theta}{\partial x}+\mu\frac{\partial\Phi}{\partial x}$, are proportional to the direction-cosines of some direction in the plane orthogonal to the tangent plane; and therefore the radius of circular curvature at any point of the geodesic lies in the plane orthogonal to the surface. Its direction, in that plane, has to be made specific.

Next, let ξ, η, ζ, υ, be a point on the curve, in the near vicinity of the point x, y, z, v, at which the tangent plane to the surface is represented by the equations

$$\Sigma(X-x)\Theta_x=0,\quad \Sigma(X-x)\Phi_x=0;$$

and let the perpendicular, of length Q and with direction-cosines l, m, n, k, be drawn from ξ, η, ζ, υ, upon this plane. Then, as in § 211, if $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{V}$, be the foot of this perpendicular, the quantity

$$\Sigma(\xi-\bar{X})^2$$

is less for this point than for any other point in the plane, and thus acquires a minimum value for the range of values of $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{V}$, which are subject to the two relations $\Sigma(\bar{X}-x)\Theta_x=0$ and $\Sigma(\bar{X}-x)\Phi_x=0$. Hence

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* See my *Calculus of Variations*, § 261.

$$\begin{aligned}\theta\Theta_x+\phi\Phi_x&=\xi-\bar{X}=lQ,\\ \theta\Theta_y+\phi\Phi_y&=\eta-\bar{Y}=mQ,\\ \theta\Theta_z+\phi\Phi_z&=\zeta-\bar{Z}=nQ,\\ \theta\Theta_v+\phi\Phi_v&=\upsilon-\bar{V}=kQ,\end{aligned}$$

where θ and ϕ are new multipliers for the minimum length of the perpendicular, a quest different from the determination of the shortest length along the surface. Now

$$\xi-\bar{X}=\xi-x-(\bar{X}-x),$$

and similarly for $\eta-\bar{Y}$, $\zeta-\bar{Z}$, $\upsilon-\bar{V}$; hence

$$\begin{aligned}\Sigma(\xi-\bar{X})\Theta_x&=\Sigma(\xi-x)\Theta_x-\Sigma(\bar{X}-x)\Theta_x=\Sigma(\xi-x)\Theta_x,\\ \Sigma(\xi-\bar{X})\Phi_x&=\Sigma(\xi-x)\Phi_x-\Sigma(\bar{X}-x)\Phi_x=\Sigma(\xi-x)\Phi_x,\end{aligned}$$

and therefore

$$\left.\begin{aligned}\Sigma(\xi-x)\Theta_x&=\theta\Sigma\Theta_x^2+\phi\Sigma\Theta_x\Phi_x\\ \Sigma(\xi-x)\Phi_x&=\theta\Sigma\Theta_x\Phi_x+\phi\Sigma\Phi_x^2\end{aligned}\right\}.$$

Let δ denote the arc-length along the curve in the surface from x, y, z, v, to $\xi, \eta, \zeta, \upsilon$, so that δ is a small quantity because the point $\xi, \eta, \zeta, \upsilon$, is taken in the near vicinity of the point x, y, z, v; then

$$\begin{aligned}\xi&=x+x'\delta+\frac{x''}{2!}\delta^2+\frac{x'''}{3!}\delta^3+\dots,\\ \eta&=y+y'\delta+\frac{y''}{2!}\delta^2+\frac{y'''}{3!}\delta^3+\dots,\\ \zeta&=z+z'\delta+\frac{z''}{2!}\delta^2+\frac{z'''}{3!}\delta^3+\dots,\\ \upsilon&=v+v'\delta+\frac{v''}{2!}\delta^2+\frac{v'''}{3!}\delta^3+\dots.\end{aligned}$$

As the direction of the tangent lies in the tangent plane, we have

$$\Sigma x'\Theta_x=0,\quad \Sigma x'\Phi_x=0;$$

hence

$$\begin{aligned}\Sigma(\xi-x)\Theta_x&=\tfrac{1}{2}\delta^2\Sigma x''\Theta_x+\tfrac{1}{6}\delta^3\Sigma x'''\Theta_x+\dots,\\ \Sigma(\xi-x)\Phi_x&=\tfrac{1}{2}\delta^2\Sigma x''\Phi_x+\tfrac{1}{6}\delta^3\Sigma x'''\Phi_x+\dots,\end{aligned}$$

and therefore

$$\left.\begin{aligned}\theta\Sigma\Theta_x^2+\phi\Sigma\Theta_x\Phi_x&=\tfrac{1}{2}\delta^2\Sigma x''\Theta_x+\tfrac{1}{6}\delta^3\Sigma x'''\Theta_x+\dots\\ \theta\Sigma\Theta_x\Phi_x+\phi\Sigma\Phi_x^2&=\tfrac{1}{2}\delta^2\Sigma x''\Phi_x+\tfrac{1}{6}\delta^3\Sigma x'''\Phi_x+\dots\end{aligned}\right\}.$$

These equations are connected with the perpendicular drawn from a point Q on the surface to the tangent plane at x, y, z, v, this point Q being taken along the geodesic.

But, from the equations of the geodesic which are

$$x''=\lambda\Theta_x+\mu\Phi_x,\quad y''=\lambda\Theta_y+\mu\Phi_y,\quad z''=\lambda\Theta_z+\mu\Phi_z,\quad v''=\lambda\Theta_v+\mu\Phi_v,$$

we have

$$\Sigma x''\Theta_x=\lambda\Sigma\Theta_x^2\quad+\mu\Sigma\Theta_x\Phi_x,$$
$$\Sigma x''\Phi_x=\lambda\Sigma\Theta_x\Phi_x+\mu\Sigma\Phi_x^2.$$

Consequently, when we take δ to be very small so that the point Q is in the immediate vicinity of the point O and powers δ^3, δ^4, ... can be neglected in comparison with δ^2, we have

$$\theta=\tfrac{1}{2}\delta^2\lambda,\quad \phi=\tfrac{1}{2}\delta^2\mu.$$

Thus

$$x''=\lambda\Theta_x+\mu\Phi_x$$
$$=\frac{2}{\delta^2}(\theta\Theta_x+\phi\Phi_x),$$

$$\tfrac{1}{2}\delta^2x''=\theta\Theta_x+\phi\Phi_x=\xi-\overline{X}=lQ;$$

and similarly

$$\tfrac{1}{2}\delta^2y''=\eta-\overline{Y}=mQ,$$
$$\tfrac{1}{2}\delta^2z''=\zeta-\overline{Z}=nQ,$$
$$\tfrac{1}{2}\delta^2v''=\upsilon-\overline{V}=kQ.$$

From these equations, several inferences follow.

The foot of the perpendicular on the tangent plane drawn from ξ, η, ζ, υ, is $\overline{X}$, $\overline{Y}$, $\overline{Z}$, $\overline{V}$. Now

$$\overline{X}=\xi-\tfrac{1}{2}\delta^2x''$$
$$=x+x'\delta-\tfrac{1}{6}x'''\delta^3+\ldots,$$

and similarly for $\overline{Y}$, $\overline{Z}$, $\overline{V}$; hence, to the second order inclusive,

$$\overline{X}-x=x'\delta,\quad \overline{Y}-y=y'\delta,\quad \overline{Z}-z=z'\delta,\quad \overline{V}-v=v'\delta.$$

Thus the projection of the chord OQ upon the tangent plane has the direction-cosines of the tangent line at O; and the length of the projection of that chord is equal, save as to small quantities of the third and higher powers of δ, to the length of the arc δ.

Next, the direction-cosines of the perpendicular, being l, m, n, k, are proportional to x'', y'', z'', v'': that is, the said perpendicular and the radius of circular curvature of the geodesic are in the same direction. Denoting this radius of circular curvature by ρ, we have

$$\delta^2=2Q\rho;$$

and therefore, in the immediate vicinity of O, the geodesic coincides with a section of the surface, by a normal plane through the tangent to the curve at the point and through the perpendicular from a consecutive point on the curve drawn to the tangent plane. This normal plane is not the orthogonal plane; its orientation depends on the tangent to the curve

NOTE 1. It is to be noted that, as before (§ 212), while the number of directions through ξ, η, ζ, v, perpendicular to the tangent plane is unlimited, there is only one direction such that a line, drawn along it, meets the plane.

NOTE 2. The quantities θ and ϕ, λ and μ, are multipliers which, initially, are undetermined; and their determination would constitute part of the detailed solution of the problem. They are, of course, distinct from the parameters required for the general specification of a point either in the tangent plane or in the orthogonal plane. We leave, as an exercise, the determination of the significance of the quantities θ and ϕ, and their relation to the quantity Q, in connection (§ 68) with the perpendiculars drawn from the point Q to the two flats which are tangential to the regions $\Theta = 0$ and $\Phi = 0$ respectively.

Critical tests for geodesics, due to Legendre, Jacobi, Weierstrass.

218. The equations, that express the essential property of a geodesic, have an entirely different form when the surface is represented parametrically. An element of arc is then given by the expression

$$ds^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2.$$

As the arc is to be made geodesic, it is convenient not to take s as an independent current variable. so a variable t is introduced, and we write

$$p_0 = \frac{dp}{dt}, \quad q_0 = \frac{dq}{dt},$$

and then we have to make

$$\int f\,dt = \int (Ep_0^2 + 2Fp_0q_0 + Gq_0^2)^{\frac{1}{2}}\,dt$$

a minimum, where E, F, G, are known functions of p and q.

We first dispose of the three tests, other than the characteristic equation which is the most important of them all. These three tests are, (i) the Legendre test which discriminates between maxima and minima for stationary values provided by the characteristic equation: (ii) the Jacobi test which imposes a limit (if there be a limit) upon the length of range that possesses the maximum or the minimum quality: and (iii) the Weierstrass test, which gives the critical discrimination for strong variations. As f, the subject of integration, satisfies the equation

$$p_0 \frac{\partial f}{\partial p_0} + q_0 \frac{\partial f}{\partial q_0} = f,^*$$

we may apply the tests in the customary Weierstrass forms

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[*] See my *Calculus of Variations*, chapter II, §§ 41–43, 55–56, 63–64, for the weak variations: *ib.* chapter VII, § 213, for the strong variations.

(i) The Legendre test, for a minimum, is satisfied. We have

$$\frac{1}{q_0^2}\frac{\partial^2 f}{\partial p_0^2} = -\frac{1}{p_0 q_0}\frac{\partial^2 f}{\partial p_0 \partial q_0} = \frac{1}{p_0^2}\frac{\partial^2 f}{\partial q_0^2} = P$$

$$= \frac{V^2}{(Ep_0^2 + 2Fp_0q_0 + Gq_0^2)^{\frac{3}{2}}},$$

where the sign of the radical, being the same as the sign of the radical in the integral, is positive. Thus the critical quantity P is positive. So far as the Legendre test is concerned, a minimum is provided.

(ii) The Jacobi test is quantitative, not qualitative. It assigns a range, extending from a point on the curve to another point called the conjugate: and the course of integration, beginning at the first point, must certainly not extend beyond the conjugate for the possession (in this case) of a minimum. As our main concern is with the current qualitative properties of the curve, any discussion of the quantitative Jacobi test will be deferred until it is actually needed in any specific instance: the construction of the critical function in the Jacobi test depends upon a knowledge of the primitive of the characteristic equations or upon equivalent knowledge.

(iii) The Weierstrass test, being the test through the Excess-Function, is satisfied. In the present instance, this Excess-Function is

$$\mathfrak{E} = (E\lambda^2 + 2F\lambda\mu + G\mu^2)^{\frac{1}{2}} - \left(\lambda\frac{\partial}{\partial p_0} + \mu\frac{\partial}{\partial q_0}\right)(Ep_0^2 + 2Fp_0q_0 + Gq_0^2)^{\frac{1}{2}};$$

here λ and μ are arbitrary values of p_0 and q_0, which represent an arbitrary direction through a point on the curve, coinciding with the tangent to the curve only when $\lambda = p_0$ and $\mu = q_0$. But

$$\mathfrak{E} = (E\lambda^2 + 2F\lambda\mu + G\mu^2)^{\frac{1}{2}}\left\{1 - \frac{E\lambda p_0 + F(\lambda q_0 + \mu p_0) + G\mu q_0}{(E\lambda^2 + 2F\lambda\mu + G\mu^2)^{\frac{1}{2}}(Ep_0^2 + 2Fp_0q_0 + Gq_0^2)^{\frac{1}{2}}}\right\}$$

$$= (E\lambda^2 + 2F\lambda\mu + G\mu^2)^{\frac{1}{2}}(1 - \cos\psi),$$

where ψ is the inclination of the arbitrary direction at the point to the tangent to the curve. When there is any variation off the curve, ψ is not zero: the radical is positive: thus $\mathfrak{E}$ is positive So far as the Weierstrass test is concerned, a minimum is provided.

There remain the characteristic equations.

Characteristic equations of geodesics in parametric form.

219. The characteristic equations, which must be satisfied if

$$\int f\,dt, = \int (Ep_0^2 + 2Fp_0q_0 + Gq_0^2)^{\frac{1}{2}}\,dt$$

is to possess a minimum, now that the other tests are met, are

$$\frac{d}{dt}\left(\frac{\partial f}{\partial p_0}\right) - \frac{\partial f}{\partial p} = 0, \quad \frac{d}{dt}\left(\frac{\partial f}{\partial q_0}\right) - \frac{\partial f}{\partial q} = 0,$$

in general: in the present instance, they are

$$\frac{d}{dt}\left\{\frac{Ep_0+Fq_0}{(Ep_0^2+2Fp_0q_0+Gq_0^2)^{\frac{1}{2}}}\right\}=\frac{1}{2}\frac{E_1p_0^2+2F_1p_0q_0+G_1q_0^2}{(Ep_0^2+2Fp_0q_0+Gq_0^2)^{\frac{1}{2}}},$$

$$\frac{d}{dt}\left\{\frac{Fp_0+Gq_0}{(Ep_0^2+2Fp_0q_0+Gq_0^2)^{\frac{1}{2}}}\right\}=\frac{1}{2}\frac{E_2p_0^2+2F_2p_0q_0+G_2q_0^2}{(Ep_0^2+2Fp_0q_0+Gq_0^2)^{\frac{1}{2}}}.$$

The two equations are easily proved to be equivalent to a single equation, through the identity $p_0\frac{\partial f}{\partial p_0}+q_0\frac{\partial f}{\partial q_0}=f$: this single equation ultimately being explicitly free from the unessential variable t. But it is convenient to retain the two equations simultaneously.

Returning to the quantities p' and q', equal to $\frac{dp}{ds}$ and $\frac{dq}{ds}$, because the arc is the convenient variable now that the characteristic equations have been formed, we have

$$\frac{d}{dt}\left\{\frac{Ep_0+Fq_0}{(Ep_0^2+2Fp_0q_0+Gq_0^2)^{\frac{1}{2}}}\right\}=\frac{d}{dt}(Ep'+Fq');$$

and so the first equation becomes

$$\frac{d}{dt}(Ep'+Fq')=\tfrac{1}{2}(E_1p'^2+2F_1p'q'+G_1q'^2)\left(\frac{ds}{dt}\right)^{-1}.$$

Hence

$$Ep''+Fq''+p'(E_1p'+E_2q')+q'(F_1p'+F_2q')=\tfrac{1}{2}(E_1p'^2+2F_1p'q'+G_1q'^2),$$

and therefore

$$Ep''+Fq''+\alpha p'^2+2\alpha'p'q'+\alpha''q'^2=0,$$

introducing the derived quantities defined in § 207. Similarly, the second equation becomes

$$Fp''+Gq''+\beta p'^2+2\beta'p'q'+\beta''q'^2=0.$$

When the two equations are resolved for p'' and q'', they yield the two equivalent equations

$$\left.\begin{aligned}p''+\Gamma p'^2+2\Gamma'p'q'+\Gamma''q'^2=0\\ q''+\Delta p'^2+2\Delta'p'q'+\Delta''q'^2=0\end{aligned}\right\},$$

which accordingly are *the equations of a geodesic curve* upon a surface in what may be regarded as their canonical form.

Ex. Verify that these two equations are equivalent to a single equation only, in virtue of the identical relation

$$Ep'^2+2Fp'q'+Gq'^2=1,$$

arising out of the definition of an arc-element; and shew that, if q be taken as a dependent variable and p as the independent variable, the single equation characteristic of a geodesic is

$$\frac{d^2q}{dp^2}=\Gamma''\left(\frac{dq}{dp}\right)^3+(2\Gamma'-\Delta'')\left(\frac{dq}{dp}\right)^2+(\Gamma-2\Delta')\frac{dq}{dp}-\Delta.$$

One important inference from the theory of differential equations may be stated here. The primitive of the two equations, expressing p and q as functions of t, involves four arbitrary constants. As t does not occur explicitly in the differential equation, one of these occurs in an expression $t+t_0$ wherever t occurs explicitly in the primitive. The other three are determinable uniquely by assigned values of p and q for an assigned value of t, (that is, an assigned point on the surface) and by the assigned value of $\frac{dq}{dp}$ for that value of t, (that is, an assigned direction in the surface through that point). In other words, a geodesic is uniquely determinate* by the assignment of a point on its course and of its direction at that point.

Ex. 1. Consider the geodesics on the cylindro-cylindrical surface given by the equations

$$x^2+y^2=a^2, \quad z^2+v^2=c^2,$$

where a and c are constants.

(i) For the parametric representation of the surface, we take

$$x=a\cos\theta, \quad y=a\sin\theta, \quad z=c\cos\phi, \quad v=c\sin\phi.$$

Then

$$ds^2=a^2d\theta^2+c^2d\phi^2,$$

so that

$$E=a^2, \quad F=0, \quad G=c^2.$$

The quantities Γ, Γ', Γ'', Δ, Δ', Δ'', vanish: thus the geodesic equations are

$$\theta''=0, \quad \phi''=0;$$

and integrals of these are

$$a\theta'=\cos\alpha, \quad c\phi'=\sin\alpha,$$

the two constants of the respective integrals being subject to the requirement that the equation

$$a^2\theta'^2+c^2\phi'^2=1$$

shall be satisfied.

The characteristic equations can be derived at once by minimising the integral

$$\int\left\{a^2\left(\frac{d\theta}{dt}\right)^2+c^2\left(\frac{d\phi}{dt}\right)^2\right\}^{\frac{1}{2}}dt.$$

For the primitive of the equations, two essential constants occur: and there is an unessential arbitrary constant, which is absorbed into the variable s. The essential arbitrary constants are β and γ in the integrals

$$a(\theta-\beta)=s\cos\alpha, \quad c(\phi-\gamma)=s\sin\alpha;$$

the substitution of these values of θ and ϕ in the expressions for x, y, z, v, leads to the primitive, that is, to the general equations of geodesics on the surface.

In the accompanying figure, the axes and the lines parallel to the axes are drawn as in the figure on p. 7; letters, indicating points, correspond in the two figures.

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[*] It is assumed, throughout, that the surface possesses no singular points and no singular lines within the considered range.

The circle $x^2+y^2=a^2$ in the plane XOY is the circle ..KhK', and the angle XOh is θ. The circle $z^2+v^2=c^2$ in the plane ZOV is the circle ..$Lh'L'$, and the angle LOh' is ϕ. The point P is $a\cos\theta$, $a\sin\theta$, $c\cos\phi$, $c\sin\phi$. The circle ..K_1PK_1' is the circle $x^2+y^2=a^2$, $z=c\cos\phi$, $v=c\sin\phi$, that is, lying in the plane $\beta h'a$ parallel to XOY, and it has its centre at h'. The circle ..L_1PL_1' is the circle $z^2+v^2=c^2$, $x=a\cos\theta$, $y=a\sin\theta$, that is, lying in the plane $\delta h\gamma$ parallel to ZOV, and it has its centre at h. These two circles K_1PK_1' and L_1PL_1' lie in the surface; at P, the two circles cut at right angles, so that a geodesic through P cuts all circles of the type K_1K_1' at the same angle α as that at which it cuts the circle K_1K_1', and cuts all circles of the type L_1L_1' at the same angle $\frac{1}{2}\pi-\alpha$ as that at which it cuts the circle L_1L_1'.

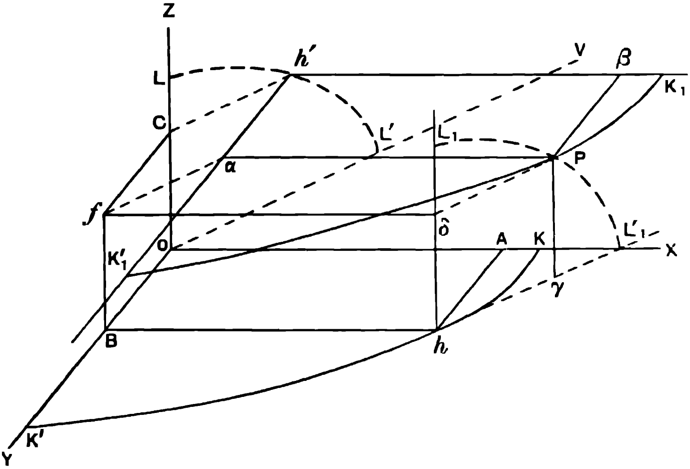

Fig 20.

(ii) We have seen (§ 218) that, for a geodesic on a surface, the Legendre test and the Weierstrass test (affecting a minimum, or a maximum, as the alternative may be) are satisfied

The Jacobi test, quantitative as to the range along a curve satisfying the characteristic equations, is individual to each curve; it has therefore to be applied to the curve in particular, not to geodesics in general. Stated geometrically* for the present instance, the requirement is that the curve shall not extend, from an initial point, up to the conjugate of that point; and the conjugate is determined as the next intersection of the curve by a consecutive curve through the initial point, when that consecutive curve is obtained by small variations of the arbitrary constants.

For the geodesic under consideration, the integral equations may be taken in the form

$$\theta=\beta+s\frac{\cos\alpha}{a}, \quad \phi=\gamma+s\frac{\sin\alpha}{c},$$

where α, β, γ, are arbitrary constants We have to find the conjugate (if any) of an initial point.

Let the initial point be given by $s=s_0$, and the (possible) conjugate by $s=s_1$. For the consecutive curve, let the arbitrary constants α, β, γ, be changed to

$$\alpha+\delta\alpha, \quad \beta+\delta\beta, \quad \gamma+\delta\gamma,$$

---

* See my *Calculus of Variations*, §§ 173, 174.

let the value of s at the initial point on this consecutive curve be $s_0+\delta s_0$, and let the value of s at the conjugate point (which, if there be such a point, lies on this consecutive curve) be $s_1+\delta s_1$. As the initial point is common to the two curves, the values of θ and the values of ϕ at that point are the same for the two curves; hence

$$\left.\begin{aligned} \delta\beta+\delta s_0\frac{\cos\alpha}{a}-s_0\frac{\sin\alpha}{a}\delta\alpha=0 \\ \delta\gamma+\delta s_0\frac{\sin\alpha}{c}+s_0\frac{\cos\alpha}{c}\delta\alpha=0 \end{aligned}\right\}.$$

As the conjugate point (if any) is common to the two curves, the two values of θ and the two values of ϕ at that point are the same for the two curves, hence

$$\left.\begin{aligned} \delta\beta+\delta s_1\frac{\cos\alpha}{a}-s_1\frac{\sin\alpha}{a}\delta\alpha=0 \\ \delta\gamma+\delta s_1\frac{\sin\alpha}{c}+s_1\frac{\cos\alpha}{c}\delta\alpha=0 \end{aligned}\right\}$$

If all these equations coexist, for some non-zero change from a curve to a distinct consecutive curve, the existence of the conjugate point will be established The equations are easily seen to require, firstly,

$$\delta s_1=\delta s_0,$$

and secondly, either $s_1-s_0=0$ or $\delta\alpha=0$: each combination precludes a distinct consecutive curve.

Hence a consecutive geodesic, through an initial point, does not intersect a first geodesic. There is no conjugate of an initial point. For such a geodesic, the Jacobi test imposes no limit on the range of an established curve

It is to be noted that the result refers to an established curve. As[*] with a helix on a circular cylinder in homaloidal triple space, an unlimited number of geodesics can be drawn on the surface joining two points on the surface; each is a minimum, and there is no limit to its range.

We leave, as an exercise, the determination of the least among all these minimum values.

(iii) To find the various curvatures, we proceed from the parametric representation, using the (first) integral equations

$$a\theta'=\cos\alpha,\quad c\phi'=\sin\alpha,$$

of the geodesics. We have

$$\begin{array}{llll} x'=-\sin\theta\cos\alpha, & y'=\cos\theta\cos\alpha, & z'=-\sin\phi\sin\alpha, & v'=\cos\phi\sin\alpha; \\ x''=-\cos\theta\dfrac{\cos^2\alpha}{a}, & y''=-\sin\theta\dfrac{\cos^2\alpha}{a}, & z''=-\cos\phi\dfrac{\sin^2\alpha}{c}, & v''=-\sin\phi\dfrac{\sin^2\alpha}{c}; \\ x'''=\sin\theta\dfrac{\cos^3\alpha}{a^2}, & y'''=-\cos\theta\dfrac{\cos^3\alpha}{a^2}, & z'''=\sin\phi\dfrac{\sin^3\alpha}{c^2}, & v'''=-\cos\phi\dfrac{\sin^3\alpha}{c^2}, \\ x^{\text{iv}}=\cos\theta\dfrac{\cos^4\alpha}{a^3}, & y^{\text{iv}}=\sin\theta\dfrac{\cos^4\alpha}{a^3}, & z^{\text{iv}}=\cos\phi\dfrac{\sin^4\alpha}{c^3}, & v^{\text{iv}}=\sin\phi\dfrac{\sin^4\alpha}{c^3}. \end{array}$$

These quantities satisfy the persistent relation

$$\Sigma x'^2=1.$$

We have

$$\frac{1}{\rho^2}=\Sigma x''^2=\frac{\cos^4\alpha}{a^2}+\frac{\sin^4\alpha}{c^2},$$

and therefore the radius of circular curvature is constant.

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[*] *Calculus of Variations*, § 75, p. 107.

Again,

$$\Sigma x'''^2 = \frac{\cos^6 \alpha}{a^4} + \frac{\sin^6 \alpha}{c^4}.$$

But, in general,

$$\Sigma x'''^2 = \frac{1}{\sigma^2 \rho^2} + \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4};$$

and, in the present instance, ρ is constant, so that

$$\frac{1}{\sigma^2 \rho^2} + \frac{1}{\rho^4} = \frac{\cos^6 \alpha}{a^4} + \frac{\sin^6 \alpha}{c^4}.$$

Consequently

$$\frac{1}{\rho\sigma} = \left(\frac{\cos^2 \alpha}{a^2} - \frac{\sin^2 \alpha}{c^2}\right) \sin \alpha \cos \alpha,$$

with an implied assumption concerning the positive value of σ.

Further, by substitution and evaluation, we have

$$\begin{vmatrix} x', & y', & z', & v' \\ x'', & y'', & z'', & v'' \\ x''', & y''', & z''', & v''' \\ x^{\text{iv}}, & y^{\text{iv}}, & z^{\text{iv}}, & v^{\text{iv}} \end{vmatrix} = \left(\frac{\cos^2 \alpha}{a^2} - \frac{\sin^2 \alpha}{c^2}\right)^2 \frac{\cos^3 \alpha \sin^3 \alpha}{ac}.$$

Now (§ 145) the value of the determinant on the left-hand side is $-(\rho^3 \sigma^2 \tau)^{-1}$; hence

$$\frac{1}{\rho^3 \sigma^2 \tau} = -\left(\frac{\cos^2 \alpha}{a^2} - \frac{\sin^2 \alpha}{c^2}\right)^2 \frac{\cos^3 \alpha \sin^3 c}{ac}$$

and therefore

$$\frac{1}{\rho\tau} = -\frac{\cos \alpha \sin \alpha}{ac}.$$

It follows that the radius of circular curvature, the radius of torsion, and the radius of tilt, are constant, each of them. There are, moreover, three disposable constants a, c, α: thus there is no identical relation connecting ρ, σ, τ.

We thus have a curve in the quadruple space, with its three curvatures constant and different from zero consequently, it belongs to the curves already discussed (§ 170). Now, save as to position and orientation, a curve in quadruple space is intrinsically unique (§ 169) by the assignment of its three curvatures; hence *every curve in quadruple space, that has all its curvatures constant, can be represented as a geodesic on a cylindro-cylindrical surface*

$$x^2 + y^2 = a^2, \quad z^2 + v^2 = c^2.$$

Note. We leave, as an exercise, the discussion of the curvatures by the use of the Frenet formulæ (§ 164).

Ex. 2. As another example, we may take the property (thus extended from homaloidal triple space to homaloidal quadruple space) that the lines of zero length, the nul lines, on any surface satisfy the equations characteristic of geodesics

To establish the property, we make the parameters of the nul lines—assumed to be distinct from one another—to be the parameters of reference for the surface In that event, we have

$$E = 0, \quad G = 0,$$

and therefore

$$V^2 = -F^2.$$

We find

$$\Gamma = \frac{F_1}{F}, \quad \Gamma' = 0, \quad \Gamma'' = 0,$$

$$\Delta = 0, \quad \Delta' = 0, \quad \Delta'' = \frac{F_2}{F};$$

and so the equations of geodesics on the surface, with t as a current variable, are

$$\frac{d^2p}{dt^2}\frac{ds}{dt}-\frac{dp}{dt}\frac{d^2s}{dt^2}+\frac{F_1}{F}\left(\frac{dp}{dt}\right)^2\frac{ds}{dt}=0,\quad \frac{d^2q}{dt^2}\frac{ds}{dt}-\frac{dq}{dt}\frac{d^2s}{dt^2}+\frac{F_2}{F}\left(\frac{dq}{dt}\right)^2\frac{ds}{dt}=0.$$

These are equivalent to a single equation, in virtue of the fundamental arc-equation

$$2F\frac{dp}{dt}\frac{dq}{dt}=\left(\frac{ds}{dt}\right)^2.$$

Manifestly all the equations are satisfied by

$$p=\text{constant},\ ds=0.\quad q=\text{constant},\ ds=0:$$

so that the nul lines can be regarded as (imaginary) geodesics on any surface.

Geodesic polar coordinates.

220. The customary expression, in terms of geodesic coordinates, for an arc of a surface in homaloidal triple space, holds also for homaloidal quadruple space. Let the infinitude of geodesics through a point on the surface be given by the curves

$$q=\text{constant},$$

which accordingly must satisfy the geodesic equations

$$p''+\Gamma p'^2+2\Gamma' p'q'+\Gamma'' q'^2=0,\quad q''+\Delta p'^2+2\Delta' p'q'+\Delta'' q'^2=0.$$

It follows, from the second equation, that

$$\Delta=0.$$

Hence (§ 207)

$$\frac{\alpha}{E}=\frac{\beta}{F},$$

that is,

$$\frac{E_1}{2E}=\frac{F_1}{F}-\frac{E_2}{2F},$$

consequently

$$\frac{\partial}{\partial p}\left(\frac{F}{E^{\frac{1}{2}}}\right)=\frac{F_1}{E^{\frac{1}{2}}}-\frac{1}{2}\frac{FE_1}{E^{\frac{3}{2}}}=\frac{E_2}{2E^{\frac{1}{2}}}=\frac{\partial E^{\frac{1}{2}}}{\partial q}.$$

There therefore exists some function θ, of p and q, such that

$$E^{\frac{1}{2}}=\frac{\partial\theta}{\partial p},\quad FE^{-\frac{1}{2}}=\frac{\partial\theta}{\partial q};$$

and so

$$\begin{aligned}ds^2&=Edp^2+2Fdp\,dq+Gdq^2\\&=d\theta^2+\left\{G-\left(\frac{\partial\theta}{\partial q}\right)^2\right\}dq^2=d\theta^2+Pdq^2,\end{aligned}$$

in form the same as for an arc in the Gauss theory of surfaces.

The coordinates are the "geodesic polar" coordinates. Along the geodesics given by $q=$ constant, the geodesic length is θ; and a geodesic circle, $\theta=$ constant, cuts the geodesics at right angles.

The circular curvature of geodesics.

221. Now consider the circular curvature of the geodesic and, in particular, its principal normal along which the radius of circular curvature is measured. The direction-cosines of that principal normal are proportional to x'', y'', z'', v''. But, for any curve on the surface,

$$x'' = x_1 p'' + x_2 q'' + x_{11} p'^2 + 2x_{12} p' q' + x_{22} q'^2,$$

and therefore, along a geodesic on the surface,

$$x'' = (x_{11} - x_1 \Gamma - x_2 \Delta) p'^2 + 2(x_{12} - x_1 \Gamma' - x_2 \Delta') p' q' + (x_{22} - x_1 \Gamma'' - x_2 \Delta'') q'^2$$
$$= \frac{l}{\rho},$$

where l, m, n, k, are the direction-cosines of the perpendicular upon the tangent plane drawn from a point neighbouring to O on the normal section of the surface through the direction p', q', and ρ is the radius of curvature of that normal section. Similarly

$$y'' = \frac{m}{\rho}, \quad z'' = \frac{n}{\rho}, \quad v'' = \frac{k}{\rho}.$$

Consequently, the principal normal of the geodesic coincides in direction with the aforesaid perpendicular; and the curvature of the geodesic at the point through the direction p', q', is equal to the curvature of the normal section of the surface.

We shall therefore substitute the geodesic for the curve of normal section of the surface. That normal section is used only in connection with the curvature at the point; and the curve of section is no further considered. The curvature of the geodesic is the same as that of the normal section: its principal normal, in conjunction with the tangent, defines the plane of normal section: and it is an organic curve on the surface, uniquely defined by its initial direction at the point. Consequently when a curve is given through a point on the surface, we shall use the geodesic tangent to the curve at that point as a curve of reference: and when a direction at a point on the surface is assigned without any continuing curve, we shall likewise use the geodesic defined by that initial direction as a curve of reference, while the principal normal to the geodesic can be regarded as the normal to the surface associated with the assigned direction through the point.

Circular and geodesic curvatures of a curve: geometric derivation

222. The relations between a curve through a point and its geodesic tangent at the point, which have been obtained implicitly in § 216 (where they related to the normal section), can also be exhibited by a simple geometrical construction originally devised by Liouville* in connection with the Gauss theory of surfaces in three dimensions. Let AB be an elementary arc of any direction AB through a point A; let ABG be the geodesic determined by the direction AB, choosing an elementary arc BG equal to AB, and let GT be the perpendicular from G on the tangent plane, so that $ABGT$ is the plane of normal section through the direction AB. Let ABC be any curve through A having AB for its tangent, A and B being consecutive points, and in the plane CBT draw TC perpendicular to BT meeting the curve in C, thus GTC is a plane perpendicular to BT, and $BC = BT = BG$ to quantities of the second order inclusive (but not inclusive of quantities of the third order).

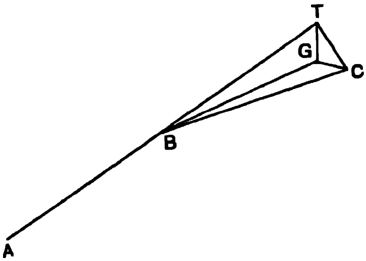

Fig. 21.

In connection with the curvatures, we take ρ to be the radius of circular curvature of the geodesic, that is, the radius of curvature of the earlier normal section. We take ρ_0 to be the radius of circular curvature of the curve ABC at the point A. We denote by $1/\gamma$ the arc-rate of deviation of the curve from the geodesic tangent, that is, the arc-rate of change of the angle GBC as the increment of that change at the point. Then, up to the second order of small quantities (but not inclusive of the third order), we have

$$2\rho_0 . CT = BT^2, \text{ for the curve},$$

$$2\rho \ . GT = BT^2, \text{ for the geodesic};$$

$$2\gamma \ . CG = BG^2 = BT^2, \text{ for the deviation.}$$

Consequently

$$\rho_0 . CT = \rho . GT = \gamma . CG.$$

Now as in § 216, let χ be the angle between the osculating plane CBT of the curve and the osculating plane GBT of the geodesic, so that

$$GTC = \chi,$$

and therefore

$$GT = CT \cos\chi, \quad CG = CT \sin\chi.$$

Thus

$$\rho_0 = \rho \cos\chi, \quad \rho_0 = \gamma \sin\chi,$$

or

$$\frac{1}{\rho} = \frac{\cos\chi}{\rho_0}, \quad \frac{1}{\gamma} = \frac{\sin\chi}{\rho_0}.$$

---

* See my *Lectures on Differential Geometry*, § 104. The construction was first given by Liouville in his edition of Monge's *Application de l'Analyse à la Géométrie*, p. 575.

The first of these results is Meusnier's theorem. Combining the two values of $\frac{\sin\chi}{\rho_0}$ which have been obtained, we find an expression for the quantity $\frac{1}{\gamma}$ which, as being the arc-rate of deviation of the curve from its geodesic tangent, may be called the *geodesic curvature of the curve* $\theta(p, q) = 0$; and the value is given by

$$\frac{1}{\gamma} = \frac{\sin\chi}{\rho_0}$$

$$= V\frac{(\theta_{11} - \theta_1\Gamma - \theta_2\Delta)\theta_2^2 - 2(\theta_{12} - \theta_1\Gamma' - \theta_2\Delta')\theta_2\theta_1 + (\theta_{22} - \theta_1\Gamma'' - \theta_2\Delta'')\theta_1^2}{(E\theta_2^2 - 2F\theta_2\theta_1 + G\theta_1^2)^{\frac{3}{2}}}.$$

Further,

$$\frac{1}{\rho} = \frac{\cos\chi}{\rho_0}$$

$$= \frac{\{\mathrm{a}\theta_2^4 - 4\mathrm{h}\theta_2^3\theta_1 + (2\mathrm{g} + 4\mathrm{b})\theta_2^2\theta_1^2 - 4\mathrm{f}\theta_2\theta_1^3 + \mathrm{c}\theta_1^4\}^{\frac{1}{2}}}{E\theta_2^2 - 2F\theta_2\theta_1 + G\theta_1^2}$$

from the result in § 213. We thus have two equations: they determine the value of χ, the inclination of the osculating plane of the curve $\theta(p, q) = 0$ to the osculating plane of the geodesic tangent (the normal section of the surface through the tangent to the curve), and they determine also the curvature of the curve $\theta(p, q) = 0$.

Geodesic curvature of a curve: analytical expression.

223. The preceding expression for the geodesic curvature of the curve $\theta(p, q) = 0$ can be obtained otherwise as follows, by formulating the analytical expression of the geometry.

When an initial direction for a geodesic is p', q', a consecutive direction along the geodesic is $p' + p''ds$, $q' + q''ds$, and so the angle $d\epsilon$ between these two directions is

$$d\epsilon = V(p'q'' - q'p'')\,ds.$$

On a curve having that same initial direction, let a consecutive direction along the curve be denoted by $p' + P''ds$, $q' + Q''ds$; the angle $d\eta$ between these two directions is

$$d\eta = V(p'Q'' - q'P'')\,ds.$$

Thus the angle of geodesic contingence of the curve

$$= d\epsilon - d\eta$$

$$= -V\{p'(Q'' - q'') - q'(P'' - p'')\}\,ds,$$

or the geodesic curvature $1/\gamma$ is given by

$$-\frac{1}{V\gamma} = p'(Q'' - q'') - q'(P'' - p'').$$

Now

$$\theta_1 p' + \theta_2 q' = 0,$$

so that

$$\frac{p'}{\theta_2} = \frac{q'}{-\theta_1} = \frac{1}{\Theta^{\frac{1}{2}}},$$

where

$$\Theta = E\theta_2{}^2 - 2F\theta_2\theta_1 + G\theta_1{}^2.$$

Further,

$$\theta_1 P'' + \theta_2 Q'' + \theta_{11} p'^2 + 2\theta_{12} p'q' + \theta_{22} q'^2 = 0;$$

or, if

$$\left.\begin{aligned} \theta_{11} - \theta_1 \Gamma \; - \theta_2 \Delta \; &= \gamma_{11} \\ \theta_{12} - \theta_1 \Gamma' \; - \theta_2 \Delta' \; &= \gamma_{12} \\ \theta_{22} - \theta_1 \Gamma'' - \theta_2 \Delta'' &= \gamma_{22} \end{aligned}\right\},$$

this equation is

$$\theta_1 (P'' - p'') + \theta_2 (Q'' - q'') = -(\gamma_{11}, \gamma_{12}, \gamma_{22} \,)\!(\, p', q')^2.$$

Again, differentiating the equation $Ep'^2 + 2Fp'q' + Gq'^2 = 1$, which holds for all directions on the surface, we have

$$(Ep' + Fq')\{P'' + (\Gamma, \Gamma', \Gamma'' \,)\!(\, p', q')^2\} + (Fp' + Gq')\{Q'' + (\Delta, \Delta', \Delta'' \,)\!(\, p', q')^2\} = 0,$$

that is,

$$(Ep' + Fq')(P'' - p'') + (Fp' + Gq')(Q'' - q'') = 0,$$

and therefore

$$(E\theta_2 - F\theta_1)(P'' - p'') + (F\theta_2 - G\theta_1)(Q'' - q'') = 0.$$

Hence

$$\frac{P'' - p''}{F\theta_2 - G\theta_1} = \frac{Q'' - q''}{-E\theta_2 + F\theta_1} = \frac{(\gamma_{11}, \gamma_{12}, \gamma_{22} \,)\!(\, p', q')^2}{\Theta}.$$

Consequently

$$\begin{aligned} -\frac{1}{V\gamma} &= p'(Q'' - q'') - q'(P'' - p'') \\ &= \frac{1}{\Theta^{\frac{1}{2}}} \{\theta_2 (Q'' - q'') + \theta_1 (P'' - p'')\} \\ &= -\frac{(\gamma_{11}, \gamma_{12}, \gamma_{22} \,)\!(\, p', q')^2}{\Theta^{\frac{1}{2}}} \\ &= -\frac{(\gamma_{11}, \gamma_{12}, \gamma_{22} \,)\!(\, \theta_2, -\theta_1)^2}{\Theta^{\frac{3}{2}}}, \end{aligned}$$

that is,

$$\frac{1}{\gamma} = V \frac{(\theta_{11} - \theta_1 \Gamma - \theta_2 \Delta,\; \theta_{12} - \theta_1 \Gamma' - \theta_2 \Delta',\; \theta_{22} - \theta_1 \Gamma'' - \theta_2 \Delta'' \,)\!(\, \theta_2, -\theta_1)^2}{(E\theta_2{}^2 - 2F\theta_2\theta_1 + G\theta_1{}^2)^{\frac{3}{2}}}.$$

By a former relation (§ 216), the expression on the right-hand side is equal to

$$\frac{\sin\chi}{\rho_0};$$

consequently, we have

$$\frac{1}{\gamma} = \frac{\sin\chi}{\rho_0},$$

in accordance with the result in § 222.

Orthogonal plane of a surface.

224. The equations of the plane orthogonal to the surface at a point x, y, z, v, already obtained in § 209, can be obtained, differently, as follows.

Let the surface be given by two equations $\Theta(x, y, z, v) = 0$ and $\Phi(x, y, z, v) = 0$, any surface direction dx, dy, dz, dv, at the point satisfies the two relations

$$\Theta_x dx + \Theta_y dy + \Theta_z dz + \Theta_v dv = 0, \quad \Phi_x dx + \Phi_y dy + \Phi_z dz + \Phi_v dv = 0.$$

When a direction $\alpha, \beta, \gamma, \delta$, is perpendicular to dx, dy, dz, dv, then

$$\alpha dx + \beta dy + \gamma dz + \delta dv = 0.$$

Should a particular direction $\alpha, \beta, \gamma, \delta$, be assigned, then the three equations determine the ratios $dx : dy : dz . dv$ uniquely: that is, in the tangent plane to the surface, there lies a single direction which is perpendicular to the assigned direction $\alpha, \beta, \gamma, \delta$; and it is given by the equations

$$\Sigma\Theta_x(X - x) = 0, \quad \Sigma\Phi_x(X - x) = 0, \quad \Sigma\alpha(X - x) = 0.$$

If however the third equation is to be satisfied, without other conditions, for all directions dx, dy, dz, dv, which satisfy the first two relations, that third equation must be an analytical consequence of the two relations; and therefore

$$\left\| \begin{matrix} \alpha, & \beta, & \gamma, & \delta \\ \Theta_x, & \Theta_y, & \Theta_z, & \Theta_v \\ \Phi_x, & \Phi_y, & \Phi_z, & \Phi_v \end{matrix} \right\| = 0.$$

But the line through the direction $\alpha, \beta, \gamma, \delta$, and the point x, y, z, v, is

$$\frac{\bar{x} - x}{\alpha} = \frac{\bar{y} - y}{\beta} = \frac{\bar{z} - z}{\gamma} = \frac{\bar{v} - v}{\delta};$$

and therefore any point on a line, which is perpendicular to every direction in the tangent plane to the surface, satisfies the equations

$$\left\| \begin{matrix} \bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\ \Theta_x, & \Theta_y, & \Theta_z, & \Theta_v \\ \Phi_x, & \Phi_y, & \Phi_z, & \Phi_v \end{matrix} \right\| = 0.$$

These equations represent a plane. Accordingly, they are the equations of the normal plane: as every direction in it is perpendicular to every direction in the tangent plane, it is the orthogonal plane at the point.

The transformation of these equations, to a form more obviously connected with the parametric expression of the surface, is immediate. They can be taken in the form

$$\bar{x} - x = \lambda\Theta_x + \mu\Phi_x, \quad \bar{y} - y = \lambda\Theta_y + \mu\Phi_y, \quad \bar{z} - z = \lambda\Theta_z + \mu\Phi_z, \quad \bar{v} - v = \lambda\Theta_v + \mu\Phi_v.$$

Now the equations $\Theta = 0$ and $\Phi = 0$ are satisfied identically when the parametric values of x, y, z, v, are inserted in them; hence

$$\Sigma x_1\Theta_x = 0, \quad \Sigma x_2\Theta_x = 0, \quad \Sigma x_1\Phi_x = 0, \quad \Sigma x_2\Phi_x = 0.$$

Accordingly

$$\Sigma(\bar{x}-x)x_1=\lambda\Sigma x_1\Theta_x+\mu\Sigma x_1\Phi_x=0,$$
$$\Sigma(\bar{x}-x)x_2=\lambda\Sigma x_2\Theta_x+\mu\Sigma x_2\Phi_x=0$$

which are the parametric equations of the normal plane. Their form shews that this normal plane is orthogonal to the tangent plane.

Osculating plane of a geodesic.

225. Further, among planes perpendicular but not orthogonal to the tangent plane at the point, there occurs the osculating plane of the geodesic through each particular direction. This osculating plane contains the tangent line to the curve having direction-cosines x', y', z', v'; and it contains the principal normal to the geodesic having the direction-cosines l, m, n, k. Consequently the equations of the plane are

$$\left\|\begin{array}{cccc}\bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v\\ x', & y', & z', & v'\\ l, & m, & n, & k\end{array}\right\|=0.$$

This plane contains the tangent to the geodesic, a line which lies also in the tangent plane; the two planes therefore intersect in a line and thus cannot be orthogonal.

But the two planes are perpendicular to one another. The necessary analytical condition (§ 83) is that the expression

$$\begin{aligned}&(x'm-y'l)(x_1y_2-y_1x_2)+(x'k-v'l)(x_1v_2-v_1x_2)\\+&(y'n-z'm)(y_1z_2-z_1y_2)+(y'k-v'm)(y_1v_2-v_1y_2)\\+&(z'l-x'n)(z_1x_2-x_1z_2)+(z'k-v'n)(z_1v_2-v_1z_2)\end{aligned}$$

should vanish. In this expression, the coefficient of l is

$$\begin{aligned}&=-y'(x_1y_2-y_1x_2)+z'(z_1x_2-x_1z_2)-v'(x_1v_2-v_1x_2)\\&=-x_1\Sigma x_2x'+x_2\Sigma x_1x'=-x_1(Fp'+Gq')+x_2(Ep'+Fq'),\end{aligned}$$

and similarly for the coefficients of m, n, k; thus the whole expression

$$=-(Fp'+Gq')\Sigma lx_1+(Ep'+Fq')\Sigma lx_2,$$

which vanishes because $\Sigma lx_1=0$, $\Sigma lx_2=0$. Thus the condition is satisfied: the two planes are perpendicular.

We thus have the family of normal (perpendicular) planes at the point, given by the osculating planes of the geodesics: and they give the normal sections of the surface for directions through the point.

This normal plane to the surface, passing through the tangent to any direction and the principal normal of the geodesic in that direction, and represented by the equations

$$\left\| \begin{matrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ x', & y', & z', & v' \\ l, & m, & n, & k \end{matrix} \right\| = 0,$$

is manifestly not orthogonal to the plane

$$\Sigma(\bar{x}-x)\,x_1 = 0, \quad \Sigma(\bar{x}-x)\,x_2 = 0,$$

which is the orthogonal plane of the surface at the point. But the two planes are perpendicular.

When the former plane is taken in its canonical form (§ 29)

$$\left.\begin{aligned} \bar{z}-z &= p(\bar{x}-x) + q(\bar{y}-y) \\ \bar{v}-v &= r(\bar{x}-x) + s(\bar{y}-y) \end{aligned}\right\},$$

the values of the canonical coordinates are

$$\frac{p}{z'm-y'n} = \frac{q}{x'n-z'l} = \frac{r}{v'm-y'k} = \frac{s}{x'k-v'l} = \frac{ps-qr}{z'k-v'n}$$
$$= \frac{1}{x'm-y'l};$$

and when the latter plane is taken in its canonical form

$$\left.\begin{aligned} \bar{z}-z &= p'(\bar{x}-x) + q'(\bar{y}-y) \\ \bar{v}-v &= r'(\bar{x}-x) + s'(\bar{y}-y) \end{aligned}\right\},$$

the values of the canonical coordinates are

$$\frac{p'}{x_1v_2-v_1x_2} = \frac{q'}{y_1v_2-v_1y_2} = \frac{r'}{z_1x_2-x_1z_2} = \frac{s'}{z_1y_2-y_1z_2} = \frac{p's'-q'r'}{y_1x_2-x_1y_2}$$
$$= \frac{1}{v_1z_2-z_1v_2}.$$

Now with the notation of § 84, we have

$$S^2(x'm-y'l)^2 = \Sigma x'^2 \Sigma l^2 - (\Sigma l x')^2 = 1,$$
$$S'^2(v_1z_2-z_1v_2)^2 = \Sigma x_1^2 \Sigma x_2^2 - (\Sigma x_1x_2)^2 = EG - F^2 = V^2,$$

so that neither S nor S' vanishes; and

$$T(x'm-y'l)(v_1z_2-z_1v_2) = \Sigma(z'm-y'n)(z_1v_2-v_1z_2).$$

In the right-hand side, the coefficient of k

$$= -y'(z_1x_2-x_1z_2) + x'(z_1y_2-y_1z_2) + z'(y_1x_2-x_1y_2)$$
$$= -\left| \begin{matrix} x', & y', & z' \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{matrix} \right| = 0,$$

because $x' = x_1 p' + x_2 q'$, and so for the others. Similarly the coefficients of l, m, n, in T vanish. Consequently

$$T = 0.$$

As the angle between the two planes is given by

$$SS' \cos \phi = T,$$

we have $\phi = \frac{1}{2}\pi$; and therefore the two planes are perpendicular.

Osculating flat of the surface along a geodesic.

226. We have seen that the flat

$$\begin{vmatrix} \bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\ x_1 \;, & y_1 \;, & z_1 \;, & v_1 \\ x_2 \;, & y_2 \;, & z_2 \;, & v_2 \\ l \;, & m \;, & n \;, & k \end{vmatrix} = 0$$

contains the tangent plane to the surface, that is, it passes through two consecutive points at O in every direction at O. Further, l, m, n, k, are the direction-cosines of the principal normal to the geodesic, thus, as the flat manifestly contains this direction, the flat contains the normal plane to the surface through the geodesic direction, that is, it contains the osculating plane of the geodesic, and therefore passes through three consecutive points on the geodesic. Thus it may be regarded as the osculating flat of the surface at O containing the directions of all geodesics at O and the osculating plane of one particular geodesic; but it is not the osculating flat of the particular geodesic which would pass through four consecutive points on the curve.

Let A, B, C, D, be the direction-cosines of the normal to the flat: then

$$\Sigma A x_1 = 0, \quad \Sigma A x_2 = 0, \quad \Sigma A l = 0.$$

But

$$\frac{l}{\rho} = (x_{11} - x_1 \Gamma - x_2 \Delta) p'^2 + 2(x_{12} - x_1 \Gamma' - x_2 \Delta') p'q' + (x_{22} - x_1 \Gamma'' - x_2 \Delta'') q'^2,$$

with corresponding expressions for m, n, k; hence the third equation can, by use of the first two equations, be expressed in the form

$$p'^2 \Sigma A x_{11} + 2p'q' \Sigma A x_{12} + q'^2 \Sigma A x_{22} = 0.$$

We introduce new quantities Ω, Ω', Ω'', according to the definitions

$$\left.\begin{aligned} \Sigma A x_{11} &= \Omega \\ \Sigma A x_{12} &= \Omega' \\ \Sigma A x_{22} &= \Omega'' \end{aligned}\right\};$$

and then the form of the third equation is

$$\Omega p'^2 + 2\Omega' p'q' + \Omega'' q'^2 = 0.$$

A flat, orthogonal to the principal normal of the geodesic.

227. Moreover, in connection with these direction-cosines A, B, C, D, it is convenient to recall the two relations

$$\Sigma l x_1 = 0, \quad \Sigma l x_2 = 0\,;$$

and, with them, to associate the relation

$$\Sigma l A = 0.$$

Hence l, m, n, k, are the direction-cosines of the normal to the flat

$$\begin{vmatrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ x_1\;, & y_1\;, & z_1\;, & v_1 \\ x_2\;, & y_2\;, & z_2\;, & v_2 \\ A\;, & B\;, & C\;, & D \end{vmatrix} = 0.$$

This flat contains the tangent plane at the point; it also contains the normal to the osculating flat associated with the assigned direction determined by p', q'.

Manifestly, owing to the relation $\Sigma l A = 0$, the two flats are perpendicular to one another. As before for A, B, C, D, we have

$$Vl = -\begin{vmatrix} y_1, & z_1, & v_1 \\ y_2, & z_2, & v_2 \\ B, & C, & D \end{vmatrix}, \quad Vm = \begin{vmatrix} z_1, & v_1, & x_1 \\ z_2, & v_2, & x_2 \\ C, & D, & A \end{vmatrix},$$

$$Vn = -\begin{vmatrix} v_1, & x_1, & y_1 \\ v_2, & x_2, & y_2 \\ D, & A, & B \end{vmatrix}, \quad Vk = \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ A, & B, & C \end{vmatrix},$$

the signs for Vl, Vm, Vn, Vk, being determined so as to accord with the sign about to be selected in the determination of A, B, C, D.

We leave, as an exercise, the verification of the fact, that the substitution of the former values of A, B, C, D, in these expressions actually leads to the values l, m, n, k.

Further, to obtain the actual values of A, B, C, D, the direction-cosines of the osculating flat, we proceed from the equations

$$\Sigma A x_1 = 0, \quad \Sigma A x_2 = 0, \quad \Sigma A l = 0:$$

thus

$$\frac{A}{\begin{vmatrix} y_1, & z_1, & v_1 \\ y_2, & z_2, & v_2 \\ m, & n, & k \end{vmatrix}} = \frac{B}{-\begin{vmatrix} z_1, & v_1, & x_1 \\ z_2, & v_2, & x_2 \\ n, & k, & l \end{vmatrix}} = \frac{C}{\begin{vmatrix} v_1, & x_1, & y_1 \\ v_2, & x_2, & y_2 \\ k, & l, & m \end{vmatrix}} = \frac{D}{-\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ l, & m, & n \end{vmatrix}}.$$

Now $\Sigma A^2 = 1$; if, therefore, $1/\Upsilon$ be the common value of these fractions,

$$\Upsilon^2 = \Sigma \begin{vmatrix} y_1, & z_1, & v_1 \\ y_2, & z_2, & v_2 \\ m, & n, & k \end{vmatrix}^2$$

$$= \begin{vmatrix} \Sigma x_1^2, & \Sigma x_1 x_2, & \Sigma x_1 l \\ \Sigma x_1 x_2, & \Sigma x_2^2, & \Sigma x_2 l \\ \Sigma x_1 l, & \Sigma x_2 l, & \Sigma l^2 \end{vmatrix} = \begin{vmatrix} E, & F, & 0 \\ F, & G, & 0 \\ 0, & 0, & 1 \end{vmatrix} = V^2.$$

Consequently, choosing $+V$ as the value of Υ, we have

$$VA = \begin{vmatrix} y_1, & z_1, & v_1 \\ y_2, & z_2, & v_2 \\ m, & n, & k \end{vmatrix}, \quad VB = -\begin{vmatrix} z_1, & v_1, & x_1 \\ z_2, & v_2, & x_2 \\ n, & k, & l \end{vmatrix};$$

$$VC = \begin{vmatrix} v_1, & x_1, & y_1 \\ v_2, & x_2, & y_2 \\ k, & l, & m \end{vmatrix}, \quad VD = -\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ l, & m, & n \end{vmatrix}.$$

These, of course, agree with the expressions (§ 47) for the direction-cosines of the normal to a flat, which contains a direction l, m, n, k, and a plane with

$$\left\| \begin{matrix} x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \end{matrix} \right\|$$

as orientation-coordinates.

The quantities Ω, Ω', Ω''.

228. Next, the quantities Ω, Ω', Ω'', do not vanish in general. When they do vanish simultaneously, the implication is that the whole configuration of the surface lies in a flat, that is, in a triple space, so that then all the properties become those of surfaces in the Gauss theory for homaloidal triple space. This result is established as follows.

When the quantities Ω, Ω', Ω'', vanish generally, we have

$$\Sigma A x_{11} = 0, \quad \Sigma A x_{12} = 0, \quad \Sigma A x_{22} = 0.$$

The two relations

$$\Sigma A x_1 = 0, \quad \Sigma A x_2 = 0,$$

always hold. From the first of these two relations, we have, by complete derivation with regard to p and to q in turn,

$$\Sigma \frac{dA}{dp} x_1 + \Sigma A x_{11} = 0, \quad \Sigma \frac{dA}{dq} x_1 + \Sigma A x_{12} = 0,$$

that is, on the present hypothesis,

$$\Sigma \frac{dA}{dp} x_1 = 0, \quad \Sigma \frac{dA}{dq} x_1 = 0.$$

Similarly, from the second of these two relations, we have

$$\Sigma \frac{dA}{dp} x_2 + \Sigma A x_{12} = 0, \quad \Sigma \frac{dA}{dq} x_2 + \Sigma A x_{22} = 0,$$

that is, on the present hypothesis,

$$\Sigma \frac{dA}{dp} x_2 = 0, \quad \Sigma \frac{dA}{dq} x_2 = 0.$$

Moreover, we always have

$$\Sigma \frac{dA}{dp} A = 0, \quad \Sigma \frac{dA}{dq} A = 0.$$

Now from the equations

$$\Sigma \frac{dA}{dp} x_1 = 0, \quad \Sigma \frac{dA}{dp} x_2 = 0, \quad \Sigma \frac{dA}{dp} A = 0,$$

we have

$$\frac{1}{l}\frac{dA}{dp} = \frac{1}{m}\frac{dB}{dp} = \frac{1}{n}\frac{dC}{dp} = \frac{1}{k}\frac{dD}{dp}, = \theta, \text{ say;}$$

and from the equations

$$\Sigma \frac{dA}{dq} x_1 = 0, \quad \Sigma \frac{dA}{dq} x_2 = 0, \quad \Sigma \frac{dA}{dq} A = 0,$$

we have

$$\frac{1}{l}\frac{dA}{dq} = \frac{1}{m}\frac{dB}{dq} = \frac{1}{n}\frac{dC}{dq} = \frac{1}{k}\frac{dD}{dq}, = \phi, \text{ say.}$$

Again, from the equations $\Sigma A x_{11} = 0$ and $\Sigma A x_{12} = 0$, we have

$$\Sigma A x_{112} + \Sigma \frac{dA}{dq} x_{11} = 0, \quad \Sigma A x_{112} + \Sigma \frac{dA}{dp} x_{12} = 0,$$

so that

$$\Sigma \frac{dA}{dq} x_{11} = \Sigma \frac{dA}{dp} x_{12},$$

hence

$$\phi \Sigma l x_{11} = \theta \Sigma l x_{12},$$

that is,

$$\phi L = \theta M,$$

where L and M are the quantities defined in § 215. Similarly, from the equations $\Sigma A x_{12} = 0$ and $\Sigma A x_{22} = 0$, we infer the relation

$$\phi M = \theta N.$$

Hence, either

$$LN - M^2 = 0,$$

and a later investigation will shew that this relation requires the surface to contain straight lines, a result to be excluded as implying the special class of ruled surfaces: or

$$\theta = 0, \quad \phi = 0.$$

Thus $\frac{dA}{dp} = \frac{dB}{dp} = \frac{dC}{dp} = \frac{dD}{dp} = 0$, $\frac{dA}{dq} = \frac{dB}{dq} = \frac{dC}{dq} = \frac{dD}{dq} = 0$; and therefore A, B, C, D, are constants. But

$$\Sigma A x_1 = 0, \quad \Sigma A x_2 = 0,$$

and therefore

$$Ax + By + Cz + Dv = \text{constant},$$

a relation satisfied by the coordinates of every point on the surface. The surface therefore lies in a flat: that is, it is a surface in some homaloidal triple space, and all its properties are deducible by the usual Gauss theory. In such an event, comparison with the Gauss theory is at once obtained by assuming a rectangular transformation of the axes of x, y, z, v, such that the containing flat can be represented by $v = 0$.

Equations of the second order satisfied by point-coordinates.

229. We now are in a position to obtain equations of the second order, which are an extension of the equations of the second order satisfied by the coordinates of a point on a surface in the Gauss theory.

We introduce four quantities θ, ϕ, ψ, χ, provisionally defined by the relations

$$\begin{aligned} x_{11} - x_1\Gamma - x_2\Delta - Ll &= \theta, \\ y_{11} - y_1\Gamma - y_2\Delta - Lm &= \phi, \\ z_{11} - z_1\Gamma - z_2\Delta - Ln &= \psi, \\ v_{11} - v_1\Gamma - v_2\Delta - Lk &= \chi. \end{aligned}$$

Then

$$\begin{aligned} \Sigma x_1\theta &= \Sigma x_1 x_{11} - \Gamma\Sigma x_1^2 - \Delta\Sigma x_1 x_2 - L\Sigma l x_1 \\ &= \alpha - E\Gamma - F\Delta = 0, \\ \Sigma x_2\theta &= \beta - F\Gamma - G\Delta = 0, \\ \Sigma l\ \theta &= \Sigma l x_{11} - \Gamma\Sigma l x_1 - \Delta\Sigma l x_2 - L = L - L = 0, \\ \Sigma A\ \theta &= \Sigma A x_{11} - \Gamma\Sigma A x_1 - \Delta\Sigma A x_2 - L\Sigma A l = \Omega, \end{aligned}$$

consequently

$$\theta = A\Omega, \quad \phi = B\Omega, \quad \psi = C\Omega, \quad \chi = D\Omega.$$

We therefore have the relations

$$\left.\begin{aligned} \xi_{11} &= x_{11} - x_1\Gamma - x_2\Delta = Ll + \Omega A \\ \eta_{11} &= y_{11} - y_1\Gamma - y_2\Delta = Lm + \Omega B \\ \zeta_{11} &= z_{11} - z_1\Gamma - z_2\Delta = Ln + \Omega C \\ \upsilon_{11} &= v_{11} - v_1\Gamma - v_2\Delta = Lk + \Omega D \end{aligned}\right\}.$$

Proceeding in the same way, we have the further set of relations

$$\left.\begin{aligned} \xi_{12} &= x_{12} - x_1\Gamma' - x_2\Delta' = Ml + \Omega' A \\ \eta_{12} &= y_{12} - y_1\Gamma' - y_2\Delta' = Mm + \Omega' B \\ \zeta_{12} &= z_{12} - z_1\Gamma' - z_2\Delta' = Mn + \Omega' C \\ \upsilon_{12} &= v_{12} - v_1\Gamma' - v_2\Delta' = Mk + \Omega' D \end{aligned}\right\},$$

and the set of relations

$$\left.\begin{aligned}\xi_{22} &= x_{22} - x_1\Gamma'' - x_2\Delta'' = Nl \;\; + \Omega'' A\\ \eta_{22} &= y_{22} - y_1\Gamma'' - y_2\Delta'' = Nm + \Omega'' B\\ \zeta_{22} &= z_{22} - z_1\Gamma'' - z_2\Delta'' = Nn \;\; + \Omega'' C\\ v_{22} &= v_{22} - v_1\Gamma'' - v_2\Delta'' = Nk \;\; + \Omega'' D\end{aligned}\right\}.$$

In the first place, these expressions for $\xi_{11}, \eta_{11}, \ldots, v_{22}$, manifestly satisfy the relations

$$\left\| \begin{matrix} \xi_{11}, & \eta_{11}, & \zeta_{11}, & v_{11} \\ \xi_{12}, & \eta_{12}, & \zeta_{12}, & v_{12} \\ \xi_{22}, & \eta_{22}, & \zeta_{22}, & v_{22} \end{matrix} \right\| = 0,$$

which have already (§ 214) been established.

Again, with these values, we have

$$\xi_{11}p'^2 + 2\xi_{12}p'q' + \xi_{22}q'^2$$
$$= l\,(Lp'^2 + 2Mp'q' + Nq'^2) + A\,(\Omega p'^2 + 2\Omega' p'q' + \Omega'' q'^2),$$

that is,

$$\frac{l}{\rho} = \frac{l}{\rho} + A\,(\Omega p'^2 + 2\Omega' p'q' + \Omega'' q'^2);$$

and therefore we have the former equation

$$\Omega p'^2 + 2\Omega' p'q' + \Omega'' q'^2 = 0.$$

This equation equally follows from the expressions for $\eta_{11}, \eta_{12}, \eta_{22}$; for $\zeta_{11}, \zeta_{12}, \zeta_{22}$, and for v_{11}, v_{12}, v_{22}. It is a fundamental relation connecting the quantities $\Omega, \Omega', \Omega''$, which belong to the assigned direction p', q', through the point.

Now multiply the equations of the first of the foregoing sets of relations by $x_{11}, y_{11}, z_{11}, v_{11}$, respectively, and add; then

$$\Sigma x_{11}^2 - \Gamma\Sigma x_1 x_{11} - \Delta\Sigma x_2 x_{11} = L\Sigma l x_{11} + \Omega\Sigma A x_{11},$$

that is,

$$\Sigma x_{11}^2 - (E\Gamma^2 + 2F\Gamma\Delta + G\Delta^2) = L^2 + \Omega^2.$$

The left-hand side is the quantity denoted (§ 208) by $\mathbf{a}$, a quantity independent of p' and q' which occur in both L and Ω: thus we have

$$L^2 + \Omega^2 = \mathbf{a}.$$

Similarly, multiplying the equations in the first set of relations by $x_{12}, y_{12}, z_{12}, v_{12}$, and adding: or multiplying the equations in the second set of relations by $x_{11}, y_{11}, z_{11}, v_{11}$, and adding: we find

$$LM + \Omega\Omega' = \mathbf{h}.$$

Multiplication of the equations in the first set of relations by $x_{22}, y_{22}, z_{22}, v_{22}$, and addition, lead to the result

$$LN + \Omega\Omega'' = \mathbf{g}.$$

And so for others: the full tale of results is

$$\begin{aligned} L^2 + \Omega^2 &= \mathbf{a}, & MN + \Omega'\Omega'' &= \mathbf{f}, \\ M^2 + \Omega'^2 &= \mathbf{b}, & NL + \Omega''\Omega &= \mathbf{g}, \\ N^2 + \Omega''^2 &= \mathbf{c}, & LM + \Omega\Omega' &= \mathbf{h}. \end{aligned}$$

The former fundamental relation (§ 214)

$$\begin{vmatrix} \mathbf{a}, & \mathbf{h}, & \mathbf{g} \\ \mathbf{h}, & \mathbf{b}, & \mathbf{f} \\ \mathbf{g}, & \mathbf{f}, & \mathbf{c} \end{vmatrix} = 0$$

can be verified at once, on the substitution of the foregoing values for **a**, **b**, **c**, **f**, **g**, **h**.

The quantities and the magnitudes of the second order.

230. The quantities L, M, N; l, m, n, k, A, B, C, D; and therefore $\Omega, \Omega', \Omega''$; all involve p' and q', while **a**, **b**, **c**, **f**, **g**, **h**, depend solely upon the position O and are independent of directions through O.

To each of the radicals $(\mathbf{bc} - \mathbf{f}^2)^{\frac{1}{2}}, (\mathbf{ca} - \mathbf{g}^2)^{\frac{1}{2}}, (\mathbf{ab} - \mathbf{h}^2)^{\frac{1}{2}}$, being real quantities, we have allotted (§ 214) a positive sign: in general, no one of these real quantities is zero. We write

$$\left.\begin{aligned} L &= \mathbf{a}^{\frac{1}{2}} \cos \alpha \\ \Omega &= \mathbf{a}^{\frac{1}{2}} \sin \alpha \end{aligned}\right\}, \quad \left.\begin{aligned} M &= \mathbf{b}^{\frac{1}{2}} \cos \beta \\ \Omega' &= \mathbf{b}^{\frac{1}{2}} \sin \beta \end{aligned}\right\}, \quad \left.\begin{aligned} N &= \mathbf{c}^{\frac{1}{2}} \cos \gamma \\ \Omega'' &= \mathbf{c}^{\frac{1}{2}} \sin \gamma \end{aligned}\right\},$$

again taking positive signs for $\mathbf{a}^{\frac{1}{2}}, \mathbf{b}^{\frac{1}{2}}, \mathbf{c}^{\frac{1}{2}}$. It appears, at once, that no two of the three magnitudes α, β, γ, are equal; we therefore assume $\alpha > \beta > \gamma$, and find

$$\left.\begin{aligned} \frac{\cos(\beta-\gamma)}{\mathbf{f}} &= \frac{\sin(\beta-\gamma)}{(\mathbf{bc}-\mathbf{f}^2)^{\frac{1}{2}}} = \frac{1}{(\mathbf{bc})^{\frac{1}{2}}} \\ \frac{\cos(\gamma-\alpha)}{\mathbf{g}} &= \frac{\sin(\gamma-\alpha)}{-(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}} = \frac{1}{(\mathbf{ca})^{\frac{1}{2}}} \\ \frac{\cos(\alpha-\beta)}{\mathbf{h}} &= \frac{\sin(\alpha-\beta)}{(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}} = \frac{1}{(\mathbf{ab})^{\frac{1}{2}}} \end{aligned}\right\}.$$

The identical relation

$$\cos^2(\beta-\gamma) + \cos^2(\gamma-\alpha) + \cos^2(\alpha-\beta) - 2\cos(\beta-\gamma)\cos(\gamma-\alpha)\cos(\alpha-\beta) = 1$$

becomes

$$\mathbf{abc} + 2\mathbf{fgh} - \mathbf{af}^2 - \mathbf{bg}^2 - \mathbf{ch}^2 = 0,$$

a relation already known to be satisfied.

From the preceding values, we have

$$\begin{aligned} M\Omega - L\Omega' &= (\mathbf{ab})^{\frac{1}{2}} \sin(\alpha-\beta) = (\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}, \\ N\Omega' - M\Omega'' &= (\mathbf{bc})^{\frac{1}{2}} \sin(\beta-\gamma) = (\mathbf{bc}-\mathbf{f}^2)^{\frac{1}{2}}, \\ L\Omega'' - N\Omega &= (\mathbf{ca})^{\frac{1}{2}} \sin(\gamma-\alpha) = -(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}; \end{aligned}$$

and therefore

$$(\mathbf{bc}-\mathbf{f}^2)^{\frac{1}{2}}\,L-(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}\,M+(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}\,N\ =0,$$

$$(\mathbf{bc}-\mathbf{f}^2)^{\frac{1}{2}}\,\Omega-(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}\,\Omega'+(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}\,\Omega''=0$$

We have already (§ 214) obtained the similar relations

$$(\mathbf{bc}-\mathbf{f}^2)^{\frac{1}{2}}\,\xi_{11}-(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}\,\xi_{12}+(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}\,\xi_{22}=0,$$

$$(\mathbf{bc}-\mathbf{f}^2)^{\frac{1}{2}}\,\eta_{11}-(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}\,\eta_{12}+(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}\,\eta_{22}=0,$$

$$(\mathbf{bc}-\mathbf{f}^2)^{\frac{1}{2}}\,\zeta_{11}-(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}\,\zeta_{12}+(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}\,\zeta_{22}=0,$$

$$(\mathbf{bc}-\mathbf{f}^2)^{\frac{1}{2}}\,\upsilon_{11}-(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}\,\upsilon_{12}+(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}\,\upsilon_{22}=0.$$

The foregoing linear relation between L, M, N, is consistent with these relations.

Similarly, we have

$$\cos(\beta-\gamma)-\cos(\gamma-\alpha)\cos(\alpha-\beta)=\sin(\beta-\alpha)\sin(\gamma-\alpha),$$

that is,

$$\mathbf{af}-\mathbf{gh}=\quad(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}};$$

and also

$$\mathbf{bg}-\mathbf{hf}=-(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}(\mathbf{bc}-\mathbf{f}^2)^{\frac{1}{2}},$$

$$\mathbf{ch}-\mathbf{fg}=\quad(\mathbf{bc}-\mathbf{f}^2)^{\frac{1}{2}}(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}},$$

in accordance with the results already (§ 214) given.

Similarly, when the values of Ω, Ω', Ω'', are substituted in the fundamental relation

$$\Omega p'^2+2\Omega' p'q'+\Omega'' q'^2=0,$$

so that it becomes

$$p'^2\mathbf{a}^{\frac{1}{2}}\sin\alpha+2p'q'\,\mathbf{b}^{\frac{1}{2}}\sin\beta+q'^2\mathbf{c}^{\frac{1}{2}}\sin\gamma=0,$$

and when we use the equations

$$\sin\beta=\sin\alpha\cos(\alpha-\beta)-\cos\alpha\sin(\alpha-\beta)$$

$$=(\mathbf{ab})^{-\frac{1}{2}}\{\mathbf{h}\sin\alpha-(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}\cos\alpha\},$$

$$\sin\gamma=(\mathbf{ca})^{-\frac{1}{2}}\{\mathbf{g}\sin\alpha-(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}\cos\alpha\},$$

we find

$$\frac{\cos\alpha}{\mathbf{a}p'^2+2\mathbf{h}p'q'+\mathbf{g}q'^2}=\frac{\sin\alpha}{2(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}p'q'+(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}q'^2}=\frac{1}{U},$$

say. Then

$$U^2=(\mathbf{a}p'^2+2\mathbf{h}p'q'+\mathbf{g}q'^2)^2+\{2(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}p'q'+(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}q'^2\}^2=\frac{\mathbf{a}}{\rho^2}.$$

Consequently

$$\left.\begin{aligned} \frac{L}{\rho} &= \frac{\mathbf{a}^{\frac12}\cos\alpha}{\rho} = \mathbf{a}p'^2 + 2\mathbf{h}p'q' + \mathbf{g}q'^2 \\ \frac{\Omega}{\rho} &= \frac{\mathbf{a}^{\frac12}\sin\alpha}{\rho} = 2(\mathbf{ab}-\mathbf{h}^2)^{\frac12}p'q' + (\mathbf{ca}-\mathbf{g}^2)^{\frac12}q'^2 \end{aligned}\right\}.$$

Similarly, we find

$$\left.\begin{aligned} \frac{M}{\rho} &= \frac{\mathbf{b}^{\frac12}\cos\beta}{\rho} = \mathbf{h}p'^2 + 2\mathbf{b}p'q' + \mathbf{f}q'^2 \\ \frac{\Omega'}{\rho} &= \frac{\mathbf{b}^{\frac12}\sin\beta}{\rho} = -(\mathbf{ab}-\mathbf{h}^2)^{\frac12}p'^2 + (\mathbf{bc}-\mathbf{f}^2)^{\frac12}q'^2 \end{aligned}\right\},$$

and

$$\left.\begin{aligned} \frac{N}{\rho} &= \frac{\mathbf{c}^{\frac12}\cos\gamma}{\rho} = \mathbf{g}p'^2 + 2\mathbf{f}p'q' + \mathbf{c}q'^2 \\ \frac{\Omega''}{\rho} &= \frac{\mathbf{c}^{\frac12}\sin\gamma}{\rho} = -(\mathbf{ca}-\mathbf{g}^2)^{\frac12}p'^2 - 2(\mathbf{bc}-\mathbf{f}^2)^{\frac12}p'q' \end{aligned}\right\}.$$

We note, as immediate corollaries from the formulæ, the two results

$$\frac{1}{\rho}(\Omega p' + \Omega' q') = q'\{(\mathbf{ab}-\mathbf{h}^2)^{\frac12}p'^2 + (\mathbf{ca}-\mathbf{g}^2)^{\frac12}p'q' + (\mathbf{bc}-\mathbf{f}^2)^{\frac12}q'^2\},$$

$$\frac{1}{\rho}(\Omega' p' + \Omega'' q') = -p'\{(\mathbf{ab}-\mathbf{h}^2)^{\frac12}p'^2 + (\mathbf{ca}-\mathbf{g}^2)^{\frac12}p'q' + (\mathbf{bc}-\mathbf{f}^2)^{\frac12}q'^2\}.$$

The quantity within the bracket on the right-hand side will recur later: we write

$$W = (\mathbf{ab}-\mathbf{h}^2)^{\frac12}p'^2 + (\mathbf{ca}-\mathbf{g}^2)^{\frac12}p'q' + (\mathbf{bc}-\mathbf{f}^2)^{\frac12}q'^2.$$

and so the two preceding results are

$$\frac{1}{\rho}(\Omega p' + \Omega' q') = q'W, \quad \frac{1}{\rho}(\Omega' p' + \Omega'' q') = -p'W.$$

Thus

$$\begin{vmatrix} \Omega\ , & \Omega' - \rho W \\ \Omega' + \rho W, & \Omega'' \end{vmatrix} = 0$$

and therefore

$$\rho^2 W^2 = \Omega'^2 - \Omega\Omega'',$$

a relation which can be verified immediately.

The quantity W obviously vanishes with Ω, Ω', Ω'', that is, when (§ 228) the surface lies wholly in a flat.

Direction-cosines of the osculating flat.

231. For an alternative set of expressions for A, B, C, D, let

$$\left\Vert \begin{matrix} x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ \xi_{11}, & \eta_{11}, & \zeta_{11}, & v_{11} \end{matrix} \right\Vert = e_{11}, f_{11}, g_{11}, h_{11};$$

$$\left\| \begin{matrix} x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ \xi_{12}, & \eta_{12}, & \zeta_{12}, & v_{12} \end{matrix} \right\| = e_{12}, f_{12}, g_{12}, h_{12},$$

$$\left\| \begin{matrix} x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ \xi_{22}, & \eta_{22}, & \zeta_{22}, & v_{22} \end{matrix} \right\| = e_{22}, f_{22}, g_{22}, h_{22}.$$

Then

$$\frac{VA}{\rho} = \left| \begin{matrix} y_1, & z_1, & v_1 \\ y_2, & z_2, & v_2 \\ \frac{m}{\rho}, & \frac{n}{\rho}, & \frac{k}{\rho} \end{matrix} \right| = e_{11}p'^2 + 2e_{12}p'q' + e_{22}q'^2,$$

and so for the others: thus

$$\left.\begin{aligned} A\frac{V}{\rho} &= (e_{11}, e_{12}, e_{22} \between p', q')^2 \\ B\frac{V}{\rho} &= (f_{11}, f_{12}, f_{22} \between p', q')^2 \\ C\frac{V}{\rho} &= (g_{11}, g_{12}, g_{22} \between p', q')^2 \\ D\frac{V}{\rho} &= (h_{11}, h_{12}, h_{22} \between p', q')^2 \end{aligned}\right\}.$$

NOTE Some relations can be established, connecting the quantities e_{lm}, f_{lm}, g_{lm}, h_{lm}, with the quantities ξ_{lm}, η_{lm}, ζ_{lm}, v_{lm}, respectively The central equations are

$$\left.\begin{aligned} \xi_{11} &= Ll + \Omega A \\ \xi_{12} &= Ml + \Omega' A \\ \xi_{22} &= Nl + \Omega'' A \end{aligned}\right\},$$

with corresponding relations for η, m, B; ζ, n, C; v, k, D From the first of these, we have

$$\begin{aligned} \frac{\xi_{11}}{\rho^2} &= \frac{L}{\rho}\frac{l}{\rho} + \frac{\Omega}{\rho}\frac{A}{\rho} \\ &= \frac{L}{\rho}\frac{l}{\rho} + \frac{1}{V}\frac{\Omega}{\rho}A\frac{V}{\rho}. \end{aligned}$$

Thus

$$\begin{aligned} &\xi_{11}\{\mathbf{a}p'^4 + 4\mathbf{h}p'^3q + (2\mathbf{g} + 4\mathbf{b})p'^2q'^2 + 4\mathbf{f}p'q'^3 + \mathbf{c}q'^4\} \\ &= (\mathbf{a}p'^2 + 2\mathbf{h}p'q' + \mathbf{g}q'^2)(\xi_{11}p'^2 + 2\xi_{12}p'q' + \xi_{22}q'^2) \\ &+ \frac{1}{V}[2(\mathbf{ab} - \mathbf{h}^2)^{\frac{1}{2}}p'q' + (\mathbf{ca} - \mathbf{g}^2)^{\frac{1}{2}}q'^2][e_{11}p'^2 + 2e_{12}p'q' + e_{22}q'^2]. \end{aligned}$$

The only relation connecting p' and q' is the non-homogeneous relation

$$Ep'^2 + 2Fp'q' + Gq'^2 = 1;$$

and the foregoing relation is homogeneous, so that the coefficients of the various combinations of p' and q' can be equated. Thus

$$\left.\begin{aligned}
\mathbf{a}\xi_{11} &= \mathbf{a}\xi_{11}\\
4\mathbf{h}\xi_{11} &= 2\mathbf{h}\xi_{11}+2\mathbf{a}\xi_{12} + 2\frac{(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}}{V}e_{11}\\
(2\mathbf{g}+4\mathbf{b})\xi_{11} &= \mathbf{g}\xi_{11}+4\mathbf{h}\xi_{12}+\mathbf{a}\xi_{22}+\frac{(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}}{V}e_{11}+4\frac{(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}}{V}e_{12}\\
4\mathbf{f}\xi_{11} &= 2\mathbf{g}\xi_{12}+2\mathbf{h}\xi_{22}+2\frac{(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}}{V}e_{12}+2\frac{(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}}{V}e_{22}\\
\mathbf{c}\xi_{11} &= \mathbf{g}\xi_{22}+\frac{(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}}{V}e_{22}
\end{aligned}\right\}.$$

Five similar equations result from the relation

$$\xi_{12}=Ml+\Omega' A,$$

and five from the relation

$$\xi_{22}=Nl+\Omega'' A.$$

Moreover, there are equations, precisely similar to these sets, in

$$\eta_{lm} \text{ and } f_{lm} \qquad \zeta_{lm} \text{ and } g_{lm} \qquad \nu_{lm} \text{ and } h_{lm}.$$

We also had the relation

$$(\mathbf{bc}-\mathbf{f}^2)^{\frac{1}{2}}\xi_{11}-(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}\xi_{12}+(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}\xi_{22}=0$$

with corresponding relations among η_{lm}, ζ_{lm}, ν_{lm}. It is easy to see that, because of the relation $\mathbf{abc}+2\mathbf{fgh}-\mathbf{af}^2-\mathbf{bg}^2-\mathbf{ch}^2=0$, all the equations are satisfied in virtue of the set

$$\left.\begin{aligned}
\mathbf{h}\xi_{11}-\mathbf{a}\xi_{12} &= \frac{(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}}{V}e_{11}\\
\mathbf{b}\xi_{11}-\mathbf{h}\xi_{12} &= \frac{(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}}{V}e_{12}\\
\mathbf{c}\xi_{11}-\mathbf{g}\xi_{22} &= \frac{(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}}{V}e_{22}\\
\mathbf{g}\xi_{11}-\mathbf{a}\xi_{22} &= \frac{(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}}{V}e_{11}\\
\mathbf{c}\xi_{12}-\mathbf{f}\xi_{22} &= \frac{(\mathbf{bc}-\mathbf{f}^2)^{\frac{1}{2}}}{V}e_{22}\\
\mathbf{f}\xi_{12}-\mathbf{b}\xi_{22} &= \frac{(\mathbf{bc}-\mathbf{f}^2)^{\frac{1}{2}}}{V}e_{12}
\end{aligned}\right\}.$$

Even these equations are not independent. thus, when the two values of e_{11} are equated, we obtain the equation in ξ_{11}, ξ_{12}, ξ_{22}, already given.

When an equation

$$\mathbf{h}\xi_{11}-\mathbf{a}\xi_{12}=\frac{(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}}{V}e_{11}$$

is combined with the corresponding equations in η and f, ζ and g, ν and h, and when the equations are squared and added, we merely have an identity.

Again, there are the relations

$$\xi_{12}=\frac{\mathbf{h}}{\mathbf{a}}\xi_{11}-\frac{(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}}}{\mathbf{a}V}e_{11},$$

$$\xi_{22}=\frac{\mathbf{g}}{\mathbf{a}}\xi_{11}-\frac{(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}}{\mathbf{a}V}e_{11};$$

and therefore

$$\frac{l}{\rho} = \xi_{11}\left(p'^2 + 2\frac{\mathbf{h}}{\mathbf{a}}p'q' + \frac{\mathbf{g}}{\mathbf{a}}q'^2\right) - \{(\mathbf{ab} - \mathbf{h}^2)^{\frac{1}{2}} 2p'q' + (\mathbf{ca} - \mathbf{g}^2)^{\frac{1}{2}} q'^2\} \frac{e_{11}}{\mathbf{a}V}$$

$$= \frac{\xi_{11}}{\mathbf{a}} \frac{L}{\rho} - \frac{\Omega}{\rho} \frac{e_{11}}{\mathbf{a}V}.$$

Thus

$$\mathbf{a}l = L\xi_{11} - \frac{\Omega}{V} e_{11},$$

with, of course, the corresponding relations

$$\mathbf{a}m = L\eta_{11} - \frac{\Omega}{V} f_{11},$$

$$\mathbf{a}n = L\zeta_{11} - \frac{\Omega}{V} g_{11},$$

$$\mathbf{a}k = L\upsilon_{11} - \frac{\Omega}{V} h_{11}$$

When these equations are squared and added, we find

$$\mathbf{a}^2 = L^2\mathbf{a} + \Omega^2\mathbf{a},$$

in virtue of $\Sigma\xi_{11}{}^2 = \mathbf{a}$, $\Sigma\xi_{11}e_{11} = 0$, $\Sigma e_{11}{}^2 = V^2\mathbf{a}$, and this is the old relation (§ 229)

Three special magnitudes of the second order.

232. We shall require the magnitudes* R, S, T, defined by the expressions

$$R = \begin{vmatrix} x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ x_{11}, & y_{11}, & z_{11}, & v_{11} \\ x_{12}, & y_{12}, & z_{12}, & v_{12} \end{vmatrix}, \quad S = \begin{vmatrix} x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ x_{11}, & y_{11}, & z_{11}, & v_{11} \\ x_{22}, & y_{22}, & z_{22}, & v_{22} \end{vmatrix}, \quad T = \begin{vmatrix} x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ x_{12}, & y_{12}, & z_{12}, & v_{12} \\ x_{22}, & y_{22}, & z_{22}, & v_{22} \end{vmatrix}.$$

In each of these, let the values of the second derivatives, as obtained in the relations of § 229, be substituted; then

$$R = \begin{vmatrix} x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ Ll + \Omega A, & Lm + \Omega B, & Ln + \Omega C, & Lk + \Omega D \\ Ml + \Omega' A, & Mm + \Omega' B, & Mn + \Omega' C, & Mk + \Omega' D \end{vmatrix}$$

$$= (L\Omega' - M\Omega) \begin{vmatrix} x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ l, & m, & n, & k \\ A, & B, & C, & D \end{vmatrix}$$

$$= -V(L\Omega' - M\Omega),$$

by the result of § 227. Hence

$$R = V(\mathbf{ab})^{\frac{1}{2}} \sin(\alpha - \beta) = V(\mathbf{ab} - \mathbf{h}^2)^{\frac{1}{2}}.$$

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* They are taken as *fundamental* magnitudes of the second order by Kommerell, *Math. Ann.*, vol. LX (1905), pp. 548–596.

Similarly for S and T: the three values are

$$\left.\begin{aligned} R &= V(\mathbf{ab}-\mathbf{h}^2)^{\frac{1}{2}} \\ S &= V(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}} \\ T &= V(\mathbf{bc}-\mathbf{f}^2)^{\frac{1}{2}} \end{aligned}\right\}.$$

Also, as

$$R = V(M\Omega - L\Omega'), \quad -S = V(N\Omega' - M\Omega''), \quad T = V(L\Omega'' - N\Omega),$$

we have

$$\left.\begin{aligned} TL - SM + RN &= 0 \\ T\Omega - S\Omega' + R\Omega'' &= 0 \end{aligned}\right\}.$$

The quantities e_{lm}, f_{lm}, g_{lm}, h_{lm}, of § 231 are connected with R, S, T, as minors in the determinantal forms of the latter. We can take

$$R = \begin{vmatrix} x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ \xi_{11}, & \eta_{11}, & \zeta_{11}, & \upsilon_{11} \\ \xi_{12}, & \eta_{12}, & \zeta_{12}, & \upsilon_{12} \end{vmatrix}, \quad S = \begin{vmatrix} x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ \xi_{11}, & \eta_{11}, & \zeta_{11}, & \upsilon_{11} \\ \xi_{22}, & \eta_{22}, & \zeta_{22}, & \upsilon_{22} \end{vmatrix}, \quad T = \begin{vmatrix} x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ \xi_{12}, & \eta_{12}, & \zeta_{12}, & \upsilon_{12} \\ \xi_{22}, & \eta_{22}, & \zeta_{22}, & \upsilon_{22} \end{vmatrix},$$

with constituents ξ_{lm}, η_{lm}, ζ_{lm}, υ_{lm}, instead of x_{lm}, y_{lm}, z_{lm}, v_{lm}. Now

$-e_{11}$, $-f_{11}$, $-g_{11}$, $-h_{11}$, are the minors of ξ_{12}, η_{12}, ζ_{12}, υ_{12}, in R,
e_{12}, f_{12}, g_{12}, h_{12}, ... ξ_{11}, η_{11}, ζ_{11}, υ_{11}, in R;

and therefore, by a theorem in determinants,

$$-\begin{vmatrix} e_{11}, & e_{12} \\ f_{11}, & f_{12} \end{vmatrix} = R\,(z_1 v_2 - v_1 z_2).$$

Similarly

$$-\begin{vmatrix} e_{11}, & e_{22} \\ f_{11}, & f_{22} \end{vmatrix} = S\,(z_1 v_2 - v_1 z_2),$$

$$-\begin{vmatrix} e_{12}, & e_{22} \\ f_{12}, & f_{22} \end{vmatrix} = T\,(z_1 v_2 - v_1 z_2).$$

Consequently

$$\begin{vmatrix} E, & F, & G \\ e_{11}, & e_{12}, & e_{22} \\ f_{11}, & f_{12}, & f_{22} \end{vmatrix} = -(z_1 v_2 - v_1 z_2)(ET - FS + GR),$$

a result which, with other like results, will be useful in discussing (§ 247) the locus of the intersection of consecutive orthogonal planes for varying directions through a point.

Similarly, by using the results (§ 214)

$$\left\| \begin{matrix} \xi_{11}, & \eta_{11}, & \zeta_{11}, & \upsilon_{11} \\ \xi_{12}, & \eta_{12}, & \zeta_{12}, & \upsilon_{12} \\ \xi_{22}, & \eta_{22}, & \zeta_{22}, & \upsilon_{22} \end{matrix} \right\| = 0,$$

we establish the relations

$$\left.\begin{aligned} Re_{22}-Se_{12}+Te_{11}&=0\\ Rf_{22}-Sf_{12}+Tf_{11}&=0\\ Rg_{22}-Sg_{12}+Tg_{11}&=0\\ Rh_{22}-Sh_{12}+Th_{11}&=0 \end{aligned}\right\},$$

which will be required in the same discussion.

Further, we have the relations

$$\begin{aligned} \Sigma e_{11}\xi_{11}&=e_{11}\xi_{11}+f_{11}\eta_{11}+g_{11}\zeta_{11}+h_{11}v_{11}=0,\\ \Sigma e_{12}\xi_{11}&=e_{12}\xi_{11}+f_{12}\eta_{11}+g_{12}\zeta_{11}+h_{12}v_{11}=R,\\ \Sigma e_{22}\xi_{11}&=e_{22}\xi_{11}+f_{22}\eta_{11}+g_{22}\zeta_{11}+h_{22}v_{11}=S, \end{aligned}$$

and similarly for the others; the whole aggregate of relations of this type is

$$\left.\begin{aligned} \Sigma e_{11}\xi_{11}&=0\\ \Sigma e_{12}\xi_{11}&=R\\ \Sigma e_{22}\xi_{11}&=S \end{aligned}\right\},\quad \left.\begin{aligned} \Sigma e_{11}\xi_{12}&=-R\\ \Sigma e_{12}\xi_{12}&=0\\ \Sigma e_{22}\xi_{12}&=T \end{aligned}\right\},\quad \left.\begin{aligned} \Sigma e_{11}\xi_{22}&=-S\\ \Sigma e_{12}\xi_{22}&=-T\\ \Sigma e_{22}\xi_{22}&=0 \end{aligned}\right\},$$

results which will be used hereafter.

CHAPTER XIII.

SURFACES IN QUADRUPLE SPACE: GEODESICS.

Quadruple frames associated with a surface direction.

233. At any point on the surface and associated with a superficial direction in the surface, three directions have already been obtained. There is the tangential direction x', y', z', v', lying in the tangent plane at the point. There is the principal normal l, m, n, k, to the surface, associated with the tangential direction. There is the normal A, B, C, D, to the flat, which contains the tangent plane to the surface at O and also contains the principal normal to the surface associated with the tangential direction.

Now these three directions are perpendicular to one another, for

$$\Sigma lx' = p'\Sigma lx_1 + q'\Sigma lx_2 = 0,$$

$$\Sigma Ax' = p'\Sigma Ax_1 + q'\Sigma Ax_2 = 0;$$

and we have, by definition (§ 226),

$$\Sigma Al = 0$$

Hence we can constitute a complete orthogonal frame, associated with the given tangential direction or, in other words, associated with a geodesic through the point. For this purpose, we take a fourth direction, perpendicular to each of the three directions; let its direction-cosines be λ, μ, ν, κ.

The two directions l, m, n, k, and A, B, C, D, are perpendicular to the tangent plane; they therefore are guiding lines for the orthogonal plane. Thus the direction λ, μ, ν, κ, perpendicular to both these guiding lines, is perpendicular to the orthogonal plane: that is, it lies in the tangent plane. The required direction must also be perpendicular to the tangential direction x', y', z', v'; and therefore it lies along a line in the tangent plane perpendicular to this tangential direction. Hence[*] its direction-cosines are given by the equations

$$\left.\begin{aligned} V\lambda &= x_2(Ep'+Fq') - x_1(Fp'+Gq') \\ V\mu &= y_2(Ep'+Fq') - y_1(Fp'+Gq') \\ V\nu &= z_2(Ep'+Fq') - z_1(Fp'+Gq') \\ V\kappa &= v_2(Ep'+Fq') - v_1(Fp'+Gq') \end{aligned}\right\}.$$

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[*] See the example in § 210.

If the four perpendicular directions thus obtained constitute the conventional setting (§ 25) of an orthogonal frame of reference, the determinant Υ, where

$$\Upsilon = \begin{vmatrix} x', & y', & z', & v' \\ l, & m, & n, & k \\ \lambda, & \mu, & \nu, & \kappa \\ A, & B, & C, & D \end{vmatrix} = \begin{vmatrix} \lambda, & \mu, & \nu, & \kappa \\ x', & y', & z', & v' \\ l, & m, & n, & k \\ A, & B, & C, & D \end{vmatrix},$$

should be equal to unity. Now

$$\begin{vmatrix} \lambda, & x' \\ \mu, & y' \end{vmatrix} = \frac{1}{V} \begin{vmatrix} x_2(Ep'+Fq') - x_1(Fp'+Gq'), & x_2q'+x_1p' \\ y_2(Ep'+Fq') - y_1(Fp'+Gq'), & y_2q'+y_1p' \end{vmatrix}$$

$$= \frac{1}{V} \begin{vmatrix} x_2, & x_1 \\ y_2, & y_1 \end{vmatrix} \begin{vmatrix} (Ep'+Fq'), & -(Fp'+Gq') \\ q', & p' \end{vmatrix}$$

$$= -\frac{1}{V} \begin{vmatrix} x_1, & x_2 \\ y_1, & y_2 \end{vmatrix},$$

and so for the other second minors, arising from the first two rows in Υ; hence

$$-V\Upsilon = \begin{vmatrix} x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ l, & m, & n, & k \\ A, & B, & C, & D \end{vmatrix}.$$

When we use the values of A, B, C, D, obtained in § 227, we have

$$-V\Upsilon = -\Sigma A \begin{vmatrix} y_1, & z_1, & v_1 \\ y_2, & z_2, & v_2 \\ m, & n, & k \end{vmatrix}$$

$$= -V\Sigma A^2 = -V,$$

so that

$$\Upsilon = 1;$$

and the orthogonal frame, constituted by the four directions, conforms to the conventional setting.

Moreover, in this orthogonal frame, the guiding lines for the tangent plane are the tangent to the normal section of the surface and a line perpendicular to that tangent Hitherto, the directions of the parametric curves at O have usually been taken as guiding lines for the tangent plane. When these parametric directions are associated with the principal normal of the geodesic and the normal to the osculating flat, we have an alternative frame of reference for the configuration; but it is not completely orthogonal.

Accordingly, there are two frames of reference, which are equivalent to one another for the present purpose. In the orthogonal frame, there are four orthogonal directions, viz. (i), the initial direction x', y', z', v', of the geodesic, (ii), the direction λ, μ, ν, κ, in the tangent plane perpendicular to this initial direction, (iii), the direction l, m, n, k, of the radius of circular curvature of the geodesic, and (iv), the direction A, B, C, D, required to complete the orthogonal frame. In the alternative frame of reference, we substitute the tangents to the parametric curves in the surface for the first two of the preceding directions, and we retain the other two directions unchanged. In each frame, we have the same two orthogonal planes, one of these is the tangent plane to the surface: the other is the plane through the radius of circular curvature of the geodesic orthogonal to the tangent plane. Further, when different directions are taken in the tangent plane, it remains unaltered; consequently, the plane orthogonal to the tangent plane remains unaltered, and therefore the different radii of circular curvature which belong to the different geodesics through these directions, all lie in this orthogonal plane. In fact, the tangent plane at any point of the surface and its orthogonal plane are two fundamental planes of reference of the surface at any point.

The analytical verification of the complanarity of all the geodesic radii of circular curvature is simple. Let

$$\begin{aligned}
&\mathrm{a}^{\frac{1}{2}}l_1 = \xi_{11}, && \mathrm{a}^{\frac{1}{2}}m_1 = \eta_{11}, && \mathrm{a}^{\frac{1}{2}}n_1 = \zeta_{11}, && \mathrm{a}^{\frac{1}{2}}k_1 = \upsilon_{11},\\
&\mathrm{b}^{\frac{1}{2}}l_{12} = \xi_{12}, && \mathrm{b}^{\frac{1}{2}}m_{12} = \eta_{12}, && \mathrm{b}^{\frac{1}{2}}n_{12} = \zeta_{12}, && \mathrm{b}^{\frac{1}{2}}k_{12} = \upsilon_{12},\\
&\mathrm{c}^{\frac{1}{2}}l_2 = \xi_{22}, && \mathrm{c}^{\frac{1}{2}}m_2 = \eta_{22}, && \mathrm{c}^{\frac{1}{2}}n_2 = \zeta_{22}, && \mathrm{c}^{\frac{1}{2}}k_2 = \upsilon_{22},
\end{aligned}$$

so that l_1, m_1, n_1, k_1, are the direction-cosines of the radius of curvature of the geodesic touching the parametric curve $q =$ constant, and l_2, m_2, n_2, k_2, are the direction-cosines of the radius of curvature of the geodesic touching the other parametric curve $p =$ constant. Now (§ 214)

$$\begin{Vmatrix} \xi_{11}, & \eta_{11}, & \zeta_{11}, & \upsilon_{11} \\ \xi_{12}, & \eta_{12}, & \zeta_{12}, & \upsilon_{12} \\ \xi_{22}, & \eta_{22}, & \zeta_{22}, & \upsilon_{22} \end{Vmatrix} = 0,$$

and therefore

$$\begin{Vmatrix} l_1, & m_1, & n_1, & k_1 \\ l_{12}, & m_{12}, & n_{12}, & k_{12} \\ l_2, & m_2, & n_2, & k_2 \end{Vmatrix} = 0.$$

But

$$\frac{l}{\rho} = \xi_{11}p'^2 + 2\xi_{12}p'q' + \xi_{22}q'^2 = \mathrm{a}^{\frac{1}{2}}l_1p'^2 + 2\mathrm{b}^{\frac{1}{2}}l_{12}p'q' + \mathrm{c}^{\frac{1}{2}}l_2q'^2,$$

with corresponding values for m, n, k; hence

$$\begin{Vmatrix} l_1, & m_1, & n_1, & k_1 \\ l\ , & m\ , & n\ , & k \\ l_2, & m_2, & n_2, & k_2 \end{Vmatrix} = 0.$$

Consequently, every direction l, m, n, k, of a geodesic radius of circular curvature lies in the plane determined at the point by the radii of circular curvature of the two geodesics which touch the parametric curve.

Again, from the equation

$$\frac{l}{\rho} = \mathbf{a}^{\frac{1}{2}} l_1 p'^2 + 2\mathbf{b}^{\frac{1}{2}} l_{12} p'q' + \mathbf{c}^{\frac{1}{2}} l_2 q'^2,$$

with the three like equations, we have

$$\begin{aligned} \frac{\Sigma l l_1}{\rho} &= \mathbf{a}^{\frac{1}{2}} p'^2 + 2\mathbf{b}^{\frac{1}{2}} p'q' \Sigma l_1 l_{12} + 2\mathbf{c}^{\frac{1}{2}} q'^2 \Sigma l_1 l_2 \\ &= \mathbf{a}^{\frac{1}{2}} p'^2 + 2\frac{\mathbf{h}}{\mathbf{a}^{\frac{1}{2}}} p'q' + 2\frac{\mathbf{g}}{\mathbf{a}^{\frac{1}{2}}} q'^2; \end{aligned}$$

and therefore

$$\frac{\mathbf{a}^{\frac{1}{2}}}{\rho} \Sigma l l_1 = \mathbf{a} p'^2 + 2\mathbf{h} p'q' + \mathbf{g} q'^2 = \frac{L}{\rho} = \frac{\mathbf{a}^{\frac{1}{2}}}{\rho} \cos\alpha.$$

Similarly

$$\frac{\mathbf{b}^{\frac{1}{2}}}{\rho} \Sigma l l_{12} = \mathbf{h} p'^2 + 2\mathbf{b} p'q' + \mathbf{f} q'^2 = \frac{M}{\rho} = \frac{\mathbf{b}^{\frac{1}{2}}}{\rho} \cos\beta,$$

$$\frac{\mathbf{c}^{\frac{1}{2}}}{\rho} \Sigma l l_2 = \mathbf{g} p'^2 + 2\mathbf{f} p'q' + \mathbf{c} q'^2 = \frac{N}{\rho} = \frac{\mathbf{c}^{\frac{1}{2}}}{\rho} \cos\gamma.$$

Hence the angles α, β, γ, of § 230 are the inclinations of the geodesic radius of curvature to the respective directions l_1, m_1, n_1, k_1; $l_{12}, m_{12}, n_{12}, k_{12}$; l_2, m_2, n_2, k_2, all of which lie in the orthogonal plane at the point.

Torsion of a geodesic.

234. The torsion and the tilt of any geodesic can be determined by the use of the orthogonal frame of the geodesics through a point, for which the tangent plane to the surface is one of the planes of reference. We already (§ 226) have had to consider the flat

$$\begin{vmatrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ x_1\ , & y_1\ , & z_1\ , & v_1 \\ x_2\ , & y_2\ , & z_2\ , & v_2 \\ l\ , & m\ , & n\ , & k \end{vmatrix} = 0.$$

This flat contains the tangent plane to the surface, the equations of which are

$$\begin{Vmatrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ x_1\ , & y_1\ , & z_1\ , & v_1 \\ x_2\ , & y_2\ , & z_2\ , & v_2 \end{Vmatrix} = 0.$$

It also contains the osculating plane of the geodesic, determined by the tangent and the principal normal, the equations of this plane being

$$\left\|\begin{matrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ x', & y', & z', & v' \\ l, & m, & n, & k \end{matrix}\right\| = 0.$$

Now the binormal at any point of a curve lies in the flat which contains two successive tangents to the curve and the principal normal to the curve at that point; and, within that flat, the direction of the binormal is perpendicular to the osculating plane of the curve. Hence in the case of the foregoing geodesic, the binormal at the point lies in the foregoing flat; and its direction in the flat is perpendicular to the foregoing osculating plane, that is, the direction of the binormal is parallel to a line in the tangent plane of the surface, this line being perpendicular to the tangent to the geodesic. The said line has, for its direction-cosines, the magnitudes denoted (§§ 210, 233) by λ, μ, ν, κ, where

$$\begin{aligned} V\lambda &= (Ex_2 - Fx_1)\,p' + (Fx_2 - Gx_1)\,q' \\ &= x_2(Ep' + Fq') - x_1(Fp' + Gq'), \end{aligned}$$

with corresponding expressions for μ, ν, κ.

With the customary notation (§ 134) for the principal lines at any point of a curve, the direction-cosines of the binormal are

$$\frac{\sigma}{\rho}(x' + \rho\rho' x'' + \rho^2 x'''),$$

and three similar expressions. Also, we have $\rho x'' = l$, $\rho y'' = m$, $\rho z'' = n$, $\rho v'' = k$, with the current significance of l, m, n, k, for geodesics; hence there are the four equations of the type

$$\left.\begin{aligned} \frac{\sigma}{\rho}(x' + \rho l') &= \frac{1}{V}\{x_2(Ep' + Fq') - x_1(Fp' + Gq')\} \\ \frac{\sigma}{\rho}(y' + \rho m') &= \frac{1}{V}\{y_2(Ep' + Fq') - y_1(Fp' + Gq')\} \\ \frac{\sigma}{\rho}(z' + \rho n') &= \frac{1}{V}\{z_2(Ep' + Fq') - z_1(Fp' + Gq')\} \\ \frac{\sigma}{\rho}(v' + \rho k') &= \frac{1}{V}\{v_2(Ep' + Fq') - v_1(Fp' + Gq')\} \end{aligned}\right\}.$$

From this set of equations various inferences can be derived. We have

$$\Sigma l x_1 = 0, \quad \Sigma l x_2 = 0;$$

hence

$$\Sigma x_1 l' = -\Sigma l(x_{11}p' + x_{12}q') = -(Lp' + Mq'),$$

$$\Sigma x_2 l' = -\Sigma l(x_{12}p' + x_{22}q') = -(Mp' + Nq'),$$

and

$$\begin{aligned}\Sigma x'l' &= \Sigma l'(x_1 p' + x_2 q') \\ &= p'\Sigma l'x_1 + q'\Sigma l'x_2 \\ &= -(Lp'^2 + 2Mp'q' + Nq'^2) = -\frac{1}{\rho}.\end{aligned}$$

First, multiply the equations by x', y', z', v', and add: the result is an equation, satisfied unconditionally.

Next, multiply by l, m, n, k, and add: again the resulting equation is satisfied unconditionally.

Next, multiply by l', m', n', k', and add: then we have

$$\begin{aligned}\frac{\sigma}{\rho}\left(-\frac{1}{\rho} + \rho\Sigma l'^2\right) &= \frac{1}{V}\{(Ep' + Fq')(\Sigma x_2 l') - (Fp' + Gq')(\Sigma x_1 l')\} \\ &= \frac{1}{V}\begin{vmatrix} Lp' + Mq', & Mp' + Nq' \\ Ep' + Fq', & Fp' + Gq' \end{vmatrix}.\end{aligned}$$

The determinant on the right-hand side we denote by Υ. Now in Frenet's curvature formulæ (§ 197) for curves in quadruple space, there are four equations

$$\frac{dc_2}{ds} = -\frac{c_1}{\rho} + \frac{c_3}{\sigma},$$

where c_1 is a typical direction-cosine of the tangent, c_2 of the principal normal, and c_3 of the binormal. Thus we can take $c_2 = l$, $c_1 = x'$, $c_3 = \lambda$; and there are then four equations of the type

$$l' = -\frac{x'}{\rho} + \frac{\lambda}{\sigma}.$$

Squaring and adding, we have

$$\Sigma l'^2 = \frac{1}{\rho^2} + \frac{1}{\sigma^2}.$$

The foregoing result now becomes

$$\frac{V}{\sigma} = \begin{vmatrix} Lp' + Mq', & Mp' + Nq' \\ Ep' + Fq', & Fp' + Gq' \end{vmatrix} = \Upsilon,$$

thus giving an expression for the *torsion of a geodesic*.

Further,

$$\frac{L}{\rho} = \mathbf{a}p'^2 + 2\mathbf{h}p'q' + \mathbf{g}q'^2,$$

with like expressions for M and for N, so that

$$\frac{1}{\rho}(Lp' + Mq') = \mathbf{a}p'^3 + 3\mathbf{h}p'^2q' + (\mathbf{g} + 2\mathbf{b})\,p'q'^2 + \mathbf{f}q'^3 = \mathbf{a}p'^3 + 3\mathbf{h}p'^2q' + 3\mathbf{k}p'q'^2 + \mathbf{f}q'^3,$$

with the definition (§ 213) of $\mathbf{k}$; and, similarly,

$$\frac{1}{\rho}(Mp' + Nq') = \mathbf{h}p'^3 + 3\mathbf{k}p'^2q' + 3\mathbf{f}p'q'^2 + \mathbf{c}q'^3.$$

Hence

$$\frac{V^2}{\rho\sigma} = \begin{vmatrix} \frac{L}{\rho}p' + \frac{M}{\rho}q', & Ep' + Fq' \\ \frac{M}{\rho}p' + \frac{N}{\rho}q', & Fp' + Gq' \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{a}p'^3 + 3\mathbf{h}p'^2q' + 3\mathbf{k}p'q'^2 + \mathbf{f}q'^3, & Ep' + Fq' \\ \mathbf{h}p'^3 + 3\mathbf{k}p'^2q' + 3\mathbf{f}p'q'^2 + \mathbf{c}q'^3, & Fp' + Gq' \end{vmatrix},$$

another expression for the torsion of the geodesic.

Tilt of a geodesic.

235. Further, we have seen that an orthogonal frame for the geodesic (or, indeed, for any curve) through the point has the four lines x', y', z', v', being the tangent: l, m, n, k, being the principal normal: λ, μ, ν, κ, being the binormal of the geodesic along the direction x', y', z', v': and A, B, C, D, for its principal axes. Thus A, B, C, D, as given in § 226, are, in effect, the direction-cosines of the trinormal of the geodesic; and we have

$$\Sigma Ax_1 = 0, \quad \Sigma Ax_2 = 0, \quad \Sigma Ax' = 0, \quad \Sigma A\lambda = 0,$$
$$\Sigma Al = 0.$$

Accordingly, reverting to Frenet's formulæ (§ 164), we select the relation

$$\frac{dc_3}{ds} = -\frac{c_2}{\sigma} + \frac{c_4}{\tau}$$

where c_4 is the typical direction-cosine of the trinormal, so that $c_4 = A, B, C, D$, in turn.

Now, typically, we have

$$Vl_3 = x_2(Ep' + Fq') - x_1(Fp' + Gq'),$$

and therefore

$$V\frac{dl_3}{ds} + l_3\frac{dV}{ds} = (Ep' + Fq')(x_{12}p' + x_{22}q') - (Fp' + Gq')(x_{11}p' + x_{12}q')$$
$$+ x_2\frac{d}{ds}(Ep' + Fq') - x_1\frac{d}{ds}(Fp' + Gq').$$

Taking the four equations for the four values of c_3, multiplying them in turn by l, m, n, k, (the four values of c_2), adding, and using the relations

$$\Sigma l_2^2 = 1, \quad \Sigma l_2l_4 = 0,$$
$$\Sigma ll_3 = 0, \quad \Sigma lx_2 = 0, \quad \Sigma lx_1 = 0, \quad \Sigma lx_{11} = L, \quad \Sigma lx_{12} = M, \quad \Sigma lx_{22} = N,$$

we have

$$\frac{1}{\sigma} = -\Sigma l\frac{dl_3}{ds}$$
$$= -\frac{1}{V}\Sigma lV\frac{dl_3}{ds}$$
$$= \frac{1}{V}\begin{vmatrix} Lp' + Mq', & Mp' + Nq' \\ Ep' + Fq', & Fp' + Gq' \end{vmatrix},$$

the expression already (§ 234) obtained for the torsion of the geodesic.

Next, take the same derived equation for the four successive values of c_3, multiply by A, B, C, D, (the four successive values of c_4), and add, then, as

$$\Sigma A l_3 = \Sigma l_3 l_4 = 0, \quad \Sigma A x_2 = 0, \quad \Sigma A x_1 = 0,$$
$$\Sigma A x_{11} = \Omega, \quad \Sigma A x_{12} = \Omega', \quad \Sigma A x_{22} = \Omega'',$$

we have

$$\begin{aligned} V \Sigma A \frac{dl_3}{ds} &= (Ep' + Fq')(\Omega' p' + \Omega'' q') - (Fp' + Gq')(\Omega p' + \Omega' q') \\ &= (Ep' + Fq')(-p'\rho W) - (Fp' + Gq')(q'\rho W) \\ &= -\rho W, \end{aligned}$$

where W has the significance given in § 230. Also, from the quoted Frenet formula, we have

$$\Sigma A \frac{dl_3}{ds} = -\frac{1}{\sigma} \Sigma A l_2 + \frac{1}{\tau} \Sigma A^2 = \frac{1}{\tau},$$

so that

$$-\frac{V}{\rho\tau} = W = (\mathbf{ab} - \mathbf{h}^2)^{\frac{1}{2}} p'^2 + (\mathbf{ca} - \mathbf{g}^2)^{\frac{1}{2}} p'q' + (\mathbf{bc} - \mathbf{f}^2)^{\frac{1}{2}} q'^2,$$

thus giving an expression for the *tilt of a geodesic.*

This result is in accordance with the observation (p. 402) that W vanishes when the surface lies wholly in one flat, so that the tilt of any curve on the surface is zero.

Summary of results: covariants $LN - M^2$, $EN - 2FM + GL$.

236. Summarising, we have results associated with the binary forms

$$\begin{aligned} U &= Ep'^2 + 2F p'q' + G q'^2 = 1, \\ U' &= Lp'^2 + 2M p'q' + N q'^2 = \frac{1}{\rho}, \\ U'' &= \Omega p'^2 + 2\Omega' p'q' + \Omega'' q'^2 = 0. \end{aligned}$$

Their three Jacobians are

$$W = J\left(\frac{U'',\ U'}{p',\ q'}\right) = \begin{vmatrix} \Omega p' + \Omega' q', & \Omega' p' + \Omega'' q' \\ Lp' + Mq', & Mp' + Nq' \end{vmatrix} = -\frac{V}{\rho\tau},$$

$$\Upsilon = J\left(\frac{U',\ U}{p',\ q'}\right) = \begin{vmatrix} Lp' + Mq', & Mp' + Nq' \\ Ep' + Fq', & Fp' + Gq' \end{vmatrix} = \frac{V}{\sigma},$$

$$\mathrm{III} = J\left(\frac{U'',\ U}{p',\ q'}\right) = \begin{vmatrix} \Omega p' + \Omega' q', & \Omega' p' + \Omega'' q' \\ Ep' + Fq', & Fp' + Gq' \end{vmatrix} = -\frac{V}{\tau}.$$

Now, between any two quadratic forms

$$\Theta_1 = a_1 p'^2 + 2h_1 p'q' + b_1 q'^2, \quad \Theta_2 = a_2 p'^2 + 2h_2 p'q' + b_2 q'^2,$$

their three invariants

$$a_1 b_1 - h_1^2, \quad a_1 b_2 - 2h_1 h_2 + b_1 a_2, \quad a_2 b_2 - h_2^2,$$

and their Jacobian

$$J = J\left(\frac{\Theta_1, \Theta_2}{p', q'}\right) = \begin{vmatrix} a_1 p' + h_1 q', & h_1 p' + b_1 q' \\ a_2 p' + h_2 q', & h_2 p' + b_2 q' \end{vmatrix},$$

there exists the relation

$$J^2 = (a_1 b_2 - 2h_1 h_2 + b_1 a_2)\,\Theta_1\Theta_2 - (a_2 b_2 - h_2^2)\,\Theta_1^2 - (a_1 b_1 - h_1^2)\,\Theta_2^2.$$

We apply this relation to the three pair-combinations of U, U', U'', in turn, writing

$$\left.\begin{array}{ll} EN - 2FM + GL = \beta, & \Omega\Omega'' - \Omega'^2 = \eta \\ E\Omega'' - 2F\Omega' + G\Omega = \gamma, & LN - M^2 = \alpha \\ L\Omega'' - 2M\Omega' + N\Omega = \delta, & EG - F^2 = V^2 \end{array}\right\}.$$

We find, firstly,

$$\begin{aligned} \Upsilon^2 &= \beta U U' - \alpha U^2 - V^2 U'^2 \\ &= \frac{\beta}{\rho} - \alpha - \frac{V^2}{\rho^2}, \end{aligned}$$

that is,

$$\begin{aligned} \frac{V^2}{\sigma^2} &= -\left\{LN - M^2 - \frac{1}{\rho}(EN - 2FM + GL) + \frac{1}{\rho^2}(EG - F^2)\right\} \\ &= -\begin{vmatrix} L - \dfrac{E}{\rho}, & M - \dfrac{F}{\rho} \\ M - \dfrac{F}{\rho}, & N - \dfrac{G}{\rho} \end{vmatrix}. \end{aligned}$$

Secondly, we find

$$\begin{aligned} W^2 &= \delta U' U'' - \eta U'^2 - \alpha U''^2 \\ &= -\frac{\eta}{\rho^2}, \end{aligned}$$

that is,

$$\eta = -\frac{V^2}{\tau^2}.$$

Thirdly, we find

$$\begin{aligned} \text{III}^2 &= \gamma U U'' - V^2 U'' - \eta U^2 \\ &= -\eta, \end{aligned}$$

leading again to the result

$$\eta = -\frac{V^2}{\tau^2}.$$

There is also the algebraical relation

$$UJ\left(\frac{U'', U'}{p', q'}\right) + U''J\left(\frac{U', U}{p', q'}\right) + U'J\left(\frac{U, U''}{p', q'}\right) = 0.$$

that is,

$$UW + U''\Upsilon - U'\,\text{III} = 0,$$

which is satisfied identically by the magnitudes involved.

Later (§ 240) it will appear that the quantity $\mathsf{g}-\mathsf{b}$, there denoted by 3θ, is an invariant of the superficial configuration. Now

$$\begin{aligned}\mathsf{g}-\mathsf{b} &= LN+\Omega\Omega''-(M^2-\Omega'^2)\\ &= LN-M^2-(\Omega'^2-\Omega\Omega'')\\ &= LN-M^2-\rho^2W^2\\ &= LN-M^2-\frac{V^2}{\tau^2},\end{aligned}$$

and therefore $LN-M^2$, a covariant, can be expressed in the form

$$LN-M^2=\mathsf{g}-\mathsf{b}+\frac{V^2}{\tau^2}=3\theta+\frac{V^2}{\tau^2}.$$

Again, the relation

$$\frac{V^2}{\sigma^2}=-\left\{LN-M^2-\frac{1}{\rho}(EN-2FM+GL)+\frac{V^2}{\rho^2}\right\}$$

has been established; hence, substituting the foregoing expression for $LN-M^2$, we have

$$\frac{EN-2FM+GL}{\rho}=3\theta+V^2\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right),$$

yielding an expression for the covariant $EN-2FM+GL$.

When the surface exists in one flat, so that there is no tilt and $1/\tau$ is zero, the foregoing formula for $LN-M^2$ becomes

$$LN-M^2=\mathsf{g}-\mathsf{b}\cdot$$

in effect, the formula which then gives the Gauss measure of curvature of the surface lying in the flat, that is, existing in a customary homaloidal triple space.

Lines of curvature on surfaces in quadruple space.

237. In the discussion of the curvature (and the lines of curvature, in particular) of surfaces in homaloidal triple space, two alternative equivalent methods of procedure are adopted.

In one of these methods, the direction of a curve of curvature on the surface is such that, at every point along the curve, the principal normal of the superficial geodesic touching the curve meets the principal normal of the similar superficial geodesic at a consecutive point. Usually two analytical conditions are required, in order to secure that a couple of lines in homaloidal quadruple space shall meet, and apparently only one possibility of satisfying these conditions is provided by a variable direction on the surface, so that *primâ facie* the method indicated would not be applicable to quadruple space. It appears, however, that the two conditions assume a different organic form when the lines are principal normals of consecutive geodesics.

With the preceding notation, the equations of the principal normal of the geodesic through the point x, y, z, v, in the direction p', q', are

$$\bar{x} = x + lD, \quad \bar{y} = y + mD, \quad \bar{z} = z + nD, \quad \bar{v} = v + kD,$$

where D is the variable parameter along the line. For the principal normal of the geodesic through a consecutive point at an arc-distance δ from x, y, z, v, along the direction p', q', the equations are

$$\begin{aligned} \bar{x} &= x + x'\delta + (l + l'\delta)\Lambda, & \bar{z} &= z + z'\delta + (n + n'\delta)\Lambda, \\ \bar{y} &= y + y'\delta + (m + m'\delta)\Lambda, & \bar{v} &= v + v'\delta + (k + k'\delta)\Lambda. \end{aligned}$$

If these two normals meet, there must be values of D and Λ which make $\bar{x}, \bar{y}, \bar{z}, \bar{v}$, the same in the two sets of equations: clearly two conditions will be required. We then have, at the common point,

$$\left.\begin{aligned} lD &= x'\delta + (l + l'\delta)\Lambda \\ mD &= y'\delta + (m + m'\delta)\Lambda \\ nD &= z'\delta + (n + n'\delta)\Lambda \\ kD &= v'\delta + (k + k'\delta)\Lambda \end{aligned}\right\}.$$

First, multiply these four relations by x', y', z', v', and add: then

$$0 = \delta + \Lambda(\Sigma x'l')\delta.$$

But, as $\Sigma x'l = 0$, we have

$$\Sigma x'l' = -\Sigma x''l = -\frac{1}{\rho};$$

and therefore

$$\Lambda = \rho.$$

Next, multiply them by l, m, n, k, and add: then, as $\Sigma ll' = 0$, we have

$$D = \Lambda\Sigma l^2 = \Lambda,$$

so that

$$D = \Lambda = \rho,$$

as is to be expected from the geometrical configuration.

Next, multiply the four relations by $\lambda, \mu, \nu, \kappa$, the direction-cosines of the line in the tangent plane perpendicular, in that plane, to the direction x', y', z', v'; then as

$$\Sigma\lambda l = 0, \quad \Sigma\lambda x' = 0,$$

we have

$$\Sigma\lambda l' = 0.$$

Now, in the Frenet formulæ, we have the relation

$$\frac{dc_2}{ds} = -\frac{c_1}{\rho} + \frac{c_3}{\sigma},$$

where $c_1 = x', y', z', v'$; $c_2 = l, m, n, k$; $c_3 = \lambda, \mu, \nu, \kappa$. Hence, as $\Sigma l_1 l_3 = 0$,

$$\frac{1}{\sigma} = \Sigma\frac{l_3^2}{\sigma} = \Sigma l_3\frac{dl_2}{ds} = \Sigma\lambda l' = 0,$$

for the geodesic in question; that is, as a condition for the concurrence of normals at successive points to the tangential geodesic, we have

$$\frac{1}{\sigma} = 0.$$

Thus the torsion vanishes, again a result to be expected from this direction: because an osculating plane is stationary when consecutive radii of circular curvature intersect.

Finally, multiply the four relations by A, B, C, D, the direction-cosines of the fourth arm of the orthogonal frame, then

$$\Sigma A l' = 0,$$

because $\Lambda\ (= \rho)$ is not zero. But we have, as typical of four equations,

$$l' = -\frac{x'}{\rho} + \frac{\lambda}{\sigma},$$

and $\Sigma A x' = 0$, $\Sigma A \lambda = 0$; thus the new condition is satisfied without any residuary relation.

Consequently, by this method, we find certain properties of the geometrical configuration—viz. that $D = \Lambda = \rho$, as is to be expected; and, as a residuary relation, we have

$$\frac{1}{\sigma} = 0$$

for the geodesic touching the curve of curvature. that is, after § 234, the direction p', q', must satisfy the equation

$$\begin{vmatrix} Lp' + Mq', & Mp' + Nq' \\ Ep' + Fq', & Fp' + Gq' \end{vmatrix} = 0,$$

which can be regarded as an equation of curves of curvature on the surface.

238. In the alternative method adopted in the discussion of the curvature of surfaces in homaloidal triple space, we use an implied definition or property that the radius of circular curvature of a geodesic touching a curve of curvature is a maximum or a minimum among such radii for geodesics, which are tangential to directions through the point. In this definition the implication is, not that ρ is a maximum or a minimum along the specific geodesic (for that would require the property $\rho' = 0$), but that ρ, regarded as a function of p' and q', acquires a maximum or a minimum by a choice of p' and q' among the possible values that are limited by the equation $Ep'^2 + 2Fp'q' + Gq'^2 = 1$. The definition thus is distinct, in quality, from the earlier definition which contemplates a property arising from continued progress along the curve; here, such continued progress does not enter into the consideration.

We proceed to use this implied property: that is, we require directions at any point on the surface, such that p' and q' provide a maximum or a minimum value of ρ among all directions through the point. The circular curvature of the geodesic is given by the relation

$$\frac{1}{\rho^2} = \mathrm{a}p'^4 + 4\mathrm{h}p'^3q' + (2\mathrm{g} + 4\mathrm{b})\,p'^2q'^2 + 4\mathrm{f}p'q'^3 + \mathrm{c}q'^4.$$

When the surface does not lie in a flat within the quadruple space, the right-hand side is not, unconditionally, the perfect square of a quadratic function of p' and q'. Accordingly, the use of the method allows us to obtain a maximum value or a minimum value of $1/\rho^2$ (not of $1/\rho$) for values of p' and q', subject to the relation $\Sigma Ep'^2 = 1$. On a real surface, the curvature is real, though it may be sometimes positive, sometimes negative; thus, among the minimum values of $1/\rho^2$, a zero value would have to be considered. A zero value of the circular curvature of a geodesic implies a direction that is asymptotic (or linear) and is not the direction of a line of curvature in the customary conception: hence, in the analysis to which this method of proceeding leads, there might arise a necessity for discriminating between the directions of lines of curvature and the directions of asymptotic lines. The necessity would certainly arise if the surface were contained in a flat.

It is desirable however to possess, at once, the central analytical relations which bear upon the maximum and minimum values of the curvature. The foregoing expression for $1/\rho^2$ has to be made a maximum or a minimum subject to the permanent condition affecting p' and q'; the equations, critical for this property, are

$$\mathrm{a}p'^3 + 3\mathrm{h}p'^2q' + (\mathrm{g} + 2\mathrm{b})\,p'q'^2 + \mathrm{f}q'^3 = \lambda\,(Ep' + Fq'),$$

$$\mathrm{h}p'^3 + (\mathrm{g} + 2\mathrm{b})\,p'^2q' + 3\mathrm{f}p'q'^2 + \mathrm{c}q'^3 = \lambda\,(Fp' + Gq'),$$

where λ is the usual multiplier, to be determined after the construction of the critical equations. Multiplying the two equations by p' and q' respectively, and adding, we have

$$\frac{1}{\rho^2} = \lambda.$$

Also we have

$$L = (\mathrm{a}p'^2 + 2\mathrm{h}p'q' + \mathrm{g}q'^2)\,\rho,$$

$$M = (\mathrm{h}p'^2 + 2\mathrm{b}p'q' + \mathrm{f}q'^2)\,\rho,$$

$$N = (\mathrm{g}p'^2 + 2\mathrm{f}p'q' + \mathrm{c}q'^2)\,\rho;$$

and therefore the two equations are

$$\left.\begin{aligned} Lp' + Mq' &= \frac{1}{\rho}\,(Ep' + Fq') \\ Mp' + Nq' &= \frac{1}{\rho}\,(Fp' + Gq') \end{aligned}\right\}.$$

These two equations appear to be the same, formally, as the critical equations for the principal curvatures of a surface in homaloidal triple space. But there is one essential difference; in triple space, L, M, N, are independent of p' and q'; in quadruple space, L, M, N, usually involve p' and q'.

If this method be pursued further, it gives the directions of maximum and minimum curvature by the quartic equation which results from the elimination of ρ and which, initially, has the form

$$\begin{vmatrix} Lp' + Mq', & Ep' + Fq' \\ Mp' + Nq', & Fp' + Gq' \end{vmatrix} = 0,$$

already found. This form veils the order of the equation, the explicit form is

$$\begin{vmatrix} \mathbf{a}p'^3 + 3\mathbf{h}p'^2q' + (\mathbf{g} + 2\mathbf{b})p'q'^2 + \mathbf{f}q'^3, & Ep' + Fq' \\ \mathbf{h}p'^3 + (\mathbf{g} + 2\mathbf{b})p'^2q' + 3\mathbf{f}p'q'^2 + \mathbf{c}q'^3, & Fp' + Gq' \end{vmatrix} = 0,$$

an equation of degree four in the ratio $p' : q'$, determining four directions.

The magnitude of the maximum or minimum radii of circular curvature of geodesics through the point is the resultant of the two critical equations when the ratio $p' : q'$ is eliminated. It is manifest from the form of these two equations that this resultant equation can be taken as equivalent to the condition that the quartic equation

$$\mathbf{a}p'^4 + 4\mathbf{h}p'^3q' + (2\mathbf{g} + 4\mathbf{b})p'^2q'^2 + 4\mathbf{f}p'q'^3 + \mathbf{c}q'^4 - \frac{1}{\rho^2}(Ep'^2 + 2Fp'q' + Gq'^2)^2 = 0,$$

in p'/q' shall have equal roots, or that the discriminant of the left-hand side (regarded as a binary quartic in homogeneous variables p' and q') shall vanish.

Equation determining maximum or minimum values of ρ
principal measures.

239. We proceed to investigate, on the basis of this last analytical property, the explicit form of the equation determining ρ.

For simplicity, we write

$$(Ep'^2 + 2Fp'q' + Gq'^2)^2 = (a', h', k', f', c' \between p', q')^4,$$

taking

$$E^2 = a', \quad EF = h', \quad EG + 2F^2 = 3k', \quad FG = f', \quad G^2 = c',$$

and, for symmetry, we add

$$F^2 = b', \quad EG = g',$$

so that $g' + 2b' = 3k'$ corresponds with the relation $\mathbf{g} + 2\mathbf{b} = 3\mathbf{k}$. We now take

$$a_0 = \mathbf{a} - \frac{a'}{\rho^2}, \quad a_1 = \mathbf{h} - \frac{h'}{\rho^2}, \quad a_2 = \mathbf{k} - \frac{k'}{\rho^2}, \quad a_3 = \mathbf{f} - \frac{f'}{\rho^2}, \quad a_4 = \mathbf{c} - \frac{c'}{\rho^2},$$

so that the quartic, whose discriminant is to vanish, becomes

$$(a_0, a_1, a_2, a_3, a_4 \between p', q')^4 = 0.$$

Let I and J denote the quadrinvariant and the cubinvariant of this quartic, so that

$$I = a_0a_4 - 4a_1a_3 + 3a_2^2,$$
$$J = a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3,$$

then Δ, the discriminant, is

$$\Delta = I^3 - 27J^2.$$

When substitution is made for a_0, a_1, a_2, a_3, a_4, we find

$$I = I_0 - \frac{1}{\rho^2} I_1 + \frac{1}{\rho^4} I_2,$$

where

$$\begin{aligned} I_0 &= \mathbf{ac} - 4\mathbf{hf} + 3\mathbf{k}^2, \\ I_1 &= \mathbf{a}c' - 4\mathbf{h}f' + 6\mathbf{k}k' - 4\mathbf{f}h' + \mathbf{c}a' \\ &= G^2\mathbf{a} - 4FG\mathbf{h} + 2(V^2 + 3F^2)\,\mathbf{k} - 4EF\mathbf{f} + E^2\mathbf{c}, \\ I_2 &= a'c' - 4h'f' + 3k'^2 \\ &= \frac{4}{3} V^4. \end{aligned}$$

Similarly, we have

$$J = J_0 - \frac{1}{\rho^2} J_1 + \frac{1}{\rho^4} J_2 - \frac{1}{\rho^6} J_3,$$

where

$$\begin{aligned} J_0 &= \mathbf{akc} + 2\mathbf{hkf} - \mathbf{af}^2 - \mathbf{ch}^2 - \mathbf{k}^3, \\ J_1 &= a'(\mathbf{kc} - \mathbf{f}^2) + 2h'(\mathbf{kf} - \mathbf{ch}) + 2f'(\mathbf{hk} - \mathbf{af}) + c'(\mathbf{ak} - \mathbf{h}^2) \\ &\qquad + k'(\mathbf{ac} + 2\mathbf{hf} - 3\mathbf{k}^2), \\ J_2 &= \mathbf{a}(k'c' - f'^2) + 2\mathbf{h}(k'f' - c'h') + 2\mathbf{f}(h'k' - a'f') + \mathbf{c}(a'k' - h'^2) \\ &\qquad + \mathbf{k}(a'c' + 2h'f' - 3k'^2), \\ J_3 &= a'k'c' + 2h'k'f' - a'f'^2 - c'h'^2 - k'^3. \end{aligned}$$

By direct substitution for a', h', k', f', c', we find

$$J_2 = \frac{1}{3} V^2 I_1,$$

$$J_3 = \frac{8}{27} V^6.$$

Modified expressions will be found for J_0 and J_1, later. Meanwhile, it is to be noted that two new quantities I_0 and I_1 arise out of I, and other two new quantities J_0 and J_1 arise out of J; of these four quantities, I_0 and J_0 are the quadrinvariant and the cubinvariant of the quartic form for $1/\rho^2$, while I_1 and J_1 are invariants intermediate between this quartic form and the quadratic form $\Sigma E p'^2 = 1$. Thus we have

$$I = I_0 - \frac{I_1}{\rho^2} + \frac{4}{3}\frac{V^4}{\rho^4},$$

$$J = J_0 - \frac{J_1}{\rho^2} + \frac{1}{3}\frac{V^2 I_1}{\rho^4} - \frac{8}{27}\frac{V^6}{\rho^6}.$$

When these values are substituted in the vanishing discriminant

$$I^3 - 27J^2,$$

the terms in ρ^{-12} and ρ^{-10} disappear, as is to be expected, and the resulting equation is found to be

$$\frac{1}{\rho^8} - \mu_1 \frac{1}{\rho^6} + \mu_2 \frac{1}{\rho^4} - \mu_3 \frac{1}{\rho^2} + \mu_4 = 0,$$

where

$$\begin{aligned} \mu_1 D_0 &= I_1^3 - 18V^2 I_1 J_1 + 8V^4 I_0 I_1 - 16V^6 J_0, \\ \mu_2 D_0 &= 3I_0 I_1^2 - 18V^2 I_1 J_0 - 27 J_1^2 + 4V^4 I_0^2, \\ \mu_3 D_0 &= 3I_0 I_1 - 54 J_0 J_1, \\ \mu_4 D_0 &= I_0^3 - 27 J_0^2, \end{aligned}$$

while D_0 is given by the expression

$$D_0 = V^4 \left(I_1^2 - 16V^2 J_1 + \frac{16}{3} V^4 I_0 \right).$$

It may be noted that the value of $\mu_4 D_0$ is the discriminant of the quartic expression for $1/\rho^2$, a quantity which should vanish (and is found to vanish) when the surface exists in a flat, because then the original quartic form for $1/\rho^2$ is an algebraical perfect square.

Thus there are four principal magnitudes $\mu_1, \mu_2, \mu_3, \mu_4$, arising out of this equation, being invariantive *measures of curvature of the surface*. These four measures are expressed in terms of the four invariants

$$I_0, I_1, J_0, J_1,$$

and, conversely, the four relations are potentially sufficient for the expressions of these four invariants in terms of the four principal measures $\mu_1, \mu_2, \mu_3, \mu_4$.

The invariants θ and Π.

240. Modified expressions can be obtained for J_0 and J_1. We already (§§ 213, 239) have a quantity $\mathbf{k}$, expressed in terms of the fundamental magnitudes $\mathbf{g}$ and $\mathbf{b}$ by the relation

$$3\mathbf{k} = \mathbf{g} + 2\mathbf{b}.$$

We now introduce the magnitude $\mathbf{g} - \mathbf{b}$ of § 208, writing

$$3\theta = \mathbf{g} - \mathbf{b},$$

so that we also have

$$\mathbf{g} = \mathbf{k} + 2\theta, \quad \mathbf{b} = \mathbf{k} - \theta.$$

Among the fundamental magnitudes $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, \mathbf{h}$, there subsists the necessary relation

$$0 = \mathbf{abc} + 2\mathbf{fgh} - \mathbf{af}^2 - \mathbf{bg}^2 - \mathbf{ch}^2;$$

hence

$$\begin{aligned} J_0 &= \mathbf{akc} + 2\mathbf{hkf} - \mathbf{af}^2 - \mathbf{ch}^2 - \mathbf{k}^3 \\ &= \mathbf{ac}(\mathbf{k} - \mathbf{b}) + 2\mathbf{hf}(\mathbf{k} - \mathbf{g}) - \mathbf{k}^3 + \mathbf{bg}^2 \\ &= \theta(\mathbf{ac} - 4\mathbf{hf}) - \mathbf{k}^3 + \mathbf{bg}^2. \end{aligned}$$

But

$$\mathbf{bg}^2 = (\mathbf{k} - \theta)(\mathbf{k}^2 + 4\theta\mathbf{k} + 4\theta^2) = \mathbf{k}^3 + 3\theta\mathbf{k}^2 - 4\theta^3;$$

and therefore

$$J_0 = \theta(\mathrm{ac} - 4\mathrm{hf} + 3\mathrm{k}^2) - 4\theta^3 = \theta I_0 - 4\theta^3,$$

a well-known form that occurs otherwise in connection with the algebra of the quartic form. We have seen that I_0 and J_0 are invariants, so that θ also is an invariant; they are expressible in terms of the principal measures of superficial curvature, and therefore θ, as an invariant, also is expressible in terms of these measures.

Again, there is an invariant Π, which can be defined by

$$\begin{aligned}\Pi &= \frac{1}{2}\left(E\frac{\partial^2}{\partial q'^2} - 2F\frac{\partial^2}{\partial p'\partial q'} + G\frac{\partial^2}{\partial p'^2}\right)W \\ &= E(\mathrm{bc} - \mathrm{f}^2)^{\frac{1}{2}} - F(\mathrm{ca} - \mathrm{g}^2)^{\frac{1}{2}} + G(\mathrm{ab} - \mathrm{h}^2)^{\frac{1}{2}} \\ &= \frac{1}{V}(ET - FS + GR),\end{aligned}$$

where W is the concomitant of §§ 230, 235. As will be seen later (§§ 252, 258), the vanishing of Π implies an exceptional property of a surface. With the relations already (§ 214) established, we have

$$\begin{aligned}\Pi^2 = E^2(\mathrm{bc} - \mathrm{f}^2) &+ F^2(\mathrm{ca} - \mathrm{g}^2) + G^2(\mathrm{ab} - \mathrm{h}^2) \\ &- 2FG(\mathrm{af} - \mathrm{gh}) - 2GE(\mathrm{bg} - \mathrm{hf}) - 2EF(\mathrm{ch} - \mathrm{fg})\end{aligned}$$

relation which also can be written in the form

$$\Pi^2 = -\begin{vmatrix} \mathrm{a}, & \mathrm{h}, & \mathrm{g}, & E \\ \mathrm{h}, & \mathrm{b}, & \mathrm{f}, & F \\ \mathrm{g}, & \mathrm{f}, & \mathrm{c}, & G \\ E, & F, & G, & 0 \end{vmatrix}.$$

Now the foregoing value of J_1 is

$$\begin{aligned}J_1 = E^2(\mathrm{kc} - \mathrm{f}^2) &+ 2EF(\mathrm{kf} - \mathrm{ch}) + 2FG(\mathrm{hk} - \mathrm{af}) + G^2(\mathrm{ak} - \mathrm{h}^2) \\ &+ \frac{1}{3}EG(\mathrm{ac} + 2\mathrm{hf} - 3\mathrm{k}^2) + \frac{2}{3}F^2(\mathrm{ac} + 2\mathrm{hf} - 3\mathrm{k}^2);\end{aligned}$$

and therefore

$$\begin{aligned}J_1 - \Pi^2 = E^2\mathrm{c}(\mathrm{k} - \mathrm{b}) &+ 2EF\mathrm{f}(\mathrm{k} - \mathrm{g}) + 2FG\mathrm{h}(\mathrm{k} - \mathrm{g}) + G^2\mathrm{a}(\mathrm{k} - \mathrm{b}) \\ &+ EG\left\{\frac{1}{3}I_0 + 2(\mathrm{bg} - \mathrm{k}^2)\right\} + F^2\left\{-\frac{1}{3}I_0 + \mathrm{g}^2 - \mathrm{k}^2\right\}.\end{aligned}$$

But

$$\mathrm{k} - \mathrm{b} = \theta, \quad \mathrm{k} - \mathrm{g} = -2\theta, \quad \mathrm{bg} - \mathrm{k}^2 = \mathrm{k}\theta - 2\theta^2, \quad \mathrm{g}^2 - \mathrm{k}^2 = 4\mathrm{k}\theta + 4\theta^2;$$

consequently

$$J_1 - \Pi^2 = \theta\,\{E^2\mathrm{c} - 4EF\mathrm{f} - 4FG\mathrm{h} + G^2\mathrm{a} + \mathrm{k}(4F^2 + 2EG)\} + \frac{1}{3}V^2 I_0 - 4V^2\theta^2,$$

and therefore

$$J_1 = \theta I_1 + \frac{1}{3}V^2(I_0 - 12\theta^2) + \Pi^2.$$

Thus Π^2, expressible in terms of J_1, I_1, I_0, θ, is an invariant; and it is therefore expressible in terms of the principal measures of superficial curvature.

Further, with these results, we have

$$D_0 V^{-4} = I_1^2 - 16V^2\left\{\theta I_1 + \frac{1}{3}V^2(I_0 - 12\theta^2) + \Pi^2\right\} + \frac{16}{3}V^4 I_0$$
$$= (I_1 - 8V^2\theta)^2 - 16V^2\Pi^2,$$

so that

$$D_0 = V^4\{(I_1 - 8V^2\theta)^2 - 16V^2\Pi^2\}.$$

Again,

$$\mu_4 D_0 = I_0^3 - 27J_0^2 = I_0^3 - 27(\theta I_0 - 4\theta^3)^2 = (I_0 - 3\theta^2)(I_0 - 12\theta^2)^2;$$

and modifications of $\mu_1 D$, $\mu_2 D$, $\mu_3 D$, are possible in which the quantity $I_1 - 8V^2\theta$ is important.

NOTE. Two remarks may be made, in passing.

It is clear that, as there now are four (and not merely two) lines of principal geodesic curvature at any point of a surface in free quadruple space, the Gauss measure for a surface in a triple homaloidal space does not correspond to a similar measure of curvature (whatever be the main measure adopted) of a surface in homaloidal space of more than three dimensions. This divergence in result might be expected from a consideration of the spherical representation surface in triple space: a comparison of the surface quantity

$$\frac{dS}{\rho_1\rho_2},$$

where dS is an element of superficial area bounded by lines of curvature, with the corresponding quantity in the spherical representation, provides an explanation (or a justification) of the Gauss measure which is not offered by any configuration in quadruple space

Further, manipulation of the equations will obviously be required if, instead of being used to express measures of curvature in terms of invariants I_0, I_1, J_0, J_1, they are required to furnish expressions for these invariants in terms of the four measures. In particular, it is desirable to have the value of the quantity 3θ, or $\mathbf{g} - \mathbf{b}$, which persists for any surface, whatever be the number of dimensions of the most limited homaloidal space containing the surface. In § 236, it has appeared that the deviation of $\mathbf{g} - \mathbf{b}$ from $LN - M^2$ (now a covariant and not an invariant, and no longer the Gauss measure V^2K for a surface in triple space) is measured by the square of the tilt of the geodesic along which $LN - M^2$ is a concomitant of the surface.

Degeneration of forms when the surface is contained in a flat.

241. After the preceding remark, it is interesting to see how, when the surface exists in a flat so as to become a surface in homaloidal triple space and therefore to have its geodesics devoid of tilt, the foregoing relations ultimately simplify into the usual relations in the Gauss theory of surfaces in that triple space.

We then have $\Omega = 0, \quad \Omega' = 0, \quad \Omega'' = 0,$
so that

$$\left.\begin{array}{ll} \mathbf{a} = L^2, & \mathbf{f} = MN \\ \mathbf{b} = M^2, & \mathbf{g} = NL \\ \mathbf{c} = N^2, & \mathbf{h} = LM \end{array}\right\}, \qquad \left.\begin{array}{l} R = 0 \\ S = 0 \\ T = 0 \end{array}\right\},$$

and $3\mathbf{k} = LN + 2M^2, \quad 3\theta = LN - M^2, \quad \Pi = 0.$

We write $LN - M^2 = \bar{K}V^2, \quad EN - 2FM + GL = \bar{H}V^2,$
using new symbols $\bar{H}$ and $\bar{K}$. Then

$$I_0 = \mathbf{ac} - 4\mathbf{hf} + 3\mathbf{k}^2 = \frac{4}{3}(LN - M^2)^2 = \frac{4}{3}V^4\bar{K}^2;$$

$$\begin{aligned} I_1 &= E^2N^2 + G^2L^2 - 4EFMN - 4FGLM + \frac{2}{3}(LN + 2M^2)(EG + 2F^2) \\ &= (EN - 2FM + GL)^2 - \frac{4}{3}(EG - F^2)(LN - M^2) \\ &= V^4\left(\bar{H}^2 - \frac{4}{3}\bar{K}\right), \end{aligned}$$

$$I_1 - 8V^2\theta = V^4(\bar{H}^2 - 4\bar{K});$$

$$\theta = \frac{1}{3}(LN - M^2) = \frac{1}{3}V^2\bar{K};$$

$$J_0 = \theta I_0 - 4\theta^3 = \frac{8}{27}V^6\bar{K}^3;$$

$$\begin{aligned} J_1 &= \theta I_1 + \frac{1}{3}V^2(I_0 - 12\theta^2) + \Pi^2 \\ &= \frac{1}{3}V^6\left(\bar{H}^2 - \frac{4}{3}\bar{K}\right)\bar{K}. \end{aligned}$$

When these values are substituted in the equation for the principal measures of superficial curvature, we find

$$\begin{aligned} D &= V^4\{(I_1 - 8V^2\theta)^2 - 16V^2\Pi^2\} \\ &= V^{12}(\bar{H}^2 - 4\bar{K})^2, \\ \mu_1 D &= V^{12}(\bar{H}^2 - 4\bar{K})^2(\bar{H}^2 - 2\bar{K}), \\ \mu_2 D &= V^{12}(\bar{H}^2 - 4\bar{K})^2\bar{K}^2, \\ \mu_3 D &= 0, \\ \mu_4 D &= 0; \end{aligned}$$

and therefore the degenerate form of the equation, giving the principal measures, is

$$V^{12}(\bar{H}^2 - 4\bar{K}^2)\left\{\frac{1}{\rho^4} - \frac{1}{\rho^2}(\bar{H}^2 - 2\bar{K}) + \bar{K}^2\right\}\frac{1}{\rho^4} = 0.$$

We thus have the zero root

$$\frac{1}{\rho^2} = 0$$

repeated: that is, we have the two zero curvatures and the two asymptotic directions at the point. Further, if $1/\rho_1^2$ and $1/\rho_2^2$ are the roots of the other factor, ρ_1 and ρ_2 are the roots of the equation

$$\frac{1}{\rho^2} - \frac{\bar{H}}{\rho} + \bar{K} = 0,$$

in accord with the Gauss result for a surface in homaloidal triple space. Also,

$$\overline{H}^2 - 4\overline{K} = \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^2,$$

so that the ignoration of the remaining factor is merely an assumption that the finite principal radii of curvature are unequal.

Note. The degenerate form of the equation for the principal measures of curvature can be deduced simply from the initial postulate that we have to seek the vanishing discriminant of

$$\mathbf{a}p'^4 + 4\mathbf{h}p'^3q' + 6\mathbf{k}p'^2q'^2 + 4\mathbf{f}p'q'^3 + \mathbf{c}q'^4 - \frac{1}{\rho^2}(Ep'^2 + 2Fp'q' + Gq'^2)^2.$$

We write

$$\Theta = Lp'^2 + 2Mp'q' + Nq'^2, \quad U = Ep'^2 + 2Fp'q' + Gq'^2;$$

and then the requirement is to secure a zero value for the discriminant of

$$\Theta^2 - \frac{1}{\rho^2}U^2,$$

that is, of

$$\left(\Theta - \frac{1}{\rho}U\right)\left(\Theta + \frac{1}{\rho}U\right)$$

Now the discriminant of the product of two binary forms is the product of the square of their resultant by the product of their respective discriminants. The discriminant of $\Theta - \frac{1}{\rho}U$ is

$$\left(\overline{K} - \frac{1}{\rho}\overline{H} + \frac{1}{\rho^2}\right)V^2,$$

and the discriminant of $\Theta + \frac{1}{\rho}U$ is

$$\left(\overline{K} + \frac{1}{\rho}\overline{H} + \frac{1}{\rho^2}\right)V^2,$$

so that the product of their discriminants is

$$V^4\left\{\frac{1}{\rho^4} - \frac{1}{\rho^2}(\overline{H}^2 - 2\overline{K}) + \overline{K}^2\right\}.$$

The resultant of $\Theta - \frac{1}{\rho}U$ and $\Theta + \frac{1}{\rho}U$

$$= \frac{1}{\rho^2}\,.\ \text{resultant of } \Theta \text{ and } U$$

$$= \frac{1}{\rho^2}\begin{vmatrix} L, & 2M, & N, & 0 \\ 0, & L, & 2M, & N \\ E, & 2F, & G, & 0 \\ 0, & E, & 2F, & G \end{vmatrix}$$

$$= \frac{1}{\rho^2}\{(LG - 2MF + NE)^2 - 4(EG - F^2)(LN - M^2)\}$$

$$= \frac{1}{\rho^2}(\overline{H}^2 - 4\overline{K})V^4.$$

Hence the discriminant of $\Theta^2 - \frac{1}{\rho^2} U^2$ is

$$V^{12}(\bar{H}^2 - 4\bar{K})^2 \left\{\frac{1}{\rho^4} - \frac{1}{\rho^2}(\bar{H}^2 - 2\bar{K}) + \bar{K}^2\right\} \frac{1}{\rho^4},$$

agreeing with the result otherwise derived.

Locus of the centres of circular curvature of geodesics through a point.

242. Given a surface in homaloidal triple space, the centre of circular curvature of any superficial geodesic through a point O lies on the normal to the surface, whatever be the direction of the geodesic at O; and for different directions through O, the centre ranges between C_1 and C_2, the two principal centres of curvature of the surface at O. When the surface is synclastic, the locus of the geodesic centre is the portion of the normal lying between C_1 and C_2, this portion being described twice each way for the full tale of directions round O. When the surface is anticlastic, the specified locus is the remainder of the normal outside the portion between C_1 and C_2: the description of the locus can be taken from C_1 to positive infinity (for a positive radius of curvature), from negative infinity to C_2 and back from C_2 to negative infinity (for a negative radius of curvature), and then from positive infinity back to C_1 (again for a positive radius of curvature), thus completing the tale of directions round O.

It is natural to investigate the locus of the centre of circular curvature of geodesics through a point on a surface, for all the superficial directions through the point, when the surface is given in homaloidal quadruple space.

For a geodesic on such a surface, determined by the direction p', q', through the point O, let x_g, y_g, z_g, v_g, be the centre of circular curvature, so that

$$x_g - x = l\rho, \quad y_g - y = m\rho, \quad z_g - z = n\rho, \quad v_g - v = k\rho,$$

then we have

$$\left.\begin{aligned}
\frac{x_g - x}{\rho^2} &= \frac{l}{\rho} = \xi_{11}p'^2 + 2\xi_{12}p'q' + \xi_{22}q'^2 \\
\frac{y_g - y}{\rho^2} &= \frac{m}{\rho} = \eta_{11}p'^2 + 2\eta_{12}p'q' + \eta_{22}q'^2 \\
\frac{z_g - z}{\rho^2} &= \frac{n}{\rho} = \zeta_{11}p'^2 + 2\zeta_{12}p'q' + \zeta_{22}q'^2 \\
\frac{v_g - v}{\rho^2} &= \frac{k}{\rho} = \upsilon_{11}p'^2 + 2\upsilon_{12}p'q' + \upsilon_{22}q'^2
\end{aligned}\right\}.$$

For a clearer indication of the curve, it is convenient to change t axes in the quadruple space. We take new axes of Z and V, so t

$$\Sigma(\bar{x} - x)x_1 = ZE^{\frac{1}{2}}, \quad \Sigma(\bar{x} - x)x_2 = VG^{\frac{1}{2}}.$$

Now

$$\Sigma x_l \xi_{11} = 0, \quad \Sigma x_l \xi_{12} = 0, \quad \Sigma x_l \xi_{22} = 0,$$

for $l = 1, 2$; and therefore

$$\frac{1}{\rho^2} \Sigma x_1 (x_g - x) = 0, \quad \frac{1}{\rho^2} \Sigma x_2 (x_g - x) = 0.$$

Hence, as two of the equations of the required locus, we have

$$Z = 0, \quad V = 0,$$

in accordance with the known proposition that all the principal normals of the geodesics lie in the orthogonal plane.

Again, in that plane, two distinct directions are given by $\xi_{11}, \eta_{11}, \zeta_{11}, \upsilon_{11}$, and $\xi_{22}, \eta_{22}, \zeta_{22}, \upsilon_{22}$; so we take axes of X and of Y, such that

$$X \mathbf{a}^{\frac{1}{2}} = \Sigma \xi_{11} (\bar{x} - x), \quad Y \mathbf{c}^{\frac{1}{2}} = \Sigma \xi_{22} (\bar{x} - x).$$

Moreover, for $\theta = \xi, \eta, \zeta, \upsilon$, we have (§ 214)

$$S\theta_{12} = T\theta_{11} + R\theta_{22},$$

and therefore

$$U = \Sigma \xi_{12} (\bar{x} - x) = \frac{1}{S} \{ T \Sigma \xi_{11} (\bar{x} - x) + R \Sigma \xi_{22} (\bar{x} - x) \}$$

$$= X \frac{T}{S} \mathbf{a}^{\frac{1}{2}} + Y \frac{R}{S} \mathbf{c}^{\frac{1}{2}}.$$

Now by direct substitution, we have

$$\Sigma \xi_{11} (x_g - x) = \rho^2 (\mathbf{a} p'^2 + 2\mathbf{h} p' q' + \mathbf{g} q'^2) = L\rho,$$

$$\Sigma \xi_{12} (x_g - x) = \rho^2 (\mathbf{h} p'^2 + 2\mathbf{b} p' q' + \mathbf{f} q'^2) = M\rho,$$

$$\Sigma \xi_{22} (x_g - x) = \rho^2 (\mathbf{g} p'^2 + 2\mathbf{f} p' q' + \mathbf{c} q'^2) = N\rho,$$

and therefore for the locus in question, now manifestly a curve in the XY plane, we have

$$\left.\begin{aligned} X \mathbf{a}^{\frac{1}{2}} = L\rho &= \rho^2 (\mathbf{a} p'^2 + 2\mathbf{h} p' q' + \mathbf{g} q'^2) \\ X \frac{T}{S} \mathbf{a}^{\frac{1}{2}} + Y \frac{R}{S} \mathbf{c}^{\frac{1}{2}} = U = M\rho &= \rho^2 (\mathbf{h} p'^2 + 2\mathbf{b} p' q' + \mathbf{f} q'^2) \\ Y \mathbf{c}^{\frac{1}{2}} = N\rho &= \rho^2 (\mathbf{g} p'^2 + 2\mathbf{f} p' q' + \mathbf{c} q'^2) \end{aligned}\right\}$$

First, for the locus itself, we take

$$\begin{aligned} X \mathbf{a}^{\frac{1}{2}} p'^2 + 2U p' q' + Y \mathbf{c}^{\frac{1}{2}} q'^2 &= \rho^2 \Sigma \mathbf{a} p'^4 \\ &= 1 \\ &= E p'^2 + 2F p' q' + G q'^2, \end{aligned}$$

so that

$$(X \mathbf{a}^{\frac{1}{2}} - E) p'^2 + 2 (U - F) p' q' + (Y \mathbf{c}^{\frac{1}{2}} - G) q'^2 = 0.$$

Also,

$$X \mathbf{a}^{\frac{1}{2}} (\mathbf{g} p'^2 + 2\mathbf{f} p' q' + \mathbf{c} q'^2) = Y \mathbf{c}^{\frac{1}{2}} (\mathbf{a} p'^2 + 2\mathbf{h} p' q' + \mathbf{g} q'^2).$$

$$(\mathbf{g}X\mathbf{a}^{\frac{1}{2}} - \mathbf{a}Y\mathbf{c}^{\frac{1}{2}})\,p'^2 + 2\,(\mathbf{f}X\mathbf{a}^{\frac{1}{2}} - \mathbf{h}Y\mathbf{c}^{\frac{1}{2}})\,p'q' + (\mathbf{c}X\mathbf{a}^{\frac{1}{2}} - \mathbf{g}Y\mathbf{c}^{\frac{1}{2}})\,q'^2 = 0.$$

We have to eliminate p' and q' between these two equations. Let

$$\left.\begin{aligned}
\lambda &= (U - F)(\mathbf{c}X\mathbf{a}^{\frac{1}{2}} - \mathbf{g}Y\mathbf{c}^{\frac{1}{2}}) - (Y\mathbf{c}^{\frac{1}{2}} - G)(\mathbf{f}X\mathbf{a}^{\frac{1}{2}} - \mathbf{h}Y\mathbf{c}^{\frac{1}{2}}) \\
&= \frac{T}{S}\,\mathbf{ac}\delta^2 - (\mathbf{c}F - \mathbf{f}G)\,X\mathbf{a}^{\frac{1}{2}} + (\mathbf{g}F - \mathbf{h}G)\,Y\mathbf{c}^{\frac{1}{2}} \\
\mu &= (Y\mathbf{c}^{\frac{1}{2}} - G)\,(\mathbf{g}X\mathbf{a}^{\frac{1}{2}} - \mathbf{a}Y\mathbf{c}^{\frac{1}{2}}) - (X\mathbf{a}^{\frac{1}{2}} - E)\,(\mathbf{c}X\mathbf{a}^{\frac{1}{2}} - \mathbf{g}Y\mathbf{c}^{\frac{1}{2}}) \\
&= -\,\mathbf{ac}\delta^2 - (\mathbf{g}G - \mathbf{c}E)\,X\mathbf{a}^{\frac{1}{2}} + (\mathbf{a}G - \mathbf{g}E)\,Y\mathbf{c}^{\frac{1}{2}} \\
\nu &= (X\mathbf{a}^{\frac{1}{2}} - E)\,(\mathbf{f}X\mathbf{a}^{\frac{1}{2}} - \mathbf{h}Y\mathbf{c}^{\frac{1}{2}}) - (U - F)\,(\mathbf{g}X\mathbf{a}^{\frac{1}{2}} - \mathbf{a}Y\mathbf{c}^{\frac{1}{2}}) \\
&= \frac{R}{S}\,\mathbf{ac}\delta^2 - (\mathbf{f}E - \mathbf{g}F)\,X\mathbf{a}^{\frac{1}{2}} + (\mathbf{h}E - \mathbf{a}F)\,Y\mathbf{c}^{\frac{1}{2}}
\end{aligned}\right\},$$

where

$$\delta^2 = X^2 + Y^2 + 2XY\cos\varpi, \quad \cos\varpi = -\frac{\mathbf{g}}{(\mathbf{ac})^{\frac{1}{2}}},$$

ϖ being the angle between the axes of X and Y in their plane. Then the eliminant in question is

$$\mu^2 - 4\lambda\nu = 0,$$

a curve of order four.

For this curve, the terms of the fourth order are

$$\mathbf{a}^2\mathbf{c}^2\left(1 - \frac{4RT}{S^2}\right)(X^2 + Y^2 + 2XY\cos\varpi)^2;$$

the terms of the third order are

$$\mathbf{ac}\,(X^2 + Y^2 + 2XY\cos\varpi)\,u_1,$$

where u_1 is linear and homogeneous in X and Y; and the remaining terms are of the form

$$u_2,$$

where u_2 is homogeneous of the second order in X and Y. Also ϖ is the angle between the coordinate axes. Hence the locus is of the lemniscate type; it has a double point (real or imaginary) at the origin, where the tangents are given by $u_2 = 0$.

243. Next, proceeding from the equations

$$X\mathbf{a}^{\frac{1}{2}} = L\rho, \quad U = M\rho, \quad Y\mathbf{c}^{\frac{1}{2}} = N\rho,$$

we have

$$XY(\mathbf{ac})^{\frac{1}{2}} - U^2 = (LN - M^2)\,\rho^2,$$

$$EY\mathbf{c}^{\frac{1}{2}} - 2FU + GX\mathbf{a}^{\frac{1}{2}} = (EN - 2FM + GL)\,\rho.$$

Hence

$$\begin{aligned}
\{XY(\mathbf{ac})^{\frac{1}{2}} - U^2\} - (EY\mathbf{c}^{\frac{1}{2}} - 2FU + GX\mathbf{a}^{\frac{1}{2}}) + V^2 \\
= (LN - M^2)\,\rho^2 - (EN - 2FM + GL)\,\rho + V^2 \\
= -\,V^2\frac{\rho^2}{\sigma^2},
\end{aligned}$$

by the property already (§ 236) established. Now for the principal directions of curvature at O, we have (§ 237)

$$\frac{1}{\sigma} = 0;$$

hence the coordinates of the four principal centres of curvature of geodesics satisfy the equation

$$\{XY(\mathbf{ac})^{\frac{1}{2}} - U^2\} - (EY\mathbf{c}^{\frac{1}{2}} - 2FU + GX\mathbf{a}^{\frac{1}{2}}) + V^2 = 0,$$

obviously a conic in the plane of the locus.

This conic will re-appear, later (§ 251), from another source.

The four principal directions of curvature yield a maximum or a minimum radius of curvature (§ 238); and therefore the radius, in each of the four instances, is a maximum or a minimum distance from O to the lemniscate—that is, the line is a normal to the curve. Thus the four centres of principal curvature of geodesics through the point are, at once, the feet of the four normals from the point to the lemniscate and also the intersections of the lemniscate with the foregoing conic in the XY plane. And it will be found that the lemniscate and the conic touch at these four points.

Centre of spherical curvature of a geodesic.

244. We know that the centre of spherical curvature of a curve is the intersection of its osculating flat by three consecutive normal flats (§ 143). When we have to deal with a geodesic on the surface, the centre of spherical curvature can be obtained as follows.

It is known (§ 142) that the centre of spherical curvature of a curve lies on a line, through the centre of circular curvature and drawn parallel to the binormal. In the case of a geodesic, the direction-cosines of the binormal (p. 412) are λ, μ, ν, κ; and therefore the equations of the line, joining the centre of circular curvature and the centre of spherical curvature, are

$$\left.\begin{aligned} \bar{x} - x &= l\rho + \lambda u \\ \bar{y} - y &= m\rho + \mu u \\ \bar{z} - z &= n\rho + \nu u \\ \bar{v} - v &= k\rho + \kappa u \end{aligned}\right\},$$

where u is the current parameter of the line.

The normal flat at the point is

$$\Sigma(\bar{x} - x)x' = 0.$$

Because $\Sigma x'l = 0$, $\Sigma x'\lambda = 0$, the line lies in this flat.

The intersection of the normal flat by a consecutive normal flat is a normal plane of the curve though it is not the orthogonal plane of the surface; it is represented by the two equations

$$\Sigma(\bar{x} - x)x' = 0, \quad \Sigma(\bar{x} - x)l = \rho.$$

Because $\Sigma l\lambda = 0$, the foregoing line lies in this normal plane of the geodesic.

The intersection of a third normal flat, consecutive to the first two, with those first two, is given by associating, with the preceding pair of equations, a third equation

$$\Sigma (\bar{x} - x) l' = \Sigma x' l + \rho' = \rho',$$

because $\Sigma x' l = 0$. Thus the point, where the foregoing line (already known to lie in the intersection of the first two normal flats) intersects the third consecutive normal flat, is given by

$$\Sigma (l\rho + \lambda u) l' = \rho'.$$

Now

$$\Sigma l l' = 0.$$

Also, the direction-cosines of the binormal to the geodesic are given, by λ, μ, ν, κ, as lying on the surface, and (§ 137) by four expressions typified by

$$\frac{\sigma}{\rho} (x' + \rho\rho' x'' + \rho^2 x'''),$$

that is, as $\rho x'' = l$, typified by

$$\frac{\sigma}{\rho} (x' + \rho l');$$

thus, as in § 234, we have four relations of the type

$$\lambda = \frac{\sigma}{\rho} (x' + \rho l').$$

Consequently, we have

$$\Sigma \lambda l' = \frac{\sigma}{\rho} \{\Sigma x' l' + \rho \Sigma l'^2\}.$$

Now, because $\Sigma x' l = 0$, we have

$$\Sigma x' l' = - \Sigma x'' l = - \frac{1}{\rho} \Sigma l^2 = - \frac{1}{\rho}.$$

Also, from one of the sets of the Frenet equations represented by

$$\frac{dc_2}{ds} = - \frac{c_1}{\rho} + \frac{c_3}{\sigma},$$

(where $c_2 = l$, $c_1 = x'$, in general), we have

$$\Sigma l'^2 = \frac{1}{\rho^2} + \frac{1}{\sigma^2}.$$

Hence

$$\Sigma \lambda l' = \frac{1}{\sigma};$$

and therefore the equation for the determination of u is

$$\frac{1}{\sigma} u = \rho',$$

that is,

$$u = \sigma \rho'.$$

We thus obtain the customary expressions (§ 142) for the centre of spherical curvature of the geodesic, regarded as a curve in quadruple space.

The association of the osculating flat of the surface along a geodesic is immediate. Its equation is (§ 226)

$$\begin{vmatrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ x_1\,, & y_1\,, & z_1\,, & v_1 \\ x_2\,, & y_2\,, & z_2\,, & v_2 \\ l\,, & m\,, & n\,, & k \end{vmatrix} = 0,$$

that is,

$$\Sigma A\,(\bar{x}-x) = 0.$$

Now

$$\Sigma Al = 0, \quad \Sigma A\lambda = 0;$$

and therefore

$$\Sigma A\,(l\rho + \lambda u) = 0,$$

that is, the binormal lies in the osculating flat in question, while the centre of spherical curvature is the intersection of the osculating flat with the three consecutive normal flats.

Centre of globular curvature of a geodesic.

245. The centre of globular curvature of the geodesic on the surface can similarly be obtained. It lies (§ 147) on a line, through the centre of spherical curvature and having the direction-cosines of the fourth axis of the orthogonal frame of the curve, hence, if ∇ denote its distance from the centre of spherical curvature along that line, the coordinates of the centre of spherical curvature are

$$\left.\begin{aligned} \bar{x}-x &= l\rho + \lambda\sigma\rho' + A\nabla \\ \bar{y}-y &= m\rho + \mu\sigma\rho' + B\nabla \\ \bar{z}-z &= n\rho + \nu\sigma\rho' + C\nabla \\ \bar{v}-v &= k\rho + \kappa\sigma\rho' + D\nabla \end{aligned}\right\},$$

where ∇ has to be determined. On the other hand, the centre of globular curvature of a curve is (§ 148) the intersection of four consecutive normal flats, which (when combined for purposes of intersection) are together represented by the four equations

$$\begin{aligned} &\Sigma\,(\bar{x}-x)\,x' = 0\,, \\ &\Sigma\,(\bar{x}-x)\,l\ = \rho\,, \\ &\Sigma\,(\bar{x}-x)\,l'\ = \rho', \\ &\Sigma\,(\bar{x}-x)\,l'' = \rho'' + \Sigma x'l' = \rho'' - \frac{1}{\rho}. \end{aligned}$$

Let the foregoing values of $\bar{x}-x$, $\bar{y}-y$, $\bar{z}-z$, $\bar{v}-v$, be substituted in these equations in succession, three of the Frenet equations being

$$c_1' = \frac{c_2}{\rho}, \quad c_2' = -\frac{c_1}{\rho} + \frac{c_3}{\sigma}, \quad c_3' = -\frac{c_2}{\sigma} + \frac{c_4}{\tau},$$

where, typically,

$$c_1 = x', \quad c_2 = l, \quad c_3 = \lambda, \quad c_4 = A.$$

The first equation $\Sigma(\bar{x}-x)x'=0$ is satisfied, because

$$\Sigma x'l=0,\quad \Sigma x'\lambda=0,\quad \Sigma x'A=0.$$

The second equation is satisfied, because

$$\Sigma l^2=1,\quad \Sigma l\lambda=0,\quad \Sigma lA=0.$$

As regards the third equation, we have proved (§ 237) that

$$\Sigma\lambda l'=\frac{1}{\sigma};$$

from the second of the quoted Frenet equations, we have

$$\Sigma Al'=\Sigma l_4 l_2'=-\frac{1}{\rho}\Sigma l_1 l_4+\frac{1}{\sigma}\Sigma l_1 l_3=0;$$

and $\Sigma ll'=0$. Thus the third equation is satisfied.

For the fourth equation, we need the values of l'', m'', n'', k'', they can be derived from the Frenet equations. We have, typically, $c_2=l$; and therefore the second Frenet equation gives

$$\begin{aligned} c_2''&=-\frac{c_1'}{\rho}+\frac{c_3'}{\sigma}+\frac{c_1\rho'}{\rho^2}-\frac{c_3\sigma'}{\sigma^2}\\ &=c_1\frac{\rho'}{\rho^2}-c_2\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)-c_3\frac{\sigma'}{\sigma^2}+c_4\frac{1}{\sigma\tau}.\end{aligned}$$

Hence

$$\begin{aligned} \Sigma ll''&=\Sigma l_2 l_2''=-\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right),\\ \Sigma\lambda l''&=\Sigma l_3 l_2''=-\frac{\sigma'}{\sigma^2},\\ \Sigma Al''&=\Sigma l_4 l_2''=\quad\frac{1}{\sigma\tau};\end{aligned}$$

and therefore the fourth equation determines ∇ by the relation

$$\rho''-\frac{1}{\rho}=-\rho\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)-\frac{\rho'\sigma'}{\sigma}+\frac{\nabla}{\sigma\tau},$$

that is,

$$\nabla=\tau\frac{d}{ds}(\sigma\rho')+\frac{\rho\tau}{\sigma}.$$

We thus have the coordinates of the centre of globular curvature of the geodesic on the surface, the quantities ρ, σ, τ, being the respective radii of circular curvature, of torsion, and of tilt of the geodesic through the direction p', q', with the values already (§§ 221, 234, 235) obtained.

The centre of spherical curvature is (§ 235) given by four equations of the form

$$\bar{x}-x=l\rho+\lambda\sigma\rho';$$

and therefore the radius of spherical curvature R is given by the usual formula

$$R^2=\rho^2+\sigma^2\rho'^2.$$

The foregoing value of ∇ is easily transformed to

$$\nabla = R\,\frac{\tau R'}{\sigma \rho'};$$

and so we verify the formula (§§ 149, 152) for the radius of globular curvature

$$\begin{aligned} S^2 &= R^2 + \nabla^2 \\ &= \rho^2 + \sigma^2 \rho'^2 + \tau^2 \left\{ \frac{\rho}{\sigma} + \frac{d}{ds}(\sigma \rho') \right\}^2 \\ &= R^2 \left\{ 1 + \left(\frac{\tau R'}{\sigma \rho'} \right)^2 \right\}. \end{aligned}$$

CHAPTER XIV.

CURVATURE OF SURFACES IN SPACE.

Line of intersection of an orthogonal plane by the consecutive normal flat.

246. The foregoing methods are the manifest extension, to a surface freely set in homaloidal quadruple space, of the methods adopted for a surface in homaloidal triple space.

There is an entirely different method of proceeding, the results of which complement the foregoing results. We have seen that, whatever direction be chosen in the tangent plane at x, y, z, v, the principal normal of the superficial geodesic through that direction lies in the orthogonal plane, the equations of which can be taken in the form

$$\left.\begin{aligned} \Sigma(\bar{x}-x)x_1 &= 0 \\ \Sigma(\bar{x}-x)x_2 &= 0 \end{aligned}\right\}$$

obviously lying in the normal flat

$$\Sigma(\bar{x}-x)x' = \Sigma(\bar{x}-x)(x_1p'+x_2q') = 0.$$

But we know two directions l, m, n, k, and A, B, C, D, each of which is perpendicular to the directions x_1, y_1, z_1, v_1, and x_2, y_2, z_2, v_2; they arise out of the orthogonal frame of the geodesic. Hence the equations of the orthogonal plane of the surface can be taken

$$\left\| \begin{matrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ l\ , & m\ , & n\ , & k \\ A\ , & B\ , & C\ , & D \end{matrix} \right\| = 0,$$

and consequently any point lying in the orthogonal plane can have its coordinates expressed in the forms

$$\begin{aligned} \bar{x}-x &= l\Lambda_1 + A\Lambda_2, & \bar{y}-y &= m\Lambda_1 + B\Lambda_2, \\ \bar{z}-z &= n\Lambda_1 + C\Lambda_2, & \bar{v}-v &= k\Lambda_1 + D\Lambda_2, \end{aligned}$$

where Λ_1 and Λ_2 are parameters.

These two forms of the equations of the orthogonal plane are equivalent to one another, and can be interchanged (paired together, of course).

The normal flat at a consecutive point along the direction p', q', at a small distance δ from x, y, z, v, is given by the equation

$$\Sigma\left\{\bar{x}-x-(x_1p'+x_2q')\delta\right\}\left\{x_1p'+x_2q'+x''\delta\right\} = 0,$$

where

$$x'' = x_1p'' + x_2q'' + x_{11}p'^2 + 2x_{12}p'q' + x_{22}q'^2.$$

This flat intersects the foregoing orthogonal plane in a line, because the intersection of any flat and any plane is a line. Of this line, two of the equations are those of the orthogonal plane itself; a simplified third equation for the line is derived by using these equations to modify the equations of the consecutive normal flat. Thus, along the line of intersection, we have

$$(p'+p''\delta)\{\Sigma(\bar{x}-x)x_1\}=0,\quad (q'+q''\delta)\{\Sigma(\bar{x}-x)x_2\}=0.$$

Moreover, for every curve, we have $\Sigma x'x''=0$, and therefore

$$\Sigma(x_1p'+x_2q')(x_1p''+x_2q''+x_{11}p'^2+2x_{12}p'q'+x_{22}q'^2)=0.$$

Using these results to modify the equation of the consecutive normal flat, we have a third equation of the line in the form

$$\{\Sigma(\bar{x}-x)(x_{11}p'^2+2x_{12}p'q'+x_{22}q'^2)\}\,\delta=\{\Sigma(x_1p'+x_2q')^2\}\,\delta,$$

that is,

$$\Sigma(\bar{x}-x)(x_{11}p'^2+2x_{12}p'q'+x_{22}q'^2)=1.$$

But, from the equations of the plane, we have

$$(\Gamma p'^2+2\Gamma'p'q'+\Gamma''q'^2)\{\Sigma(\bar{x}-x)x_1\}=0,$$

$$(\Delta p'^2+2\Delta'p'q'+\Delta''q'^2)\{\Sigma(\bar{x}-x)x_2\}=0;$$

and therefore the third equation of the line becomes

$$\Sigma(\bar{x}-x)\{(x_{11}-x_1\Gamma-x_2\Delta)p'^2+2(x_{12}-x_1\Gamma'-x_2\Delta')p'q' + (x_{22}-x_1\Gamma''-x_2\Delta'')q'^2\}=1,$$

that is,

$$\Sigma(\bar{x}-x)\,l=\rho,$$

where l, m, n, k, ρ, belong to the geodesic at the initial point.

This third equation can be obtained more briefly by taking the equation of the consecutive normal flat in the form

$$\Sigma(\bar{x}-x-x'\delta)(x'+x''\delta)=0,$$

which, when combined with the equation of the initial flat, gives

$$\Sigma(\bar{x}-x)\,x''=\Sigma x'^2=1,$$

that is, again the equation

$$\Sigma(\bar{x}-x)\,l=\rho.$$

Thus the three equations of the line of intersection of the initial orthogonal plane by the consecutive normal flat are

$$\Sigma(\bar{x}-x)x_1=0,\quad \Sigma(\bar{x}-x)x_2=0,\quad \Sigma(\bar{x}-x)\,l=\rho.$$

But

$$\Sigma lx_1=0,\quad \Sigma lx_2=0,\quad \Sigma l^2=1;$$

and therefore the equations can be taken in the form

$$\Sigma\{(\bar{x}-x-l\rho)x_1\}=0,\quad \Sigma\{(\bar{x}-x-l\rho)x_2\}=0,\quad \Sigma\{(\bar{x}-x-l\rho)\,l\}=0.$$

When these three equations, linear and homogeneous in the four quantities of the type $\bar{x} - x - l\rho$, are resolved, they become

$$\frac{\bar{x} - x - l\rho}{A} = \frac{\bar{y} - y - m\rho}{B} = \frac{\bar{z} - z - n\rho}{C} = \frac{\bar{v} - v - k\rho}{D}.$$

Thus the intersection of the orthogonal plane by a consecutive normal flat is a straight line, which passes through the centre of circular curvature of the geodesic at the point and is parallel to the fourth axis of the orthogonal frame, that is, in a direction perpendicular to the tangent plane of the surface and perpendicular also to the principal normal of the geodesic.

It may be remarked that this line (which, of course, lies in the orthogonal plane) is the locus of the points

$$\bar{x} = x + l\Lambda_1 + A\Lambda_2,$$

with like expressions for $\bar{y}$, $\bar{z}$, $\bar{v}$, in the orthogonal plane, when the parameter Λ_1 is specifically equal to ρ, and Λ_2 remains the current parameter of the line.

NOTE. Various forms can be given to the equations of the orthogonal plane, two of which already have occurred, viz., the form

$$\left.\begin{aligned}\Sigma(\bar{x} - x)\, x_1 = 0\\ \Sigma(\bar{x} - x)\, x_2 = 0\end{aligned}\right\}$$

and the form

$$\begin{Vmatrix} \bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\ l\ , & m\ , & n\ , & k \\ A\ , & B\ , & C\ , & D \end{Vmatrix} = 0.$$

We have had relations

$$\Sigma x_1 l = 0, \quad \Sigma x_1 \xi_{11} = 0, \quad \Sigma x_1 \xi_{22} = 0,$$

so that

$$\frac{x_1}{\begin{vmatrix} m\ , & n\ , & k \\ \eta_{11}, & \zeta_{11}, & \upsilon_{11} \\ \eta_{22}, & \zeta_{22}, & \upsilon_{22} \end{vmatrix}} = \frac{y_1}{-\begin{vmatrix} n\ , & k\ , & l \\ \zeta_{11}, & \upsilon_{11}, & \xi_{11} \\ \zeta_{22}, & \upsilon_{22}, & \xi_{22} \end{vmatrix}} = \frac{z_1}{\begin{vmatrix} k\ , & l\ , & m \\ \upsilon_{11}, & \xi_{11}, & \eta_{11} \\ \upsilon_{22}, & \xi_{22}, & \eta_{22} \end{vmatrix}} = \frac{v_1}{-\begin{vmatrix} l\ , & m, & n \\ \xi_{11}, & \eta_{11}, & \zeta_{11} \\ \xi_{22}, & \eta_{22}, & \zeta_{22} \end{vmatrix}}.$$

Similarly we have had relations

$$\Sigma x_2 l = 0, \quad \Sigma x_2 \xi_{11} = 0, \quad \Sigma x_2 \xi_{22} = 0,$$

so that x_2, y_2, z_2, v_2, are formally in the same ratios as x_1, y_1, z_1, v_1. But, in fact, these ratios are not the same; hence

$$\begin{Vmatrix} l\ , & m, & n\ , & k \\ \xi_{11}, & \eta_{11}, & \zeta_{11}, & \upsilon_{11} \\ \xi_{22}, & \eta_{22}, & \zeta_{22}, & \upsilon_{22} \end{Vmatrix} = 0.$$

Also, from the central equations typified by

$$\xi_{11} = Ll + \Omega A, \quad \xi_{22} = \Lambda l + \Omega'' A,$$

we have

$$\begin{Vmatrix} A, & B, & C, & D \\ \xi_{11}, & \eta_{11}, & \zeta_{11}, & \upsilon_{11} \\ \xi_{22}, & \eta_{22}, & \zeta_{22}, & \upsilon_{22} \end{Vmatrix} = 0.$$

Thus the equations of the orthogonal plane can be taken

$$\left\Vert \begin{matrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ \xi_{11}\ , & \eta_{11}\ , & \zeta_{11}\ , & \nu_{11} \\ \xi_{22}\ , & \eta_{22}\ , & \zeta_{22}\ , & \nu_{22} \end{matrix} \right\Vert = 0.$$

This form is an expression of the property that the orthogonal plane contains the principal normals to all the geodesics through x, y, z, v, and therefore, in particular, contains the principal normals to the geodesics touching the direction q=constant and the direction p=constant.

Intersection of two consecutive orthogonal planes: first form of result.

247. Thus far, we have taken account only of the intersection of the orthogonal plane by the consecutive normal flat. Now let account be taken of the intersection of the orthogonal plane by the consecutive orthogonal plane. As two planes usually intersect in a point, this point of intersection will be a point (usually a specific point) on the foregoing line, so that we shall have, usually, a specific value for Λ_2.

Instead of utilising the equations of the line, we can obtain the result otherwise, as follows. The equations of the original orthogonal plane are

$$\left.\begin{aligned} \Sigma(\bar{x}-x)x_1 &= 0 \\ \Sigma(\bar{x}-x)x_2 &= 0 \end{aligned}\right\},$$

and those of a consecutive orthogonal plane are

$$\Sigma\{\bar{x}-x-(x_1p'+x_2q')\delta\}\{x_1+(x_{11}p'+x_{12}q')\delta\}=0,$$
$$\Sigma\{\bar{x}-x-(x_1p'+x_2q')\delta\}\{x_2+(x_{12}p'+x_{22}q')\delta\}=0,$$

which are

$$\Sigma(\bar{x}-x)x_1+\{\Sigma(\bar{x}-x)(x_{11}p'+x_{12}q')-(Ep'+Fq')\}\delta=0,$$
$$\Sigma(\bar{x}-x)x_2+\{\Sigma(\bar{x}-x)(x_{12}p'+x_{22}q')-(Fp'+Gq')\}\delta=0.$$

Hence the two equations, which are to be combined with the equations of the initial orthogonal plane in order to provide its point of intersection with a consecutive orthogonal plane, are

$$\left.\begin{aligned} \Sigma(\bar{x}-x)(x_{11}p'+x_{12}q') &= Ep'+Fq' \\ \Sigma(\bar{x}-x)(x_{12}p'+x_{22}q') &= Fp'+Gq' \end{aligned}\right\}.$$

To obtain the coordinates of the point of intersection in question, we now take the equations of the initial orthogonal plane in the form

$$\left\Vert \begin{matrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ l\ , & m\ , & n\ , & k \\ A\ , & B\ , & C\ , & D \end{matrix} \right\Vert = 0,$$

which are also expressible by equations of the type

$$\bar{x}=x+l\Lambda_1+A\Lambda_2,$$

with Λ_1 and Λ_2 as the parameters of the plane. For the required point of intersection, these parameters must be determined so that the coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{v}$, satisfy the two additional equations

$$\Sigma(\bar{x}-x)(x_{11}p'+x_{12}q') = Ep'+Fq',$$

$$\Sigma(\bar{x}-x)(x_{12}p'+x_{22}q') = Fp'+Gq'.$$

Accordingly, let the foregoing values of $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{v}$, be substituted. We have

$$\Sigma l x_{11} = L, \quad \Sigma l x_{12} = M, \quad \Sigma l x_{22} = N$$

$$\Sigma A x_{11} = \Omega, \quad \Sigma A x_{12} = \Omega', \quad \Sigma A x_{22} = \Omega'',$$

consequently the equations become

$$(Lp' + Mq')\Lambda_1 + (\Omega p' + \Omega' q')\Lambda_2 = Ep' + Fq',$$

$$(Mp' + Nq')\Lambda_1 + (\Omega' p' + \Omega'' q')\Lambda_2 = Fp' + Gq'.$$

Multiply the two equations by p', q', and add: then, because

$$\rho(Lp'^2 + 2Mp'q' + Nq'^2) = 1,$$

$$\Omega p'^2 + 2\Omega' p'q' + \Omega'' q'^2 = 0,$$

$$Ep'^2 + 2Fp'q' + Gq'^2 = 1,$$

we at once have

$$\Lambda_1 = \rho,$$

as is to be expected.

Again, from equations already (§ 230) established, we have

$$\Omega p' + \Omega' q' = q'\rho W, \quad \Omega' p' + \Omega'' q' = -p'\rho W,$$

where

$$W = (\mathbf{ab} - \mathbf{h}^2)^{\frac{1}{2}} p'^2 + (\mathbf{ca} - \mathbf{g}^2)^{\frac{1}{2}} p'q' + (\mathbf{bc} - \mathbf{f}^2)^{\frac{1}{2}} q'^2,$$

so, now, the equations are

$$\rho(Lp' + Mq') + \Lambda_2 W\rho q' = Ep' + Fq',$$

$$\rho(Mp' + Nq') - \Lambda_2 W\rho p' = Fp' + Gq'.$$

Consequently,

$$\Lambda_2 W = \begin{vmatrix} Ep' + Fq', & Fp' + Gq' \\ Lp' + Mq', & Mp' + Nq' \end{vmatrix}$$

$$= -\Upsilon,$$

using the notation of § 234. We thus have a value of Λ_2.

We have seen (§§ 234, 235) that, if $1/\sigma$ and $1/\tau$ are the torsion and the tilt of the geodesic through the direction p', q', their values are given by

$$\frac{1}{\sigma} = \frac{1}{V}\Upsilon, \quad \frac{1}{\rho\tau} = -\frac{W}{V};$$

hence we have two alternative expressions for Λ_2, viz.

$$\Lambda_2 = -\frac{\Upsilon}{W}, \quad \Lambda_2 = \frac{\rho\tau}{\sigma}.$$

To the first of these values we shall return later. Meanwhile, *the coordinates of the intersection of two consecutive orthogonal planes* have been obtained in the form

$$\left.\begin{aligned} \bar{x} &= x + l\rho + A\frac{\rho\tau}{\sigma} \\ \bar{y} &= y + m\rho + B\frac{\rho\tau}{\sigma} \\ \bar{z} &= z + n\rho + C\frac{\rho\tau}{\sigma} \\ \bar{v} &= v + k\rho + D\frac{\rho\tau}{\sigma} \end{aligned}\right\},$$

where $1/\rho$, $1/\sigma$, $1/\tau$ are the curvatures of the geodesic through the direction in the tangent plane.

This point is called the *orthogonal centre*; and the line joining the orthogonal centre to x', y', z', v', is called the *orthogonal radius*. There is an orthogonal centre, with an associated orthogonal radius, for every direction p', q', on the surface.

It will be noted that the orthogonal centre is usually distinct from the centre of circular curvature of the geodesic.

Length of an orthogonal radius.

248. Let ρ_0 be the length of an orthogonal radius; and let l_0, m_0, n_0, k_0, be its direction-cosines. Then we have

$$\left.\begin{aligned} l_0\rho_0 &= \bar{x} - x = l\rho + A\frac{\rho\tau}{\sigma} \\ m_0\rho_0 &= \bar{y} - y = m\rho + B\frac{\rho\tau}{\sigma} \\ n_0\rho_0 &= \bar{z} - z = n\rho + C\frac{\rho\tau}{\sigma} \\ k_0\rho_0 &= \bar{v} - v = k\rho + D\frac{\rho\tau}{\sigma} \end{aligned}\right\};$$

and therefore

$$\rho_0^2 = \rho^2 + \frac{\rho^2\tau^2}{\sigma^2}.$$

Also

$$\frac{\rho}{\rho_0} = \Sigma l l_0, \quad \frac{\rho\tau}{\sigma\rho_0} = \Sigma A l_0,$$

thus giving the respective inclinations of the orthogonal radius to the principal normal and to the trinormal of the geodesic, all these lines lying in the orthogonal plane.

Intersection of two consecutive orthogonal planes: second form of result: the locus of the orthogonal centre is a conic.

249. Before using the expressions obtained (§ 247) for the coordinates of the orthogonal centre, we obtain another form derived* by taking the equations of the two consecutive orthogonal planes, together, in the form

$$\left.\begin{aligned} \Sigma\,(\bar{x}-x)\,x_1 &= 0 \\ \Sigma\,(\bar{x}-x)\,x_2 &= 0 \\ \Sigma\,(\bar{x}-x)\,(x_{11}p'+x_{12}q') &= Ep'+Fq' \\ \Sigma\,(\bar{x}-x)\,(x_{12}p'+x_{22}q') &= Fp'+Gq' \end{aligned}\right\}.$$

The determinant of the coefficients of $\bar{x}-x$, $\bar{y}-y$, $\bar{z}-z$, $\bar{v}-v$, is

$$\begin{vmatrix} x_{11}p'+x_{12}q', & y_{11}p'+y_{12}q', & z_{11}p'+z_{12}q', & v_{11}p'+v_{12}q' \\ x_{12}p'+x_{22}q', & y_{12}p'+y_{22}q', & z_{12}p'+z_{22}q', & v_{12}p'+v_{22}q' \\ x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \end{vmatrix},$$

which is equal to

$$p'^2 \begin{vmatrix} x_{11}, & y_{11}, & z_{11}, & v_{11} \\ x_{12}, & y_{12}, & z_{12}, & v_{12} \\ x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \end{vmatrix} + p'q' \begin{vmatrix} x_{11}, & y_{11}, & z_{11}, & v_{11} \\ x_{22}, & y_{22}, & z_{22}, & v_{22} \\ x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \end{vmatrix} + q'^2 \begin{vmatrix} x_{12}, & y_{12}, & z_{12}, & v_{12} \\ x_{22}, & y_{22}, & z_{22}, & v_{22} \\ x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \end{vmatrix}.$$

The values of the coefficients of p'^2, $p'q'$, q'^2, have already (§ 232) been obtained: they are, respectively,

$$R, = V(\mathrm{ab}-\mathrm{h}^2)^{\frac{1}{2}}, \quad S, = V(\mathrm{ca}-\mathrm{g}^2)^{\frac{1}{2}}; \quad T, = V(\mathrm{bc}-\mathrm{f}^2)^{\frac{1}{2}}.$$

Thus the determinant in question is equal to

$$\begin{aligned} &Rp'^2+Sp'q'+Tq'^2 \\ &\qquad = V\{(\mathrm{ab}-\mathrm{h}^2)^{\frac{1}{2}}p'^2+(\mathrm{ca}-\mathrm{g}^2)^{\frac{1}{2}}p'q'+(\mathrm{bc}-\mathrm{f}^2)^{\frac{1}{2}}q'^2\} \\ &\qquad = VW, \end{aligned}$$

with the significance of W as defined in § 230. Hence

$$VW(\bar{x}-x) = \begin{vmatrix} Ep'+Fq', & y_{11}p'+y_{12}q', & z_{11}p'+z_{12}q', & v_{11}p'+v_{12}q' \\ Fp'+Gq', & y_{12}p'+y_{22}q', & z_{12}p'+z_{22}q', & v_{12}p'+v_{22}q' \\ 0, & y_1, & z_1, & v_1 \\ 0, & y_2, & z_2, & v_2 \end{vmatrix},$$

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[*] See the memoir by Kommerell, "Riemann'sche Flachen in ebenen Raum von vier Dimensionen," *Math. Ann.*, vol. LX (1905), pp. 548–596.

or, with the notation of § 231,

$$\left.\begin{aligned}
&VW(\bar{x}-x)=(Ep'+Fq')(e_{12}p'+e_{22}q')-(Fp'+Gq')(e_{11}p'+e_{12}q')\\
&\text{and, similarly,}\\
&VW(\bar{y}-y)=(Ep'+Fq')(f_{12}p'+f_{22}q')-(Fp'+Gq')(f_{11}p'+f_{12}q')\\
&VW(\bar{z}-z)=(Ep'+Fq')(g_{12}p'+g_{22}q')-(Fp'+Gq')(g_{11}p'+g_{12}q')\\
&VW(\bar{v}-v)=(Ep'+Fq')(h_{12}p'+h_{22}q')-(Fp'+Gq')(h_{11}p'+h_{12}q')
\end{aligned}\right\}.$$

These are the coordinates of the limiting position of the point of intersection of the orthogonal plane at O with the orthogonal plane at a consecutive point along a direction p', q', through O. This point is the *orthogonal centre* of the surface for that direction, and its distance from O is the *orthogonal radius* of the surface for that direction.

It will, however, be observed that this orthogonal radius does not (save exceptionally) coincide, either in magnitude or in direction, with the radius of circular curvature of the superficial geodesic through the direction.

One immediate inference can be made, as regards the locus of this orthogonal centre for different directions through the point x, y, z, v, of the surface.

For this locus, we have the equations

$$\Sigma(\bar{x}-x)x_1=0,\quad \Sigma(\bar{x}-x)x_2=0,$$

which represent the orthogonal plane. Also, it is convenient (for subsequent purposes) to modify the other two equations, in association with this pair, and on account of the relations

$$\xi_{11}=x_{11}-x_1\Gamma-x_2\Delta,\quad \xi_{12}=x_{12}-x_1\Gamma'-x_2\Delta',\quad \xi_{22}=x_{22}-x_1\Gamma''-x_2\Delta'',$$

they can be written

$$\left.\begin{aligned}
\Sigma(\bar{x}-x)(\xi_{11}p'+\xi_{12}q')&=Ep'+Fq'\\
\Sigma(\bar{x}-x)(\xi_{12}p'+\xi_{22}q')&=Fp'+Gq'
\end{aligned}\right\}.$$

Hence

$$[\{\Sigma(\bar{x}-x)\xi_{11}\}-E]p'+[\{\Sigma(\bar{x}-x)\xi_{12}\}-F]q'=0,$$

$$[\{\Sigma(\bar{x}-x)\xi_{12}\}-F]p'+[\{\Sigma(\bar{x}-x)\xi_{22}\}-G]q'=0;$$

consequently, the coordinates of any point on the locus satisfy the equation

$$\begin{vmatrix}
\{\Sigma(\bar{x}-x)\xi_{11}\}-E, & \{\Sigma(\bar{x}-x)\xi_{12}\}-F\\
\{\Sigma(\bar{x}-x)\xi_{12}\}-F, & \{\Sigma(\bar{x}-x)\xi_{22}\}-G
\end{vmatrix}=0,$$

which represents a configuration (a region) of the second degree. The locus in question is the section of this region by the foregoing orthogonal plane; it is therefore a curve of the second degree in that plane, that is, the locus is a conic.

A return, later (§§ 251, 252), will be made to the consideration of this conic.

Agreement of the two sets of expressions for the orthogonal centre.

250. It is important to verify that these expressions for the coordinates of the orthogonal centre are in analytical agreement with the forms in § 247, viz.

$$\bar{x} - x = l\rho + A\Lambda_2, \quad \bar{y} - y = m\rho + B\Lambda_2, \quad \bar{z} - z = n\rho + C\Lambda_2, \quad \bar{v} - v = k\rho + D\Lambda_2,$$

where $\Lambda_2 \sigma = \rho\tau$.

In the first place, because

$$\Sigma l x_1 = 0, \quad \Sigma l x_2 = 0, \quad \Sigma A x_1 = 0, \quad \Sigma A x_2 = 0,$$

the earlier equations give

$$\Sigma (\bar{x} - x) x_1 = 0, \quad \Sigma (\bar{x} - x) x_2 = 0.$$

On the other hand, we have at once

$$x_1 e_{lm} + y_1 f_{lm} + z_1 g_{lm} + v_1 h_{lm} = 0,$$

$$x_2 e_{lm} + y_2 f_{lm} + z_2 g_{lm} + v_2 h_{lm} = 0,$$

for the combinations $lm = 11$, 12, 22; hence the later equations also give

$$\Sigma (\bar{x} - x) x_1 = 0, \quad \Sigma (\bar{x} - x) x_2 = 0.$$

It therefore remains to prove that the values of ρ and Λ_2 make the earlier results agree with the results given by the later equations. From these later equations we find, by actual substitution of the values of the quantities e, f, g, h, from § 231,

$$VW\Sigma l(\bar{x} - x) = -(Fp'^2 + Gp'q') \begin{vmatrix} l, & m, & n, & k \\ x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ \xi_{11}, & \eta_{11}, & \zeta_{11}, & \nu_{11} \end{vmatrix}$$

$$+ (Ep'^2 - Gq'^2) \begin{vmatrix} l, & m, & n, & k \\ x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ \xi_{12}, & \eta_{12}, & \zeta_{12}, & \nu_{12} \end{vmatrix}$$

$$+ (Ep'q' + Fq'^2) \begin{vmatrix} l, & m, & n, & k \\ x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ \xi_{22}, & \eta_{22}, & \zeta_{22}, & \nu_{22} \end{vmatrix}.$$

The first determinant on the right-hand side (§ 226)

$$= V(A\xi_{11} + B\eta_{11} + C\zeta_{11} + D\nu_{11}) = V\Sigma A x_{11} = V\Omega.$$

The second and the third, respectively, are equal to $V\Omega'$ and $V\Omega''$. Thus

$$\begin{aligned} VW\Sigma l(\bar{x}-x) &= V\{-\Omega(Fp'^2+Gp'q')+\Omega'(Ep'^2-Gq'^2)+\Omega''(Ep'q'+Fq'^2)\} \\ &= V\begin{vmatrix} Ep'+Fq', & Fp'+Gq' \\ \Omega p'+\Omega' q', & \Omega' p'+\Omega'' q' \end{vmatrix} \\ &= V\begin{vmatrix} Ep'+Fq', & Fp'+Gq' \\ -W\rho q', & W\rho p' \end{vmatrix} \\ &= VW\rho; \end{aligned}$$

in agreement with the result from the earlier form

$$\Sigma l(\bar{x}-x)=\Sigma l(l\rho+A\Lambda_2)=\rho.$$

Proceeding similarly from the same later equations, we find

$$\begin{aligned} VW\Sigma A(\bar{x}-x) = -(Fp'^2+Gp'q') &\begin{vmatrix} A, & B, & C, & D \\ x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ \xi_{11}, & \eta_{11}, & \zeta_{11}, & v_{11} \end{vmatrix} \\ +(Ep'^2-Gq'^2) &\begin{vmatrix} A, & B, & C, & D \\ x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ \xi_{12}, & \eta_{12}, & \zeta_{12}, & v_{12} \end{vmatrix} \\ +(Ep'q'+Fq'^2) &\begin{vmatrix} A, & B, & C, & D \\ x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ \xi_{22}, & \eta_{22}, & \zeta_{22}, & v_{22} \end{vmatrix}. \end{aligned}$$

Now, by § 227, the first determinant on the right-hand side

$$=V(l\xi_{11}+m\eta_{11}+n\zeta_{11}+kv_{11})=V\Sigma lx_{11}=VL.$$

The second and the third, respectively, are equal to VM and VN. Thus

$$\begin{aligned} VW\Sigma A(\bar{x}-x) &= V\{-L(Fp'^2+Gp'q')+M(Ep'^2-Gq'^2)+N(Ep'q'+Fq'^2)\} \\ &= V\begin{vmatrix} Ep'+Fq', & Fp'+Gq' \\ Lp'+Mq', & Mp'+Nq' \end{vmatrix} \\ &= -V\Upsilon, \end{aligned}$$

with the notation of § 234. This result is in accordance with the result from the earlier form, viz.

$$\Sigma A(\bar{x}-x)=\Sigma A(l\rho+A\Lambda)=\Lambda_2;$$

for $\Lambda_2\sigma=\rho\tau$, while (§ 236)

$$\Upsilon\sigma=V,\quad -W\rho\tau=V.$$

The conic-locus of orthogonal centres.

251. The coordinates of the orthogonal centre, that is, the point of intersection of the orthogonal plane at a point O of the surface, with the consecutive orthogonal plane at a consecutive point in a direction p', q', through O, have been given in two forms. In the form (§ 249)

$$VW(\bar{x}-x)=(Ep'+Fq')(e_{12}p'+e_{22}q')-(Fp'+Gq')(e_{11}p'+e_{12}q'),$$

with three others, where

$$VW=Rp'^2+Sp'q'+Tq'^2,$$

the expressions indicate, associated with the point O, a locus of such points for values of p' and q' subject to the relation $\Sigma Ep'^2=1$. Manifestly, they express the four coordinates $\bar{x}-x$, $\bar{y}-y$, $\bar{z}-z$, $\bar{v}-v$, as functions of a single parameter: thus the locus is a curve. Again, we have noted that the two relations

$$\Sigma x_1(\bar{x}-x)=0, \quad \Sigma x_2(\bar{x}-x)=0,$$

are satisfied, and these are the equations of the orthogonal plane; hence the locus lies (as obviously is to be expected) in the orthogonal plane and therefore, being a curve, it is a plane curve. We have seen (§ 249) that the locus is a conic: we shall require a more direct expression for the equation of the conic.

For brevity, we write

$$\bar{\xi}=\bar{x}-x, \quad \bar{\eta}=\bar{y}-y, \quad \bar{\zeta}=\bar{z}-z, \quad \bar{v}=\bar{v}-v,$$

$$\left.\begin{aligned}\xi_1&=Ee_{12}-Fe_{11}\\ \xi_2&=Ee_{22}-Ge_{11}\\ \xi_3&=Fe_{22}-Ge_{12}\end{aligned}\right\},\quad \left.\begin{aligned}\eta_1&=Ef_{12}-Ff_{11}\\ \eta_2&=Ef_{22}-Gf_{11}\\ \eta_3&=Ff_{22}-Gf_{12}\end{aligned}\right\},\quad \left.\begin{aligned}\zeta_1&=Eg_{12}-Fg_{11}\\ \zeta_2&=Eg_{22}-Gg_{11}\\ \zeta_3&=Fg_{22}-Gg_{12}\end{aligned}\right\},\quad \left.\begin{aligned}v_1&=Eh_{12}-Fh_{11}\\ v_2&=Eh_{22}-Gh_{11}\\ v_3&=Fh_{22}-Gh_{12}\end{aligned}\right\},$$

and then, inserting the value of VW, viz. $Rp'^2+Sp'q'+Tq'^2$, the coordinates are given by

$$\left.\begin{aligned}[\xi]&=(R\bar{\xi}-\xi_1)p'^2+(S\bar{\xi}-\xi_2)p'q'+(T\bar{\xi}-\xi_3)q'^2=0\\ [\eta]&=(R\bar{\eta}-\eta_1)p'^2+(S\bar{\eta}-\eta_2)p'q'+(T\bar{\eta}-\eta_3)q'^2=0\\ [\zeta]&=(R\bar{\zeta}-\zeta_1)p'^2+(S\bar{\zeta}-\zeta_2)p'q'+(T\bar{\zeta}-\zeta_3)q'^2=0\\ [v]&=(R\bar{v}-v_1)p'^2+(S\bar{v}-v_2)p'q'+(T\bar{v}-v_3)q'^2=0\end{aligned}\right\}.$$

Now

$$\begin{aligned}\Sigma x_1\xi_1&=E(x_1e_{12}+y_1f_{12}+z_1g_{12}+v_1h_{12})-F(x_1e_{11}+y_1f_{11}+z_1g_{11}+v_1h_{11})\\ &=0;\end{aligned}$$

and similarly,

$$\Sigma x_2\xi_1=0, \quad \Sigma x_1\xi_2=0, \quad \Sigma x_2\xi_2=0, \quad \Sigma x_1\xi_3=0, \quad \Sigma x_2\xi_3=0.$$

Hence, multiplying the equations by x_1, y_1, z_1, v_1, and adding; and by x_2, y_2, z_2, v_2, and adding; we have, in turn,

$$VW\Sigma x_1\bar{\xi} = 0, \quad VW\Sigma x_2\bar{\xi} = 0,$$

that is, as before,

$$\Sigma x_1\bar{\xi} = 0, \quad \Sigma x_2\bar{\xi} = 0.$$

Again, from the results in § 232, we have

$$\begin{aligned} \Sigma\xi_1\xi_{11} &= E\Sigma e_{12}\xi_{11} - F\Sigma e_{11}\xi_{11} = ER, \\ \Sigma\xi_2\xi_{11} &= E\Sigma e_{22}\xi_{11} - G\Sigma e_{11}\xi_{11} = ES, \\ \Sigma\xi_3\xi_{11} &= F\Sigma e_{22}\xi_{11} - G\Sigma e_{12}\xi_{11} = FS - GR, \end{aligned}$$

or, if (§ 240) we write

$$V\Pi = ET - FS + GR,$$

the last equation is

$$\Sigma\xi_3\xi_{11} = ET - V\Pi.$$

Similarly

$$\left.\begin{aligned} \Sigma\xi_1\xi_{12} &= FR \\ \Sigma\xi_2\xi_{12} &= FS + V\Pi \\ \Sigma\xi_3\xi_{12} &= FT \end{aligned}\right\}, \quad \left.\begin{aligned} \Sigma\xi_1\xi_{22} &= GR - V\Pi \\ \Sigma\xi_2\xi_{22} &= GS \\ \Sigma\xi_3\xi_{22} &= GT \end{aligned}\right\}.$$

Hence, multiplying the four equations by ξ_{11}, η_{11}, ζ_{11}, υ_{11}, and adding; by ξ_{12}, η_{12}, ζ_{12}, υ_{12}, and adding; and by ξ_{22}, η_{22}, ζ_{22}, υ_{22}, and adding; we obtain, in turn,

$$R(\Sigma\bar{\xi}\xi_{11} - E)p'^2 + S(\Sigma\bar{\xi}\xi_{11} - E)p'q' + T(\Sigma\bar{\xi}\xi_{11} - E)q'^2 = -V\Pi q'^2,$$

that is,

$$(\Sigma\bar{\xi}\xi_{11} - E)W = -\Pi q'^2\ ;$$

and, similarly,

$$\begin{aligned} (\Sigma\bar{\xi}\xi_{12} - F)W &= \ \ \Pi p'q', \\ (\Sigma\bar{\xi}\xi_{22} - G)W &= -\Pi p'^2. \end{aligned}$$

It is to be noted that these are not three independent equations; for the first is

$$\xi_{11}[\xi] + \eta_{11}[\eta] + \zeta_{11}[\zeta] + \upsilon_{11}[\upsilon] = 0,$$

the second is

$$\xi_{12}[\xi] + \eta_{12}[\eta] + \zeta_{12}[\zeta] + \upsilon_{12}[\upsilon] = 0\,;$$

and therefore, from these two, by the use of the relations (§ 214)

$$\begin{Vmatrix} \xi_{11}, & \eta_{11}, & \zeta_{11}, & \upsilon_{11} \\ \xi_{12}, & \eta_{12}, & \zeta_{12}, & \upsilon_{12} \\ \xi_{22}, & \eta_{22}, & \zeta_{22}, & \upsilon_{22} \end{Vmatrix},$$

we have the third equation

$$\xi_{22}[\xi] + \eta_{22}[\eta] + \zeta_{22}[\zeta] + \upsilon_{22}[\upsilon] = 0.$$

This fact is in accordance with the property already indicated, that there remain only two equations additional to the two deductions

$$\Sigma x_1\bar{\xi} = 0, \quad \Sigma x_2\bar{\xi} = 0,$$

already made from the equations.

Usually the quantity Π, an invariant of the configuration, does not vanish. Consequently, eliminating p' and q', we have

$$(\Sigma\bar{\xi}\xi_{11}-E)(\Sigma\bar{\xi}\xi_{22}-G)=(\Sigma\bar{\xi}\xi_{12}-F)^2,$$

for W is not an evanescent quantity. This is a region of the second order: its section by the plane $\Sigma x_1\bar{\xi}=0$, $\Sigma x_2\bar{\xi}=0$,—a section which constitutes the locus—is therefore a plane curve of the second order, that is, a conic.

To find the fate of the conic should the invariant Π vanish, we proceed as follows. We have

$$\begin{aligned}T\xi_2-S\xi_3&=T(Ee_{22}-Ge_{11})-S(Fe_{22}-Ge_{12})\\&=e_{22}(ET-FS+GR)-G(Re_{22}-Se_{12}+Te_{11})\\&=e_{22}(ET-FS+GR),\end{aligned}$$

by the results in § 232; that is, when Π vanishes,

$$T\xi_2-S\xi_3=0.$$

Similarly, in the same event,

$$S\xi_1-R\xi_2=0;$$

and therefore, on this hypothesis,

$$\frac{\xi_1}{R}=\frac{\xi_2}{S}=\frac{\xi_3}{T}=\xi_0,$$

where ξ_0 denotes the common value of the fractions. We likewise find

$$\frac{\eta_1}{R}=\frac{\eta_2}{S}=\frac{\eta_3}{T}=\eta_0,$$

$$\frac{\zeta_1}{R}=\frac{\zeta_2}{S}=\frac{\zeta_3}{T}=\zeta_0,$$

$$\frac{v_1}{R}=\frac{v_2}{S}=\frac{v_3}{T}=v_0.$$

The four equations then become

$$W(\bar{\xi}-\xi_0)=0,\quad W(\bar{\eta}-\eta_0)=0,\quad W(\bar{\zeta}-\zeta_0)=0,\quad W(\bar{v}-v_0)=0$$

that is, the conic-locus degenerates to a point

$$\bar{\xi}=\xi_0,\quad \bar{\eta}=\eta_0,\quad \bar{\zeta}=\zeta_0,\quad \bar{v}=v_0,$$

in the circumstance that the relation

$$V\Pi=ET-FS+GR=0$$

should hold, exceptionally.

In the general case, that is, when Π does not vanish so that the conic does not degenerate into a point, the conic thus associated with the point O on the surface is styled, by Kommerell[*], the *characteristic* of the surface at that point.

The orthogonal plane has an envelope only if the invariant Π *vanishes.*

252. Before proceeding to the properties of the characteristic conic, especially in association with the lines of curvature on the surface, one negative inference may be made.

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[*] See p. 654 of the memoir already (p. 442, foot-note) cited.

Tangent planes to the surface at two consecutive points intersect (not solely in a point, the customary intersection of two planes, but) in a line; it is the line passing through their respective points of contact. The tangent planes drawn at all points consecutive to a point O (that is, in all directions through O) intersect in the point common to all these lines—that is, in the point O; in other words, the envelope of the tangent plane is the surface itself.

But the preceding investigation shews that, in general, orthogonal planes at two consecutive points on the surface meet only in a point and not a line; and when different points, consecutive to O, are taken on the surface, the different orthogonal planes intersect the initial orthogonal plane in different points, these points lying on the characteristic conic. We therefore infer that, in general, the orthogonal plane of a surface does not possess an envelope.

This result can also be obtained analytically. The equations of the orthogonal plane are

$$\Sigma(\bar{x}-x)\,x_1=0, \quad \Sigma(\bar{x}-x)\,x_2=0,$$

involving the two independent parameters p and q. If this plane had an envelope, the equations of the envelope would be given by combining the first equation with

$$\left.\begin{aligned}\Sigma(\bar{x}-x)\,x_{11}&=\Sigma x_1^2 &&=E\\ \Sigma(\bar{x}-x)\,x_{12}&=\Sigma x_2x_1&&=F\end{aligned}\right\},$$

and, simultaneously, the second equation with

$$\left.\begin{aligned}\Sigma(\bar{x}-x)\,x_{12}&=\Sigma x_1x_2&&=F\\ \Sigma(\bar{x}-x)\,x_{22}&=\Sigma x_2^2 &&=G\end{aligned}\right\}:$$

that is, the equations of the envelope would be given by the five equations

$$\Sigma(\bar{x}-x)\,x_1=0, \quad \Sigma(\bar{x}-x)\,x_2=0,$$

$$\Sigma(\bar{x}-x)\,x_{11}=E, \quad \Sigma(\bar{x}-x)\,x_{12}=F, \quad \Sigma(\bar{x}-x)\,x_{22}=G.$$

Usually these equations cannot coexist: hence, usually, the orthogonal plane does not possess an envelope.

The condition, which must be satisfied in order that the five equations may coexist, is

$$\begin{vmatrix} x_1, & y_1, & z_1, & v_1, & 0\\ x_2, & y_2, & z_2, & v_2, & 0\\ x_{11}, & y_{11}, & z_{11}, & v_{11}, & E\\ x_{12}, & y_{12}, & z_{12}, & v_{12}, & F\\ x_{22}, & y_{22}, & z_{22}, & v_{22}, & G \end{vmatrix}=0:$$

that is,

$$ET-FS+GR=0,$$

where R, S, T, have the former significance (§ 232). The quantity on the left-hand side is equal to $V\Pi$, where Π is the invariant defined in § 240; and, therefore the invariant Π must vanish if the orthogonal plane has an envelope.

But, in the event of the condition

$$\Pi = 0$$

being satisfied, the characteristic conic degenerates to the point

$$X\mathbf{a}^{\frac{1}{2}} = E, \quad Y\mathbf{c}^{\frac{1}{2}} = G,$$

in the orthogonal plane; and the envelope of the orthogonal plane then is the surface which is the locus of this point in the quadruple space.

The characteristic conic: its asymptotes and principal axes.

253. To consider the characteristic conic, it is convenient to change the four coordinate axes of space with the same transformations as before (§ 242), because the curve lies in the orthogonal plane, as does the lemniscate-locus of the centres of circular curvature of geodesics.

The new coordinate axes are given by the transformations

$$\Sigma(\bar{x} - x)\, x_1 = ZE^{\frac{1}{2}}, \quad \Sigma(\bar{x} - x)\, x_2 = VG^{\frac{1}{2}},$$

$$\Sigma(\bar{x} - x)\, \xi_{11} = X\mathbf{a}^{\frac{1}{2}}, \quad \Sigma(\bar{x} - x)\, \xi_{22} = Y\mathbf{c}^{\frac{1}{2}}.$$

The ZV plane is orthogonal to the XY plane; but the axes of X and Y in their plane are not perpendicular, nor are the axes of Z and V in their plane, their respective inclinations being ω and ϖ, where

$$(EG)^{\frac{1}{2}} \cos\omega = -F, \quad (\mathbf{ac})^{\frac{1}{2}} \cos\varpi = -\mathbf{g}.$$

Also we have

$$\Sigma(\bar{x} - x)\, \xi_{12} = \frac{1}{S}(TX\mathbf{a}^{\frac{1}{2}} + RY\mathbf{c}^{\frac{1}{2}}) = U,$$

where U is merely used for brevity. Then the equations of the conic (§ 251) can be taken as

$$\left.\begin{aligned} X\mathbf{a}^{\frac{1}{2}} - E &= -\frac{\Pi}{W} q'^2 \\ U - F &= \ \ \frac{\Pi}{W} p'q' \\ Y\mathbf{c}^{\frac{1}{2}} - G &= -\frac{\Pi}{W} p'^2 \end{aligned}\right\}.$$

But

$$S(U - F) = T(X\mathbf{a}^{\frac{1}{2}} - E) + R(Y\mathbf{c}^{\frac{1}{2}} - G) + V\Pi;$$

and therefore the equation of the conic, now lying in the plane of XY, becomes

$$S^2 (X\mathbf{a}^{\frac{1}{2}} - E)(Y\mathbf{c}^{\frac{1}{2}} - G) = \{T(X\mathbf{a}^{\frac{1}{2}} - E) + R(Y\mathbf{c}^{\frac{1}{2}} - G) + V\Pi\}^2.$$

Let X_0, Y_0, be the centre of this conic—it is a parabola only if $S^2 = 4RT$, a condition not generally satisfied—and let

$$\mu = \frac{\Pi}{S^2 - 4RT},$$

then we have

$$X_0 a^{\frac{1}{2}} - E = 2R\mu, \quad U_0 - F = S\mu, \quad Y_0 c^{\frac{1}{2}} - G = 2T\mu.$$

The parametric equations* of the conic become

$$\left.\begin{aligned} (X - X_0)\,a^{\frac{1}{2}} + 2R\mu &= -\frac{\Pi}{W}\,q'^2 \\ \frac{1}{S}\{T(X - X_0)\,a^{\frac{1}{2}} + R\,(Y - Y_0)\,c^{\frac{1}{2}}\} + S\mu &= \frac{\Pi}{W}\,p'q' \\ (Y - Y_0)\,c^{\frac{1}{2}} + 2T\mu &= -\frac{\Pi}{W}\,p'^2 \end{aligned}\right\},$$

while the Cartesian equation is

$$-aT^2(X - X_0)^2 + (ac)^{\frac{1}{2}}(S^2 - 2RT)(X - X_0)(Y - Y_0) - cR^2(Y - Y_0)^2 = \frac{S^2\Pi^2}{S^2 - 4RT}.$$

(i) The asymptotes of this conic are

$$-aT^2(X - X_0)^2 + (ac)^{\frac{1}{2}}(S^2 - 2RT)(X - X_0)(Y - Y_0) - cR^2(Y - Y_0)^2 = 0;$$

or, if an asymptote be

$$(X - X_0)\,a^{\frac{1}{2}} = \gamma\,(Y - Y_0)\,c^{\frac{1}{2}},$$

the values of γ are given by

$$(\gamma T + R)^2 = \gamma S^2.$$

Now for any point on the conic, we have

$$\frac{(X - X_0)\,a^{\frac{1}{2}} + 2R\mu}{-q'^2} = \frac{1}{S}\,\frac{T(X - X_0)\,a^{\frac{1}{2}} + R(Y - Y_0)\,c^{\frac{1}{2}} + S^2\mu}{p'q'} = \frac{(Y - Y_0)\,c^{\frac{1}{2}} + 2T\mu}{-p'^2},$$

and therefore at a great distance from the centre—that is, in an asymptotic direction through the centre—

$$\frac{(X - X_0)\,a^{\frac{1}{2}}}{-q'^2} = \frac{1}{S}\,\frac{T(X - X_0)\,a^{\frac{1}{2}} + R\,(Y - Y_0)\,c^{\frac{1}{2}}}{p'q'} = \frac{(Y - Y_0)\,c^{\frac{1}{2}}}{-p'^2},$$

or

$$\frac{\gamma}{-q'^2} = \frac{1}{S}\,\frac{\gamma T + R}{p'q'} = \frac{1}{-p'^2}.$$

These verify the relation $(\gamma T + R)^2 = \gamma S^2$, they also give

$$Rp'^2 + Sp'q' = -T\gamma p'^2 = -Tq'^2,$$

that is,

$$Rp'^2 + Sp'q' + Tq'^2 = 0.$$

The left-hand side is the quantity denoted by VW; hence the directions of *the asymptotes of the conic are given by* the relation

$$W = 0.$$

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* The elimination of $X - X_0$ and $Y - Y_0$ merely leads to the known relation (§§ 230, 232)

$$VW = Rp'^2 + Sp'q' + Tq'^2.$$

(ii) The angles between the asymptotes are bisected by the axes of the conic; hence the directions of these axes are (§ 206) given by the equation

$$\begin{vmatrix} 2Rp'+Sq', & Sp'+2Tq' \\ Ep'+Fq', & Fp'+\ Gq' \end{vmatrix} = 0.$$

Now (§ 230)

$$2Rp'q' + Sq'^2 = \frac{\Omega}{\rho}, \quad Sp'^2 + 2Tp'q' = -\frac{\Omega''}{\rho};$$

hence the foregoing equation is

$$\frac{\Omega}{q'\rho}(Fp' + Gq') + \frac{\Omega''}{p'\rho}(Ep' + Fq') = 0,$$

or, as ρ is finite along the axes, this equation effectively is

$$\frac{1}{p'q'}\{(G\Omega + E\Omega'')\,p'q' + F(\Omega p'^2 + \Omega'' q'^2)\} = 0,$$

that is,

$$G\Omega - 2F\Omega' + E\Omega'' = 0.$$

Thus the directions of *the axes of the conic are given by* the relation

$$G\Omega - 2F\Omega' + E\Omega'' = 0.$$

(iii) For brevity, we write

$$X - X_0 = x, \quad Y - Y_0 = y,$$

$$A\lambda = -\mathbf{a}T^2, \quad H\lambda = (\mathbf{ac})^{\frac{1}{2}}(\tfrac{1}{2}S^2 - RT), \quad B\lambda = -\mathbf{c}R^2,$$

where

$$\lambda = \frac{S^2\Pi^2}{S^2 - 4RT} = S^2\Pi\mu;$$

and then the equation of the conic is

$$Ax^2 + 2Hxy + By^2 = 1.$$

(iv) The principal semi-axes, α and β, of the conic are given by

$$\frac{1}{\alpha^2\beta^2} = \frac{AB - H^2}{\sin^2\varpi}$$

$$= \frac{1}{4}\frac{\mathbf{a}^2\mathbf{c}^2V^2(4RT - S^2)^3}{S^4\Pi^4},$$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{A + B - 2H\cos\varpi}{\sin^2\varpi}$$

$$= \frac{\mathbf{ac}(S^2 - 4RT)}{S^2\Pi^2}(\mathbf{g} - \mathbf{b}).$$

The equation of the axes of the conic in its plane is

$$x^2(H - A\cos\varpi) + (B - A)\,xy - (H - B\cos\varpi)\,y^2 = 0.$$

The parametric equation, determining the axial directions, has already (ii, above) been given.

The principal orthogonal centres coincide with the principal centres of circular curvature of geodesics, being the feet of the normals to the conic.

254. This conic is the locus of the orthogonal centre; and the length of the line from O to any point on the conic is the actual orthogonal radius in the direction of the line. When this radius is a maximum or a minimum, it cuts the conic at right angles: that is, the line then is a normal to the conic drawn from O. Now, from any point, four normals can be drawn to a conic (unless the conic be a parabola, when there are only three); thus there are four principal orthogonal radii, given by the four normals. We proceed to prove that these coincide, in magnitude and in direction, with the radii of circular curvature of the four principal geodesics through the point.

The direction of the tangent to a curve $f(X, Y)$, at any point X, Y, on the curve, is given by the equation

$$\frac{\partial f}{\partial X}dX + \frac{\partial f}{\partial Y}dY = 0.$$

The line, which joins this point to the origin and has

$$\overline{Y}X - \overline{X}Y = 0$$

for its equation, is perpendicular to the tangent (and therefore is a normal to the curve at X, Y) if the relation

$$1 + \left(\frac{Y}{X} + \frac{dY}{dX}\right)\cos\varpi + \frac{Y}{X}\frac{dY}{dX} = 0$$

is satisfied, ϖ being the inclination of the axes. This relation is

$$(X + Y\cos\varpi)\frac{\partial f}{\partial Y} - (Y + X\cos\varpi)\frac{\partial f}{\partial X} = 0.$$

Let the curve be the characteristic conic

$$(X\mathbf{a}^{\frac{1}{2}} - E)(Y\mathbf{c}^{\frac{1}{2}} - G) = (U - F)^2.$$

For brevity, we write

$$X\mathbf{a}^{\frac{1}{2}} - E = x, \quad Y\mathbf{c}^{\frac{1}{2}} - G = y, \quad U - F = z,$$

where

$$Sz = Tx + Ry + V\Pi;$$

then the equation of the conic becomes

$$xy = z^2.$$

We have

$$\frac{\partial f}{\partial X} = \mathbf{a}^{\frac{1}{2}}\left(y - 2\frac{T}{S}z\right), \quad \frac{\partial f}{\partial Y} = \mathbf{c}^{\frac{1}{2}}\left(x - 2\frac{R}{S}z\right).$$

Moreover, $(\mathbf{ac})^{\frac{1}{2}}\cos\varpi = -\mathbf{g}$; and therefore

$$X + Y\cos\varpi = \mathbf{a}^{-\frac{1}{2}}\left\{(x+E) - \frac{\mathbf{g}}{\mathbf{c}}(y+G)\right\}, \quad Y + X\cos\varpi = \mathbf{c}^{-\frac{1}{2}}\left\{(y+G) - \frac{\mathbf{g}}{\mathbf{a}}(x+E)\right\}$$

Hence the equation connected with the feet of the four normals to the conic is

$$\Xi = (Sx - 2Rz)\{\mathbf{c}(x+E) - \mathbf{g}(y+G)\} - (Sy - 2Tz)\{\mathbf{a}(y+G) - \mathbf{g}(x+E)\} = 0,$$

and the feet of the four normals are the intersections of the conics

$$xy = z^2, \quad \Xi = 0.$$

255. It will now be shewn that the condition $\Xi = 0$, attaching to points on the characteristic conic, is the equivalent of

$$\Upsilon = 0,$$

being the equation which determines the directions of the lines of curvature on the surface. As usual (§ 234),

$$\begin{aligned} \Upsilon = &\begin{vmatrix} Lp' + Mq', & Mp' + Nq' \\ Ep' + Fq', & Fp' + Gq' \end{vmatrix} \\ = &\, L(Fp'^2 + Gp'q') + M(Gq'^2 - Ep'^2) - N(Ep'q' + Fq'^2). \end{aligned}$$

In establishing the comparison between the expressions Ξ and Υ, we shall need the relations

$$S\mathbf{h} = T\mathbf{a} + R\mathbf{g}, \quad S\mathbf{b} = T\mathbf{h} + R\mathbf{f}, \quad S\mathbf{f} = T\mathbf{g} + R\mathbf{c},$$

$$V\Pi = ET - FS + GR,$$

connected with the fundamental magnitudes; and we shall use the parametric equations (p. 450) of the characteristic conic

$$x = -\frac{\Pi}{W}q'^2, \quad z = \frac{\Pi}{W}p'q', \quad y = -\frac{\Pi}{W}p'^2.$$

We proceed from the expression for Ξ. We have

$$V\Pi\{\mathbf{c}(x+E) - \mathbf{g}(y+G)\} = V\Pi(\mathbf{c}x - \mathbf{g}y) + (\mathbf{c}E - \mathbf{g}G)(Sz - Tx - Ry).$$

On the right-hand side, the coefficient of x

$$= \quad \mathbf{c}V\Pi - T(\mathbf{c}E - \mathbf{g}G) = S(G\mathbf{f} - F\mathbf{c}),$$

and the coefficient of y

$$= -\mathbf{g}V\Pi - R(\mathbf{c}E - \mathbf{g}G) = S(F\mathbf{g} - E\mathbf{f});$$

hence

$$V\Pi\{\mathbf{c}(x+E) - \mathbf{g}(y+G)\} = S\{\mathbf{g}(Fy - Gz) + \mathbf{f}(Gx - Ey) + \mathbf{c}(Ez - Fx)\}.$$

Similarly, we have

$$V\Pi\{\mathbf{a}(y+G) - \mathbf{g}(x+E)\} = S\{\mathbf{a}(Gz - Fy) + \mathbf{h}(Ey - Gx) + \mathbf{g}(Fx - Ez)\}.$$

Consequently, as the expression for Ξ, we find

$$\begin{aligned} \frac{V\Pi}{S}\Xi = &\quad (Fy - Gz)\{\mathbf{g}(Sx - 2Rz) + \mathbf{a}(Sy - 2Tz)\} \\ &+ (Gx - Ey)\{\mathbf{f}(Sx - 2Rz) + \mathbf{h}(Sy - 2Tz)\} \\ &+ (Ez - Fx)\{\mathbf{c}(Sx - 2Rz) + \mathbf{g}(Sy - 2Tz)\}. \end{aligned}$$

Now

$$\mathbf{g}(Sx - 2Rz) + \mathbf{a}(Sy - 2Tz) = S(\mathbf{a}y - 2\mathbf{h}z + \mathbf{g}x)$$

$$= -S\frac{\Pi}{W}(\mathbf{a}p'^2 + 2\mathbf{h}p'q' + \mathbf{g}q'^2)$$

$$= -\frac{S\Pi}{W\rho}L,$$

by the results on p. 372, and, similarly,

$$\mathbf{f}(Sx - 2Rz) + \mathbf{h}(Sy - 2Tz) = -\frac{S\Pi}{W\rho}M,$$

$$\mathbf{c}(Sx - 2Rz) + \mathbf{g}(Sy - 2Tz) = -\frac{S\Pi}{W\rho}N.$$

Hence

$$\frac{V\Pi}{S}\Xi = -\frac{S\Pi}{W\rho}\{L(Fy - Gz) + M(Gx - Ey) + N(Ez - Fx)\}$$

$$= \frac{S\Pi^2}{W^2\rho}\{L(Fp'^2 + Gp'q') + M(Gq'^2 - Fp'^2) - N(Ep'q' + Fq'^2)\}$$

$$= \frac{S\Pi^2}{W^2\rho}\Upsilon;$$

and therefore

$$\Xi = \frac{S^2\Pi}{VW^2\rho}\Upsilon.$$

Now the feet of the four normals, determining the principal orthogonal radii, have been shewn to satisfy the equation

$$\Xi = 0,$$

that is, they arise from directions on the surface given by

$$\Upsilon = 0,$$

which are the directions of the four principal geodesics and therefore determine the four principal radii of circular curvature.

For these four directions, giving four points on the characteristic conic, we have $\Lambda_2 = 0$, where (§ 247) Λ_2 is the distance between the orthogonal centre and the centre of circular curvature of the geodesic. Hence the four principal orthogonal radii at a point coincide in magnitude and in direction with the respective four radii of circular curvature of the principal geodesics; and the directions on the surface determining these orthogonal radii are the same as those determining the geodesic radii. Further, the lemniscate-locus of geodesic centres and the conic-locus of orthogonal centres lie in the orthogonal plane: the four radii from O are normals, at their other extremities, both to the lemniscate and to the conic: and therefore the lemniscate and the conic touch one another at these four principal centres. Moreover, the four centres are the intersection of the conic $\Xi = 0$ and the characteristic conic.

The lemniscate-locus is the pedal of the conic-locus.

256. We have seen that (§ 242) the locus of the geodesic centres of circular curvature is a curve, of the lemniscate type, and that (§ 249) the locus of the orthogonal centres is a conic, both curves lying in the orthogonal plane. If, for any direction through O, the geodesic centre is x_g, y_g, z_g, v_g, and the orthogonal centre is x_0, y_0, z_0, v_0, we have

$$x_g - x = l\rho, \quad x_0 - x = l\rho + A\Lambda_2,$$

and therefore

$$x_0 - x_g = A\Lambda_2;$$

and, similarly,

$$y_0 - y_g = B\Lambda_2,$$

$$z_0 - z_g = C\Lambda_2,$$

$$v_0 - v_g = D\Lambda_2.$$

Also, the line with direction-cosines A, B, C, D, lies in the orthogonal plane and is perpendicular to the line with direction-cosines l, m, n, k. Hence, if G be the geodesic centre, Γ the orthogonal centre, $OG\Gamma$ is a right-angled triangle, with the right angle at G; the lengths of the sides OG, $G\Gamma$, $O\Gamma$, are respectively ρ, $\rho\tau/\sigma$, ρ_0.

Now the locus of G is the lemniscate-curve, and the locus of Γ is a conic and the angle $OG\Gamma$ is a right angle. It is a known proposition* that the pedal of a conic, taken with respect to any point, is a lemniscate-curve; hence there arises the suggestion that, in the present instance, the lemniscate-curve is the pedal, with respect to O, of the characteristic conic. This suggestion is verified by proving that the tangent at Γ to the characteristic conic is a line with direction-cosines A, B, C, D, in space, as follows.

The variables X, Y, are defined by the relations

$$\Sigma(\bar{x} - x)\,\xi_{11} = X\mathbf{a}^{\frac{1}{2}}, \quad \Sigma(\bar{x} - x)\,\xi_{22} = Y\mathbf{c}^{\frac{1}{2}}.$$

Thus along a line parallel to the axis of Y, the variable X is constant, that is, the axis of Y is perpendicular to the line with direction-cosines proportional to ξ_{11}, η_{11}, ζ_{11}, υ_{11}; and, similarly, the axis of X is perpendicular to

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* It is easy to verify that the polar equation of the pedal (that is, the locus of the foot of the perpendicular on the tangent) of a conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

the axes being rectangular and the pedal being taken from the origin, is

$$\begin{vmatrix} a, & h, & g, & \cos\theta \\ h, & b, & f, & \sin\theta \\ g, & f, & c, & -r \\ \cos\theta, & \sin\theta, & -r, & 0 \end{vmatrix} = 0.$$

the line with direction-cosines proportional to $\xi_{22}, \eta_{22}, \zeta_{22}, \upsilon_{22}$. Moreover, the angle ϖ between the axes of X and Y is

$$\pi - \cos^{-1}\frac{\Sigma\xi_{11}\xi_{22}}{(\mathrm{ac})^{\frac{1}{2}}},$$

that is,

$$-(\mathrm{ac})^{\frac{1}{2}}\cos\varpi = \mathrm{g}.$$

On the other hand, the lines with direction-cosines l, m, n, k, and A, B, C, D, are perpendicular to one another. Let the line A, B, C, D, make an angle ϕ with the axis of X; then the line l, m, n, k, makes an angle ϕ with the line $\xi_{22}, \eta_{22}, \zeta_{22}, \upsilon_{22}$, so that

$$\cos\phi = \Sigma l\frac{\xi_{22}}{\mathrm{c}^{\frac{1}{2}}} = N\mathrm{c}^{-\frac{1}{2}}.$$

In that case, the line A, B, C, D, makes an angle $\varpi - \phi$ with the axis of Y, and the line l, m, n, k, makes an angle $\pi - (\varpi - \phi)$ with the line $\xi_{11}, \eta_{11}, \zeta_{11}, \upsilon_{11}$, so that

$$-\cos(\varpi - \phi) = \Sigma l\frac{\xi_{11}}{\mathrm{a}^{\frac{1}{2}}} = L\mathrm{a}^{-\frac{1}{2}}.$$

Consequently, the angle ϕ in question is determined by the equation

$$-\frac{\cos(\varpi - \phi)}{\cos\phi} = \frac{L\mathrm{a}^{-\frac{1}{2}}}{N\mathrm{c}^{-\frac{1}{2}}}.$$

Next, if ψ denote the angle made with the axis of X by the tangent to a curve $f(X, Y) = 0$, we have

$$\frac{dY}{dX} = \frac{\sin\psi}{\sin(\varpi - \psi)},$$

and therefore

$$\frac{\cos(\varpi - \psi)}{\cos\psi} = \frac{\dfrac{dY}{dX} + \cos\varpi}{1 + \dfrac{dY}{dX}\cos\varpi} = -\frac{(\mathrm{ac})^{\frac{1}{2}}\dfrac{\partial f}{\partial X} + \mathrm{g}\dfrac{\partial f}{\partial Y}}{(\mathrm{ac})^{\frac{1}{2}}\dfrac{\partial f}{\partial Y} + \mathrm{g}\dfrac{\partial f}{\partial X}}.$$

The equation of the characteristic conic is

$$xy = z^2,$$

in the notation of § 254. Hence, when the curve $f(X, Y) = 0$ is the characteristic conic, we have

$$(\mathrm{ac})^{\frac{1}{2}}\frac{\partial f}{\partial X} + \mathrm{g}\frac{\partial f}{\partial Y} = \mathrm{ac}^{\frac{1}{2}}\left(y - 2\frac{T}{S}z\right) + \mathrm{gc}^{\frac{1}{2}}\left(x - 2\frac{R}{S}z\right)$$

$$= \mathrm{c}^{\frac{1}{2}}(\mathrm{a}y - 2\mathrm{h}z + \mathrm{g}x),$$

owing to the relation $S\mathbf{h} = T\mathbf{a} + R\mathbf{g}$. When the parametric equations of the conic are used in the form given in § 255, we find

$$(\mathbf{ac})^{\frac{1}{2}} \frac{\partial f}{\partial X} + \mathbf{g} \frac{\partial f}{\partial Y} = - \frac{\Pi}{W} \mathbf{c}^{\frac{1}{2}} (\mathbf{a}p'^2 + 2\mathbf{h}p'q' + \mathbf{g}q'^2) = - \frac{\Pi}{W\rho} L\mathbf{c}^{\frac{1}{2}}.$$

Proceeding similarly, and using the relation $S\mathbf{f} = T\mathbf{g} + R\mathbf{c}$, we find

$$(\mathbf{ac})^{\frac{1}{2}} \frac{\partial f}{\partial Y} + \mathbf{g} \frac{\partial f}{\partial X} = - \frac{\Pi}{W\rho} N\mathbf{a}^{\frac{1}{2}}.$$

Hence, the angle made with the axis of X by the tangent to the characteristic conic is given by the equation

$$\frac{\cos(\varpi - \psi)}{\cos\psi} = - \frac{L\mathbf{c}^{\frac{1}{2}}}{N\mathbf{a}^{\frac{1}{2}}}.$$

Thus

$$\frac{\cos(\varpi - \phi)}{\cos\phi} = \frac{\cos(\varpi - \psi)}{\cos\psi},$$

and therefore

$$\phi = \psi.$$

Hence the tangent at Γ to the conic is the line ΓG; and the lemniscate-curve locus of the geodesic centre is the pedal, with respect to O, of the locus of the orthogonal centre

As there are four normals (either all real, or two of them imaginary) from any point to a conic, it is clear that there are four points of contact between the conic and this pedal lemniscate—a result already (§ 255) inferred.

Moreover, the principal values of ρ_0 are the lengths of these normals: and we have seen that these are the radii of curvature of the four principal geodesics, the equation for which has already (§ 239) been obtained. Because

$$VWl_0\rho_0 = VW(x_0 - x) = (Ep' + Fq')(e_{12}p' + e_{22}q') - (Fp' + Gq')(e_{11}p' + e_{12}q'),$$

with three like equations, we have

$$\begin{aligned} V^2W^2\rho_0^2 = {} & (Ep' + Fq')^2 \Sigma (e_{12}p' + e_{22}q')^2 \\ & - 2(Ep' + Fq')(Fp' + Gq') \Sigma (e_{12}p' + e_{22}q')(e_{11}p' + e_{12}q') \\ & + (Fp' + Gq')^2 \Sigma (e_{11}p' + e_{12}q')^2 \\ = {} & V^2 [(Ep' + Fq')^2 (\mathbf{b}p'^2 + 2\mathbf{f}p'q' + \mathbf{c}q'^2) \\ & - 2(Ep' + Fq')(Fp' + Gq') \{\mathbf{h}p'^2 + (\mathbf{b} + \mathbf{g}) p'q' + \mathbf{f}q'^2\} \\ & + (Fp' + Gq')^2 (\mathbf{a}p'^2 + 2\mathbf{h}p'q' + \mathbf{b}q'^2)]. \end{aligned}$$

It is unnecessary to form the equation which determines the maximum and the minimum values of ρ_0; the equation for these principal values of ρ_0 is the same as the cited equation for the principal values of ρ.

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[*] This result is due to Kommerell, (§ 8 of the memoir quoted on p. 442), who obtained it otherwise.

Two invariantive relations derived through the characteristic conic.

257. Two invariantive relations, derivable through the equations of the characteristic conic, may be noted here.

Denoting the inner extremity of an orthogonal radius by x_0, y_0, z_0, v_0, we have four equations of the type

$$x_0 - x = l\rho + A\Lambda_2.$$

Hence we have

$$\Sigma A\,(x_0 - x) = \Lambda_2 = -\frac{\Upsilon}{W};$$

and therefore one form of the equation ($\Upsilon = 0$) of the directions of lines of principal curvature is

$$\Sigma A\,(x_0 - x) = 0.$$

This, however, is only a difference in form; substitution of the values of A, B, C, D, from § 231, and of $x_0 - x$, $y_0 - y$, $z_0 - z$, $v_0 - v$, from § 249, merely leads to the equation $\Upsilon = 0$, after reductions.

Next, multiplying the four equations in § 248 by ξ_{11}, η_{11}, ζ_{11}, v_{11}, we have

$$\begin{aligned}\Sigma\xi_{11}\,(x_0 - x) &= \rho\Sigma l\xi_{11} + \Lambda_2\Sigma A\,\xi_{11}\\ &= \rho L + \Lambda_2\,\Omega,\end{aligned}$$

or, if X, Y, be the point on the conic in the XY plane,

$$\Sigma\xi_{11}\,(x_0 - x) = X\mathfrak{a}^{\frac{1}{2}} = E - \frac{\Pi}{W}\,q'^2;$$

thus

$$\rho L + \Lambda_2\Omega \;\; - E = -\frac{\Pi}{W}\,q'^2.$$

Similarly

$$\rho M + \Lambda_2\Omega' \; - F = \;\; \frac{\Pi}{W}\,p'q',$$

$$\rho N + \Lambda_2\Omega'' - G = -\frac{\Pi}{W}\,p'^2.$$

Consequently

$$(\rho L + \Lambda_2\Omega - E)(\rho N + \Lambda_2\Omega'' - G) = (\rho M + \Lambda_2\Omega' - F)^2.$$

Hence

$$\begin{aligned}&(\rho L - E)(\rho N - G) - (\rho M - F)^2\\ &\quad + \Lambda_2\,\{\Omega\,(N\rho - G) - 2\Omega'\,(M\rho - F) + \Omega''(L\rho - E)\} + \Lambda_2^2\,(\Omega\Omega'' - \Omega'^2) = 0.\end{aligned}$$

The first line (§ 236) is equal to

$$-V^2\,\frac{\rho^2}{\sigma^2}.$$

Also

$$\Lambda_2 = \frac{\rho\tau}{\sigma}, \quad \Omega'^2 - \Omega\Omega'' = \rho^2 W^2 \pm \frac{V^2}{\tau^2},$$

and therefore the last term also is equal to

$$-V^2\frac{\rho^2}{\sigma^2}.$$

The relation therefore becomes

$$\rho\,(N\Omega-2M\Omega'+L\Omega'')-(G\Omega-2F\Omega'+E\Omega'')=2V^2\frac{\rho}{\sigma\tau},$$

an equation connecting the two covariants

$$N\Omega-2M\Omega'+L\Omega'' \text{ and } G\Omega-2F\Omega'+E\Omega''.$$

Lastly, taking the same initial equations in the form (p. 443)

$$VW\rho_0 l_0=(Ep'+Fq')(e_{12}p'+e_{22}q')-(Fp'+Gq')(e_{11}p'+e_{12}q')$$

with three others, multiplying by ξ_{11}, η_{11}, ζ_{11}, v_{11}, and adding, we have, by the results of § 251,

$$\begin{aligned} VW\rho_0\Sigma l_0\xi_{11} &= (Ep'+Fq')(Rp'+Sq')-(Fp'+Gq')Rq' \\ &= ERp'^2+ESp'q'+(FS-GR)q'^2 \\ &= EVW-V\Pi q'^2 \end{aligned}$$

Now

$$\Sigma(x_0-x)x_1=0,\quad \Sigma(x_0-x)x_2=0,$$

so that

$$\Sigma l_0x_1=0,\quad \Sigma l_0x_2=0,$$

results to be expected from the fact that the line l_0, m_0, n_0, k_0, lies in the orthogonal plane. Hence

$$\Sigma l_0\xi_{11}=\Sigma l_0x_{11}=L_0,$$

introducing a quantity L_0 bearing to ρ_0 the same relation as L bears to ρ. We similarly introduce quantities M_0, N_0: that is, there are three quantities L_0, M_0, N_0, such that

$$\Sigma l_0x_{11}=L_0,\quad \Sigma l_0x_{12}=M_0,\quad \Sigma l_0x_{22}=N_0.$$

Our foregoing formula thus becomes

$$W(\rho_0L_0-E)=-\Pi q'^2;$$

and we obtain, similarly,

$$\begin{aligned} W(\rho_0M_0-F) &= \Pi p'q', \\ W(\rho_0N_0-G) &= -\Pi p'^2, \end{aligned}$$

leading to the simple relations

$$\frac{\rho_0L_0-E}{q'^2}=\frac{\rho_0M_0-F}{-p'q'}=\frac{\rho_0N_0-G}{p'^2}.$$

It follows that there are two covariants $GL_0-2FM_0+EN_0$, $L_0N_0-M_0^2$, such that

$$\left.\begin{aligned} \rho_0(GL_0-2FM_0+EN_0) &= 2V^2-\frac{\Pi\rho\tau}{V} \\ \rho_0^2(L_0N_0-M_0^2) &= V^2-\frac{\Pi\rho\tau}{V} \end{aligned}\right\}.$$

Consecutive orthogonal planes cannot intersect in a line.

258. Thus far, we have dealt with the customary circumstance that the two orthogonal planes, at O and at a consecutive point along the direction p', q', should intersect only in a point. But it is conceivable that, for a particular direction or for a limited number of particular directions, the two orthogonal planes should intersect in a straight line: the possibility must be investigated.

We have seen (§ 249) that the four equations of these two consecutive orthogonal planes can be taken in the form

$$\left.\begin{aligned} \Sigma(\bar{x}-x)\,x_1 &= 0 \\ \Sigma(\bar{x}-x)\,x_2 &= 0 \\ \Sigma(\bar{x}-x)(x_{11}p'+x_{12}q') &= Ep'+Fq' \\ \Sigma(\bar{x}-x)(x_{12}p'+x_{22}q') &= Fp'+Gq' \end{aligned}\right\}.$$

If, then, the two planes can intersect in a line, and not merely in a point, the four equations must be equivalent to only three independent equations. Now (§ 247) the first two of these equations are satisfied by

$$\bar{x}-x=l\Lambda_1+A\Lambda_2,\quad \bar{y}-y=m\Lambda_1+B\Lambda_2,$$
$$\bar{z}-z=n\Lambda_1+C\Lambda_2,\quad \bar{v}-v=k\Lambda_1+D\Lambda_2,$$

where Λ_1 and Λ_2 are two variables, left undetermined by the first two equations. When these values are substituted in the other two equations, they give

$$\left.\begin{aligned} (Lp'+Mq')\,\Lambda_1+(\Omega p'+\Omega' q')\,\Lambda_2 &= Ep'+Fq' \\ (Mp'+Nq')\,\Lambda_1+(\Omega' p'+\Omega'' q')\,\Lambda_2 &= Fp'+Gq' \end{aligned}\right\}.$$

In the circumstance, postulated as a possibility, that the four initial equations shall be equivalent to only three independent equations, these two equations must be equivalent to only one; and therefore they cannot determine the two magnitudes Λ_1 and Λ_2. Consequently, we must have

$$\left\|\begin{matrix} Lp'+Mq', & \Omega p'+\Omega' q', & Ep'+Fq' \\ Mp'+Nq', & \Omega' p'+\Omega'' q', & Fp'+Gq' \end{matrix}\right\|=0,$$

two analytical conditions.

One of these analytical conditions is

$$\left|\begin{matrix} Lp'+Mq', & Ep'+Fq' \\ Mp'+Nq', & Fp'+Gq' \end{matrix}\right|=0,$$

the equation which gives the directions of maximum or minimum circular curvature of superficial geodesics In the notation of § 234, it is represented by the equation

The other analytical condition can be taken as

$$\begin{vmatrix} Lp' + Mq', & \Omega p' + \Omega' q' \\ Mp' + Nq', & \Omega' p' + \Omega'' q' \end{vmatrix} = 0;$$

or, as (§ 230)

$$\begin{aligned} M\Omega - L\Omega' &= (\mathbf{ab} - \mathbf{h}^2)^{\frac{1}{2}} = R \div V, \\ N\Omega' - M\Omega'' &= (\mathbf{bc} - \mathbf{f}^2)^{\frac{1}{2}} = T \div V, \\ L\Omega'' - N\Omega &= (\mathbf{ca} - \mathbf{g}^2)^{\frac{1}{2}} = S \div V, \end{aligned}$$

and, as V is finite, the condition is

$$Rp'^2 + Sp'q' + Tq'^2 = 0,$$

that is, it can be written

$$W = 0.$$

Thus the possibility postulated would require that a direction or directions at O should exist so as to allow coexistent conditions

$$\Upsilon = 0, \quad W = 0.$$

This result can also be obtained otherwise. The two equations in Λ_1 and Λ_2, as in § 247, certainly determine Λ_1, giving

$$\Lambda_1 = \rho.$$

As the two equations are hypothetically to be equivalent to one only, they must leave Λ_2 undetermined. If the equations were independent equations, they would give (§ 247)

$$\Lambda_2 = -\frac{\Upsilon}{W};$$

consequently, the postulated possibility can only emerge if the equations

$$\Upsilon = 0, \quad W = 0,$$

are simultaneously satisfied.

Now the equation $\Upsilon = 0$ is definite and uniquely significant, as regards the circular curvature of superficial geodesics; it gives the directions of the maximum or minimum values of that curvature.

Two alternatives occur in connection with the vanishing of the magnitude W. It may be the fact that W is evanescent on account of intrinsic magnitudes of the surface, so that

$$\mathbf{bc} - \mathbf{f}^2 = 0, \quad \mathbf{ca} - \mathbf{g}^2 = 0, \quad \mathbf{ab} - \mathbf{h}^2 = 0.$$

Then (§ 230)

$$\Omega = 0 \quad \Omega' = 0, \quad \Omega'' = 0;$$

the surface then (§ 228) lies in a flat in the homaloidal quadruple space: that is, it is a surface in homaloidal triple space. We thus return to the Gauss theory of surfaces in that triple space: the surface ceases to be a general surface in quadruple space.

Consequently, there remains for consideration the alternative that the quantity W is not evanescent on account of the intrinsic magnitudes of the surface. The relation

$$W = 0$$

is then a significant equation: we have to enquire whether it can coexist with the equation

$$\Upsilon = 0.$$

We proceed to modify the expression for Υ, viz.

$$\begin{aligned}\Upsilon &= \begin{vmatrix} Lp' + Mq', & Mp' + Nq' \\ Ep' + Fq', & Fp' + Gq' \end{vmatrix} \\ &= L(Fp'^2 + Gp'q') + M(Gq'^2 - Ep'^2) - N(Ep'q' + Fq'^2).\end{aligned}$$

We have had relations (§ 214)

$$R = V(\mathbf{ab} - \mathbf{h}^2)^{\frac{1}{2}}, \qquad S = V(\mathbf{ca} - \mathbf{g}^2)^{\frac{1}{2}}, \qquad T = V(\mathbf{bc} - \mathbf{f}^2)^{\frac{1}{2}},$$

$$ST = V^2(\mathbf{ch} - \mathbf{fg}), \quad -TR = V^2(\mathbf{bg} - \mathbf{fh}), \quad RS = V^2(\mathbf{af} - \mathbf{gh}),$$

$$T\mathbf{a} - S\mathbf{h} + R\mathbf{g} = 0, \quad T\mathbf{h} - S\mathbf{b} + R\mathbf{f} = 0, \quad T\mathbf{g} - S\mathbf{f} + R\mathbf{c} = 0.$$

Further,

$$\begin{aligned} L &= (\mathbf{a}p'^2 + 2\mathbf{h}p'q' + \mathbf{g}q'^2)\,\rho, \\ M &= (\mathbf{h}p'^2 + 2\mathbf{b}p'q' + \mathbf{f}q'^2)\,\rho, \\ N &= (\mathbf{g}p'^2 + 2\mathbf{f}p'q' + \mathbf{c}q'^2)\,\rho; \end{aligned}$$

and therefore

$$TL - SM + RN = 0.$$

We also have had an invariant Π, where

$$V\Pi = ET - FS + GR.$$

Accordingly,

$$\begin{aligned} R\Upsilon &= RL(Fp'^2 + Gp'q') + RM(Gq'^2 - Ep'^2) + (TL - SM)(Ep'q' + Fq'^2) \\ &= L(RFp'^2 + RGp'q' + TEp'q' + TFq'^2) + M(RGq'^2 - REp'^2 - SEp'q' - SFq'^2) \\ &= (Lp'q' + Mq'^2)\,\Pi + (LF - ME)(Rp'^2 + Sp'q' + Tq'^2) \\ &= (LF - ME)\,VW + (Lp'q' + Mq'^2)\,V\Pi. \end{aligned}$$

Similarly

$$\begin{aligned} S\Upsilon &= (LG - NE)\,VW + (Nq'^2 - Lp'^2)\,V\Pi, \\ T\Upsilon &= (MG - NF)\,VW - (Mp'^2 + Np'q')\,V\Pi. \end{aligned}$$

We cannot simultaneously have the relations

$$Lp'q' + Mq'^2 = 0, \quad Nq'^2 - Lp'^2 = 0, \quad Mp'^2 + Np'q' = 0,$$

except in the case when

$$\frac{1}{\rho} = 0,$$

and we have seen that, for general (non-ruled) surfaces, the circular curvature of a geodesic does not vanish. Consequently, if the equations

$$\Upsilon = 0, \quad W = 0,$$

can coexist, we must have the invariantive relation

$$V\Pi = ET - FS + GR = 0.$$

This relation is satisfied, as an evanescent relation, when the surface exists merely in a homaloidal triple space.

If this relation is satisfied, along a particular curve on the surface, or if it is satisfied for the whole range of a surface, the characteristic conic degenerates into a point, for the particular curve or for the whole range of the surface. But the implication is that the surface is not completely general: all the invariants and covariants

$$\left\| \begin{array}{ccccc} E, & \mathrm{a}, & \mathrm{h}, & \mathrm{g}, & L \\ F, & \mathrm{h}, & \mathrm{b}, & \mathrm{f}, & M \\ G, & \mathrm{g}, & \mathrm{f}, & \mathrm{c}, & N \end{array} \right\|$$

then vanish.

Hence, for a general surface, we infer that $\Upsilon = 0$, $W = 0$, do not coexist for possible directions through any point, and therefore that *consecutive orthogonal planes intersect in a point only, not in a line.*

Summary of curvature properties of a surface.

259. As regards the curvature at any point O of a surface in free quadruple homaloidal space, we thus have the following inferences.

(i) For every direction through O, there is the centre of circular curvature of the superficial geodesic through that direction, say the geodesic centre, G.

(ii) For every direction through O, there is the orthogonal centre Γ.

(iii) The triangle $OG\Gamma$ is right-angled at G: the line $G\Gamma$ is parallel to the normal to the flat which osculates the surface along the geodesic: and

$$OG = \rho, \quad G\Gamma = \frac{\rho\tau}{\sigma}, \quad O\Gamma^2 = \rho^2 + \frac{\rho^2\tau^2}{\sigma^2},$$

where $1/\rho$, $1/\sigma$, $1/\tau$, are the circular curvature, the torsion, and the tilt of the geodesic.

(iv) The locus of the geodesic centre is a lemniscate-curve, and the locus of the orthogonal centre is a conic, both in the orthogonal plane at O.

The lemniscate-locus is the pedal, with respect to O, of the conic-locus; and the two loci touch in four points.

(v) Through O there are four directions, being lines of curvature, with various properties: among others, they provide the maximum and the minimum values of the radii of circular curvature of geodesics.

(vi) These maximum and minimum radii of circular curvature of geodesics are (in magnitude and in direction) also the maximum and the minimum values of the orthogonal radii.

(vii) The four directions of the lines of curvature correspond to the four normals drawn from O to the conic; and the four radii of curvature of the corresponding four principal geodesics are the lengths of these four normals.

A surface in n-fold space has no orthogonal centre, when $n > 4$:
the centres of circular curvature of its geodesics.

260. It is important to notice that a surface possesses an orthogonal centre only when the surface exists in a space of four dimensions, either freely in homaloidal space of that range, or within the restrictive range of an amplitude of four dimensions in some wider homaloidal space.

Consider, in particular, a surface in homaloidal space of n dimensions. In that space, a surface (and, therefore also, a plane) is represented by $n-2$ equations. In homaloidal space of four dimensions, the orthogonal centre, for any direction on the surface, is the point of intersection of two planes, orthogonal to the tangent plane of the surface, drawn at consecutive points: or, alternatively, it is the point of intersection of two consecutive configurations orthogonal to the tangent plane.

By the first of the derivations, we should have $n-2$ equations for the first orthogonal plane and $n-2$ further equations for the second orthogonal plane, that is, $2n-4$ equations in all. As the space is of n dimensions, there are only n quantities of the type $\bar{x}-x$, and therefore, as $2n-4>n$ when n is greater than 4, the equations possess no common solution—there is no orthogonal centre.

By the second of the derivations, the complementary orthogonal configuration of $n-2$ dimensions has, for its equations,

$$\Sigma(\bar{x}-x)x_1=0, \quad \Sigma(\bar{x}-x)x_2=0.$$

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* For surfaces in free homaloidal space of n dimensions, reference may be made to a monograph by E. E. Levi, "Saggio sulla teoria delle superficie a due dimensioni immersa in un iperspazio," *Ann. R. Scuola Norm. Sup., Pisa*, vol. x (1908), No. 2, p. 98.

The consecutive orthogonal configuration, also of $n-2$ dimensions, is represented by two further equations. Consequently, the amplitude, represented by the intersection of the two configurations, is of $(2n-4)-n$, that is, $n-4$, dimensions: it is not a point when $n>4$.

On the other hand, whatever be the number of dimensions (not less than four) of the homaloidal space in which a surface exists freely, there is always one (and there is only one) centre of circular curvature for a geodesic through any direction on a surface; there always are four lines of curvature at any point on the surface, with the four corresponding principal values (maximum or minimum values) of that circular curvature. The analysis leading to this result, for a homaloidal space of n dimensions, is almost identical with the analysis leading to the result in homaloidal quadruple space: and, in brief outline, is as follows.

Let the point-variables, the coordinates of a point in the n-dimensional space, be $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{u}$, $\bar{v}$, $\bar{w}$, ...; and let θ denote any one of these variables. There are magnitudes, for a surface, denoted by Γ, Γ', Γ'', Δ, Δ', Δ'', the same in form for the general space as for the quadruple space, save that the symmetrical summations extend over the n variables, instead of only four. Let l, m, n, k, i, j, ... denote direction-cosines of the radius of circular curvature of a geodesic through a direction p', q' on the surface, and let ρ be the length of this radius; then, if t be the direction-cosine corresponding to the variable θ, we find

$$\begin{aligned}\frac{t}{\rho} &= (\theta_{11}-\theta_1\Gamma-\theta_2\Delta)\,p'^2+2\,(\theta_{12}-\theta_1\Gamma'-\theta_2\Delta')\,p'q'+(\theta_{22}-\theta_1\Gamma''-\theta_2\Delta'')\,q'^2\\ &= \vartheta_{11}p'^2+2\vartheta_{12}p'q'+\vartheta_{22}q'^2,\end{aligned}$$

for all the n variables θ and the n associated directions t. Hence, writing (again as for four dimensions, but with n-tuple summation instead of quadruple summation)

$$\begin{aligned}&\Sigma\vartheta_{11}^2 = \mathrm{a}, \quad \Sigma\vartheta_{12}^2 = \mathrm{b}, \quad \Sigma\vartheta_{22}^2 = \mathrm{c},\\ &\Sigma\vartheta_{12}\vartheta_{22} = \mathrm{f}, \quad \Sigma\vartheta_{11}\vartheta_{22} = \mathrm{g}, \quad \Sigma\vartheta_{11}\vartheta_{12} = \mathrm{h},\end{aligned}$$

we have

$$\frac{1}{\rho^2} = \mathrm{a}p'^4+4\mathrm{h}p'^3q'+(2\mathrm{g}+4\mathrm{b})\,p'^2q'^2+4\mathrm{f}p'q'^3+\mathrm{c}q'^4.$$

Consequently, as before, there are four principal directions (lines of curvature), giving maximum or minimum values of ρ on the surface, the values of p' and q' being connected by a relation

$$Ep'^2+2Fp'q'+Gq'^2=1.$$

Hence, for the consideration of the geometry of a surface in free homaloidal space of n dimensions ($n>4$), the pivot of investigation is the centre of circular curvature of a geodesic. The orthogonal centre of a surface has significance only in quadruple space.

Orthogonal centre for m-fold amplitudes in homaloidal 2m-fold space.

261. An orthogonal centre can arise for amplitudes of other dimensions in homaloidal spaces also of other dimensions. The centre of a plane curve is its orthogonal centre, in this use of the term; it is a unique point. We have seen that a surface, in free quadruple homaloidal space, has an orthogonal centre for any direction on the surface; its locus for different directions is a conic, in a plane orthogonal to the tangent plane. It is easy to see, as follows, that a (triple) region, in free sextuple homaloidal space, has an orthogonal centre for any direction in the region, and that its locus for different directions is a cubic surface, in a flat orthogonal to the flat which is tangential to the region.

We denote any point in the sextuple space by coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{u}$, $\bar{v}$, $\bar{w}$; the parameters for the region by p, q, r, and use the customary derivation for derivatives with respect to the parameters. The equations of a tangential flat are

$$\left\| \begin{array}{cccc} \bar{x}-x, & \bar{y}-y, & \dots, & \bar{w}-w \\ x_1\ , & y_1\ , & \dots, & w_1 \\ x_2\ , & y_2\ , & \dots, & w_2 \\ x_3\ , & y_3\ , & \dots, & w_3 \end{array} \right\| = 0:$$

and therefore the equations of the orthogonal flat are

$$\Sigma\,(\bar{x}-x)\,x_1 = 0, \quad \Sigma\,(\bar{x}-x)\,x_2 = 0, \quad \Sigma\,(\bar{x}-x)\,x_3 = 0.$$

For the point-intersection of this flat by a consecutive orthogonal flat in a direction p', q', r', we associate, with these three equations, the three further equations

$$\Sigma\,(\bar{x}-x)\,(x_{11}p' + x_{12}q' + x_{13}r') = Ap' + Hq' + Gr',$$

$$\Sigma\,(\bar{x}-x)\,(x_{12}p' + x_{22}q' + x_{23}r') = Hp' + Bq' + Fr',$$

$$\Sigma\,(\bar{x}-x)\,(x_{13}p' + x_{23}q' + x_{33}r') = Gp' + Fq' + Cr',$$

where A, B, C, F, G, H, are fundamental primary magnitudes for the region, as in § 262. Thus there are six equations, to determine the six coordinates of the point of intersection, associated with a direction p', q', r', in the region.

The locus of this point of intersection, for different directions p', q', r', lies in the flat

$$\Sigma\,(\bar{x}-x)\,x_1 = 0, \quad \Sigma\,(\bar{x}-x)\,x_2 = 0, \quad \Sigma\,(\bar{x}-x)\,x_3 = 0.$$

Further, it satisfies the equation

$$\left| \begin{array}{ccc} \{\Sigma\,(\bar{x}-x)\,x_{11}\} - A, & \{\Sigma\,(\bar{x}-x)\,x_{12}\} - H, & \{\Sigma\,(\bar{x}-x)\,x_{13}\} - G \\ \{\Sigma\,(\bar{x}-x)\,x_{12}\} - H, & \{\Sigma\,(\bar{x}-x)\,x_{22}\} - B, & \{\Sigma\,(\bar{x}-x)\,x_{23}\} - F \\ \{\Sigma\,(\bar{x}-x)\,x_{13}\} - G, & \{\Sigma\,(\bar{x}-x)\,x_{23}\} - F, & \{\Sigma\,(\bar{x}-x)\,x_{33}\} - C \end{array} \right| = 0,$$

a configuration of the third degree. The locus in question is therefore the intersection of this configuration by a flat: that is, the locus is a cubic surface in homaloidal triple space.

Manifestly, there is an orthogonal centre for each direction in an amplitude of m dimensions existing freely in a homaloidal space of $2m$ dimensions; and its locus is an amplitude, of order m and of $m-1$ dimensions, existing in a homaloidal amplitude of m dimensions.

The same result holds for an amplitude of m dimensions existing in a curved amplitude of $2m$ dimensions, when the latter is primary (§ 422) to a homaloidal space of $2m+1$ dimensions.

END OF VOLUME I